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EXISTENCE THEORY FOR NONLINEAR LANGEVIN-TYPE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. This paper is devoted to establishing the solvability of a class of nonlinear Langevin fractional differential equations subject to boundary conditions and formulated in terms of the generalized Caputo proportional fractional derivative. The study is conducted in a general Banach space setting. By combining the Kuratowski measure of noncompactness with Mönch's fixed point theorem, we derive sufficient criteria ensuring the existence of solutions. A concrete example is presented at the end to confirm the effectiveness and practical relevance of the obtained results.

*Keywords.*Differential equations, generalized Caputo proportional fractional derivative, measure of noncompactness, fixed point theorem, Banach space.
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1. Introduction

Fractional calculus has emerged as a powerful mathematical framework that offers new perspectives for describing and analyzing complex phenomena in nature. Owing to its broad applicability, it has attracted considerable interest from both physicists and engineers, leading to substantial developments in theoretical and applied research. It has been widely recognized that fractional differentiation and integration operators provide an effective tool for modeling the memory and hereditary properties exhibited by a wide range of materials and dynamical systems. A substantial body of theoretical and experimental evidence has established that fractional differential equations provide an accurate and efficient framework for describing a wide variety of real-world phenomena. In particular, they have proven to be highly effective in modeling heat transfer processes and anomalous diffusion in thermal systems [25], electrical and electromagnetic phenomena [21, 33], and the hereditary behavior of viscoelastic and rheological materials [7]. Owing to their ability to capture memory effects and nonlocal interactions, fractional models have become an indispensable tool in many branches of science and engineering. For further developments and recent advances in the

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theory and applications of fractional differential equations, we refer the reader to [15, 20, 26, 31, 35, 36, 37]. In recent years, significant attention has been directed toward the analysis of Langevin-type fractional differential equations. The classical Langevin equation, introduced by Paul Langevin in 1908, provides a fundamental mathematical description of stochastic dynamics in fluctuating media and constitutes one of the cornerstones of the theory of Brownian motion [13, 38]. By incorporating fractional-order operators, this model is extended to account for memory and hereditary effects, resulting in a more versatile framework for the representation of fractal and anomalous processes. In particular, the fractional Langevin equation generalizes the standard formulation and gives rise to fractional Gaussian processes characterized by two independent parameters [12, 27].

The mathematical theory of fractional Langevin equations has undergone substantial development in recent years. For instance, in [2], the existence of solutions was established for nonlinear Langevin equations subject to multistrip boundary conditions and nonlocal multipoint constraints. In [17], the authors investigated an antiperiodic boundary value problem involving a Langevin equation with two distinct fractional orders. Additional contributions and further theoretical results on fractional Langevin differential equations can be found in [3, 6, 32].
Motivated by the aforementioned studies and the growing interest in fractional Langevin models, this work is devoted to investigating the existence of solutions for the following nonlinear boundary value problem associated with a Langevin-type fractional differential equation involving the generalized Caputo proportional fractional derivative.

$$ {}_{\eta}^{c}D^{\vartheta,h}({}_{\eta}^{c}D^{\epsilon,h}+\varkappa)(y(t)-H(t,y(t)))=A(y(t))+F(t,y(t)),\ t\in\mathbf{J}=[a,b], \tag{1.1}$$

$$ (y(t)-H(t,y(t)))_{t=a}=(y(t)-H(t,y(t)))_{t=b}=0, \tag{1.2}$$

$$ {}_{\eta}^{c}D^{\epsilon,h}(y(t)-H(t,y(t)))_{t=a}=\mu,\ {}_{\eta}^{c}D^{\epsilon,h}(y(t)-H(t,y(t)))_{t=b}=\upsilon, \tag{1.3}$$

where $a,b\in\mathbb{R}$ with $(a>0), 1<\vartheta<2, 0<\epsilon<1$, $h:\mathbf{J}\to\mathbb{R}_{+}^{*}$ is a increasing differentiable function, the proportionality parameter $\eta\in(0,1]$, ${}_{\eta}^{c}D^{\vartheta,h}$ is the generalized Caputo proportional fractional derivative of order ϑ, $\varkappa\in\mathbb{R}$, and $H,F\in C(\mathbf{J}\times E,E)$ are given functions, A is a bounded linear operator on E and E is a Banach space with the norm $\|.\|$.
To investigate the existence of solutions of the problem (1.1)-(1.3), we use Mönch's fixed point theorem combined with the technique of measures of non-compactness, which constitutes a fundamental approach for establishing solutions to differential equations, for more details see Akhmerov *et al.* [4], Alvàrez [5], Benchohra *et al.* [11], Banaś *et al.* [8, 9, 10], Guo *et al.* [19], Mönch [30], Mönch and Von Harten [22].

2. Preliminaries

In this section, we first state the following definitions, lemmas and some notation. By $C(\mathbf{J},E)$ we denote the Banach space of all continuous functions from $\mathbf{J}$

into E with the norm
$$\|y\|_\infty = \text{Sup}\{\ |y(t)\| : t \in \mathbf{J}\}.$$
Let $L^1(\mathbf{J}, E)$ be the Banach space of measurable functions $y : \mathbf{J} \to E$ which are Bochner integrable, equipped with the norm
$$\|y\|_{L^1} = \int_a^b \|y(t)\|dt.$$
Moreover, for a given set V of functions $v : \mathbf{J} \to E$, let us denote by
$$V(t) = \{v(t),\ v \in V\},\ t \in \mathbf{J}$$
and
$$V(\mathbf{J}) = \{v(t),\ v \in V,\ t \in \mathbf{J}\}.$$

Definition 2.1. A function $f : \mathbf{J} \to E$ is called strongly measurable if there exists a sequence of simple functions $(f_n)_n$, such that
$$\lim_{n\to\infty} \|f_n(t) - f(t)\| = 0.$$

Definition 2.2. A function $f : \mathbf{J} \to E$ is Bochner integrable on $\mathbf{J}$ if it is strongly measurable and
$$\lim_{n\to\infty} \int_{\mathbf{J}} \|f_n(t) - f(t)\|dt = 0,$$
for any sequence of simple functions $(f_n)_n$.
If f is Bochner integrable, then
$$\left\| \int_a^b f(t)dt \right\| \leq \int_a^b \|f(t)\|dt.$$
We refer to [29, 34] for more details.

Definition 2.3. ([8]) A map $f : \mathbf{J} \times E \to E$ is said to be Carathéodory if

(i)**:** $t \longmapsto f(t, y)$ is measurable for all $y \in E$,

(ii)**:** $y \longmapsto f(t, y)$ is continuous for almost each $t \in \mathbf{J}$.

If, in addition,

(iii)**:** for each $r > 0$, there exists $h_r \in L^1(\mathbf{J}, \mathbb{R}_+)$ such that
$$|f(t, y) \leq h_r(t) \ \text{ for all } \ |y| \leq r \ \text{ and almost each } t \in \mathbf{J},$$
then we say that the map is L^1-Carathéodory.

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 2.4. ([8]). Let X be a Banach space and Ω_X the bounded subsets of X. The Kuratowski measure of noncompactness is the map $\Upsilon : \Omega_X \to [0, \infty]$ defined by
$$\Upsilon(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^n B_i \ \textit{and}\ diam(B_i) \leq \epsilon\};\ \textit{here}\ B \in \Omega_X.$$

Properties: The Kuratowski measure of noncompactness satisfies the following properties (for more details see [8])

(a): $\Upsilon(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).

(b): $\Upsilon(B) = \Upsilon(\overline{B})$.
(c): $A \subset B \Rightarrow \Upsilon(A) = \Upsilon(B)$.
(d): $\Upsilon(A + B) \leq \Upsilon(A) + \Upsilon(B)$.
(e): $\Upsilon(cB) = |c|\Upsilon(B); c \in \mathbb{R}$.
(f): $\Upsilon(convB) = \Upsilon(B)$.

Definition 2.5. ([14]) Let $N : F \to E$ be a continuous bounded operator. Then, N is Υ-Lipschitz if there exists a constant $\theta > 0$ such that for all $V \subset F$ we have

$$\Upsilon(N(V)) \leq \theta \Upsilon(V).$$

Lemma 2.6. *([19]) If $V \subset C(\mathbf{J}; E)$ is a bounded and equicontinuous set, then*

(i): *the function $t \to \Upsilon(V(t))$ is continuous on $\mathbf{J}$, and*

$$\Upsilon_c(V) = \sup_{a \leq t \leq b} \Upsilon(V(t)),$$

(ii): $\Upsilon(\int_a^b x(s)ds : x \in V) \leq \int_a^b \Upsilon(V(s))ds,$

where

$$V(s) = \{x(s) : x \in V\}, \ \ s \in \mathbf{J}.$$

For completeness we recall the definition of Caputo derivative of fractional order.

Definition 2.7. ([23],[24]) Let $0 < \eta < 1$, $\vartheta > 0$, and $\Phi \in L^1([a, b], E)$. The left-sided generalized proportional fractional integral with respect to h of order ϑ is defined by

$$({}_\eta I_{a+}^{\vartheta;h}\Phi)(t) = \frac{1}{\eta^\vartheta\,\Gamma(\vartheta)} \int_a^t e^{\frac{\eta-1}{\eta}(h(t)-h(s))}\, h'(s)\,(h(t) - h(s))^{\vartheta-1}\,\Phi(s)\,ds,$$

for $t \in [a, b]$.The Euler gamma function is given by

$$\Gamma(\vartheta) = \int_0^{+\infty} e^{-\tau}\,\tau^{\vartheta-1}\,d\tau, \quad \vartheta > 0.$$

Definition 2.8. ([23],[24]) Let $0 < \eta < 1$, and let $\zeta, \rho : [0, 1] \times \mathbb{R} \to [0, +\infty)$ be continuous functions such that

$$\lim_{\eta\to0^+} \zeta(\eta, t) = 0, \quad \lim_{\eta\to1^-} \zeta(\eta, t) = 1,$$
$$\lim_{\eta\to0^+} \rho(\eta, t) = 1, \quad \lim_{\eta\to1^-} \rho(\eta, t) = 0,$$

and $\zeta(\eta, t) \neq 0$, $\rho(\eta, t) \neq 0$ for each $\eta \in [0, 1]$, $t \in \mathbb{R}$.

Then, the proportional derivative of order η with respect to h of the function Φ is given by

$$({}_\eta D_h\Phi)(t) = \rho(\eta, t)\,\Phi(t) + \zeta(\eta, t)\,\frac{\Phi'(t)}{h'(t)}.$$

In particular, if $\zeta(\eta, t) = \eta$ and $\rho(\eta, t) = 1 - \eta$, then

$$({}_\eta D_h\Phi)(t) = (1 - \eta)\,\Phi(t) + \eta\,\frac{\Phi'(t)}{h'(t)}.$$

Definition 2.9. ([23],[24]) Let $\eta \in (0,1]$. The left-sided generalized Caputo proportional fractional derivative of order $n-1<\vartheta<n$ is defined by

$$({}^{c}_{\eta}D^{\vartheta;h}_{a+}\Phi)(t)=\frac{1}{\eta^{n-\vartheta}\,\Gamma(n-\vartheta)}\int_a^t e^{\frac{\eta-1}{\eta}(h(t)-h(s))}\,h'(s)\,(h(t)-h(s))^{n-\vartheta-1}\,({}_{\eta}D^{n,h}\Phi)(s)\,ds,$$

where $n=\lfloor\vartheta\rfloor+1$ and

$${}_{\eta}D^{n,h}=\underbrace{{}_{\eta}D_h\,{}_{\eta}D_h\cdots{}_{\eta}D_h}_{n\text{ times}}.$$

Lemma 2.10. *([23],[24]) Let $t\in\mathbf{J}$, $\eta\in(0,1]$, $\vartheta,\theta>0$, and $\Phi\in L^1(\mathbf{J},E)$. Then we have*

$${}_{\eta}I^{\vartheta;h}_{a+}\big({}_{\eta}I^{\theta;h}_{a+}\Phi(t)\big)={}_{\eta}I^{\theta;h}_{a+}\big({}_{\eta}I^{\vartheta;h}_{a+}\Phi(t)\big)={}_{\eta}I^{\vartheta+\theta;h}_{a+}\Phi(t).$$

Throughout this paper, as a simplification, we set

$$\Omega^{\theta}_{h}(t,a)=e^{\frac{\eta-1}{\eta}\,(h(t)-h(a))}\,(h(t)-h(a))^{\theta}. \tag{2.1}$$

Lemma 2.11. *([23],[24]) Let $\vartheta>0$, $\theta>0$, and $\eta\in(0,1]$. Then we have:*

(i) $\left({}_{\eta}I^{\vartheta;h}_{a+}\left(e^{\frac{\eta-1}{\eta}(h(\tau)-h(a))}(h(\tau)-h(a))^{\theta-1}\right)\right)(t)=\frac{\Gamma(\theta)}{\eta^{\vartheta}\Gamma(\vartheta+\theta)}e^{\frac{\eta-1}{\eta}(h(t)-h(a))}(h(t)-h(a))^{\vartheta+\theta-1}$;

(ii) $\left({}^{c}_{\eta}D^{\vartheta;h}_{a+}\left(e^{\frac{\eta-1}{\eta}(h(\tau)-h(a))}(h(\tau)-h(a))^{\theta-1}\right)\right)(t)=\frac{\eta^{\vartheta}\Gamma(\theta)}{\Gamma(\theta-\vartheta)}e^{\frac{\eta-1}{\eta}(h(t)-h(a))}(h(t)-h(a))^{\theta-\vartheta-1}$.

Lemma 2.12. *([23],[24]) Let $\vartheta>0$, $\eta\in(0,1]$, and $\Phi\in L^1(\mathbf{J},E)$. Then, we have*

$$\lim_{t\to a}\left({}_{\eta}I^{\vartheta;h}_{a+}\Phi(t)\right)=0.$$

Lemma 2.13. *([28]) Let $\eta\in(0,1]$, $n-1<\vartheta<n$, where $n=\lfloor\vartheta\rfloor+1$. Then, we have*

$${}_{\eta}I^{\vartheta;h}_{a+}\left({}^{C}_{\eta}D^{\vartheta;h}_{a+}\Phi(t)\right)=\Phi(t)-\sum_{k=0}^{n-1}\frac{({}_{\eta}D^{k;h}\Phi)(a)}{\eta^{k}\Gamma(k+1)}\,\Omega^{k}_{h}(t,a).$$

Theorem 2.14. *(**Mönch's** fixed point theorem)([1],[30]). Let D be a bounded, closed and convex subset of a Banach space such that $0\in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V=\overline{conv}N(V)\ or\ V=N(V)\cup\{0\}\Rightarrow\Upsilon(V)=0, \tag{2.2}$$

holds for every subset V of D, then Nhas a fixed point.

3. Main Result

First of all, we define what we mean by a solution of the problem (1.1)-(1.3).

Definition 3.1. A function $y\in(\mathbf{J},E)$ is said to be a solution of (1.1)-(1.3) if y satisfies

$$\begin{gathered}{}^{c}_{\eta}D^{\vartheta,h}({}^{c}_{\eta}D^{\epsilon,h}+\varkappa)(y(t)-H(t,y(t)))=A(y(t))+F(t,y(t)),\,for\ each\ t\in\mathbf{J},\\(y(t)-H(t,y(t)))_{t=a}=(y(t)-H(t,y(t)))_{t=b}=0,\\{}^{c}_{\eta}D^{\eta,h}(y(t)-H(t,y(t)))_{t=a}=\mu,{}^{c}_{\eta}D^{\eta,h}(y(t)-H(t,y(t)))_{t=b}=\upsilon.\end{gathered}$$

To prove the existence of solutions to (1.1)-(1.3), we need the following lemmas.

Lemma 3.2. *Let* $\mathbf{J}=[a,b]$, $1<\vartheta<2, 0<\epsilon<1$, *let* $F,H\in C(\mathbf{J}\times E\to E)$ *and* A *is a bounded linear operator on* E, y *is a solution of problem (1.1)-(1.3) if and only if :*

$$y(t)={}_\eta I^{\vartheta+\epsilon,h}_{a+}[A(y(t))+F(t,y(t))]-\varkappa_\eta I^{\epsilon,h}_{a+}(y(t)-H(t,y(t)))+\mu({}_\eta I^{\epsilon,h}_{a+})e^{\frac{\eta-1}{\eta}(h(t)-h(a))}$$
$$+\Pi e^{\frac{\eta-1}{\eta}}(h(t)-h(a))^{\epsilon+1}+H(t,y(t))$$

where

$$\Pi=\frac{\nu-{}_\eta I^{\vartheta+\epsilon,h}_{a+}(A(y(b))+F(b,y(b)))e^{\frac{\eta-1}{\eta}(h(b)-h(a))}}{\eta^{\epsilon}\Gamma(\epsilon+2)e^{\frac{\eta-1}{\eta}(h(b)-h(a))}(h(b)-h(a))} \tag{3.1}$$

Proof. Consider that $y(.)$ is a solution of the nonlinear boundary value Langevin fractional differential equation (1.1)–(1.3). Applying the operator ${}_\eta I^{\vartheta,h}_{a+}(.)$ on both sides of the problem (1.1)–(1.3) and using Lemma (2.13), we get

$${}^c_\eta D^{\epsilon,h}_{a+}(y(t)-H(t,y(t)))={}_\eta I^{\vartheta,h}_{a+}(A(y(t))+F(t,y(t)))-\varkappa(y(t)-H(t,y(t)))+C_0e^{\frac{\eta-1}{\eta}(h(t)-h(a))} \tag{3.2}$$

$$+C_1\frac{e^{\frac{\eta-1}{\eta}(h(t)-h(a))}}{\eta}(h(t)-h(a)). \tag{3.3}$$

Now, applying the operator ${}_\eta I^{\epsilon,h}_{a+}(.)$ on both sides of the integral equation (1.1)–(1.3) and using Lemma (2.13), we get

$$y(t)-H(t,y(t))={}_\eta I^{\vartheta+\epsilon,h}_{a+}(A(y(t))+F(t,y(t)))-\varkappa_\eta I^{\epsilon,h}_{a+}(y(t)-H(t,y(t))) \tag{3.4}$$

$$+C_0({}_\eta I^{\epsilon,h}_{a+})e^{\frac{\eta-1}{\eta}(h(t)-h(a))}+C_1({}_\eta I^{\epsilon,h}_{a+})\left(\frac{e^{\frac{\eta-1}{\eta}(h(t)-h(a))}}{\eta}(h(t)-h(a))\right) \tag{3.5}$$

$$+C_2e^{\frac{\eta-1}{\eta}(h(t)-h(a))}, \tag{3.6}$$

with C_0,C_1 and $C_2\in\mathbb{R}$.
Using Lemma (2.11) (i), the integral equation (3.4) becomes

$$y(t)-H(t,y(t))={}_\eta I^{\vartheta+\epsilon,h}_{a+}(A(y(t))+F(t,y(t)))-\varkappa_\eta I^{\epsilon,h}_{a+}(y(t)-H(t,y(t))) \tag{3.7}$$

$$+C_0({}_\eta I^{\epsilon,h}_{a+})e^{\frac{\eta-1}{\eta}(h(t)-h(a))}+C_1\frac{e^{\frac{\eta-1}{\eta}(h(t)-h(a))}}{\eta^{\epsilon+1}\Gamma(\epsilon+1)}(h(t)-h(a))^{\epsilon+1} \tag{3.8}$$

$$+C_2e^{\frac{\eta-1}{\eta}(h(t)-h(a))}. \tag{3.9}$$

We now evaluate the constants C_0,C_1, and C_2. In the integral equation (3.7), putting $t=a$, using the Lemma (2.10), and the condition $(y(t)-H(t,y(t))_{t=a}=0$, we get

$$C_2=(y(t)-H(t,y(t))_{t=a}=0.$$

Now we proceed to determine C_0, by applying the operator ${}^c_\eta D^{\epsilon,h}_{a^+}(.)$ on both sides of the integral equation (3.7) with $C_2 = 0$ and using Lemma (2.11) (ii), we get the integral equation (3.2), putting $t = a$ in this equation, using Lemma (2.12), and the initial condition ${}^c_\eta D^{\epsilon,h}_{a^+}(y(t) - H(t, y(t))_{t=a} = \mu$, we get

$$C_0 =^c_\eta D^{\epsilon,h}_{a^+}(y(t) - H(t, y(t))_{t=a} = \mu.$$

This time, putting $t = b$ in the integral equation (3.2) with ($C_0 = \mu$) and using the initial condition ${}^c_\eta D^{\epsilon,h}_{a^+}(y(t) - H(t, y(t))_{t=b} = \nu$, we get

$$\begin{aligned}{}^c_\eta D^{\epsilon,h}_{a^+}(y(t) - H(t, y(t))_{t=b} &=_\eta I^{\vartheta,h}_{a+}(A(y(t)) + F(t, y(t)))_{t=b} - \varkappa(y(t) - H(t, y(t)))_{t=b} \\ &\quad + \mu e^{\frac{\eta-1}{\eta}(h(t)-h(a))} + C_1 \frac{e^{\frac{\eta-1}{\eta}(h(b)-h(a))}}{\eta}(h(b) - h(a)).\end{aligned}$$

This implies that

$$\nu =_\eta I^{\vartheta,h}_{a+}(A(y(b)) + F(b, y(b))) + \mu e^{\frac{\eta-1}{\eta}(h(b)-h(a))} + C_1 \frac{e^{\frac{\eta-1}{\eta}(h(b)-h(a))}}{\eta}(h(b) - h(a)).$$

Therefore

$$C_1 = \eta \left(\frac{\nu -_\eta I^{\vartheta,h}_{a+}(A(y(b)) + F(b, y(b))) - \mu e^{\frac{\eta-1}{\eta}(h(b)-h(a))}}{e^{\frac{\eta-1}{\eta}(h(b)-h(a))}(h(b) - h(a))} \right).$$

Substituting C_0, C_1, and C_2 in (3.7) we obtain

$$\begin{aligned} y(t) - H(t, y(t)) &=_\eta I^{\vartheta+\epsilon,h}_{a+}(A(y(t)) + F(t, y(t))) - \varkappa_\eta I^{\epsilon,h}_{a+}(y(t) - H(t, y(t))) \\ &\quad + \mu({}_\eta I^{\epsilon,h}_{a+})e^{\frac{\eta-1}{\eta}(h(t)-h(a))} \\ &\quad + \frac{\nu -_\eta I^{\vartheta+\epsilon,h}_{a+}(A(y(b)) + F(b, y(b)))e^{\frac{\eta-1}{\eta}(h(b)-h(a))}}{\eta^\epsilon \Gamma(\epsilon + 2)e^{\frac{\eta-1}{\eta}(h(b)-h(a))}(h(b) - h(a))} e^{\frac{\eta-1}{\eta}(h(t)-h(a))}(h(t) - h(a))^{\epsilon+1}. \end{aligned}$$

This implies that

$$\begin{aligned} y(t) &=_\eta I^{\vartheta+\epsilon,h}_{a+}(A(y(t)) + F(t, y(t))) - \varkappa_\eta I^{\epsilon,h}_{a+}(y(t) - H(t, y(t))) \\ &\quad + \mu({}_\eta I^{\epsilon,h}_{a+})e^{\frac{\eta-1}{\eta}(h(t)-h(a))} + \Pi e^{\frac{\eta-1}{\eta}(h(t)-h(a))}(h(t) - h(a))^{\epsilon+1} + H(t, y(t)), \end{aligned}$$

where Π is given by (3.1).
The converse result follows immediately by direct computation. □

We now introduce the following set of conditions

($\mathfrak{C}$1) The functions $F, H \in C(\mathbf{J} \times E, E)$ satisfy the Carathéodory conditions.
($\mathfrak{C}$2) the function $H : \mathbf{J} \times E \to E$ is Υ-Lipschitz, i.e., there existe a constant $\theta > 0$, such that for all $N \subset E$

$$\Upsilon(H(t, N)) \leq \theta \Upsilon(N)$$

($\mathfrak{C}$3) There exists $\xi > 0$ such that for each $y \in C(\mathbf{J}, E)$ and $t \in \mathbf{J}$ we have

$$\|H(t, y)\| \leq \xi \|y\|.$$

($\mathfrak{C}_4$) There exists $\omega \in C(\mathbf{J}, \mathbb{R}^+)$ and $\psi \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that for each ψ being nondecreasing

$$\|H(t, y)\| \leq \omega(t)\psi(\|y\|).$$

($\mathfrak{C}_5$) For every bounded subset $N \subseteq E$ and for each $t \in \mathbf{J}$ we have:

$$\Upsilon(F(t, N)) \leq \omega(t)\Upsilon(N).$$

Theorem 3.3. *Assume that assumptions ($\mathfrak{C}_1$)-($\mathfrak{C}_5$) hold. If*

$$\overline{\omega} = \frac{(\|A\| + \omega^*)(h(b) - h(a))^{\vartheta+\epsilon}}{\eta^{\vartheta+\epsilon}\Gamma(\vartheta + \epsilon + 1)} + \frac{|\varkappa|(h(b) - h(a))^{\epsilon}(1 + \theta)}{\eta^{\epsilon}\Gamma(\epsilon + 1)} + \theta < 1, \tag{3.10}$$

then the problem (1.1)-(1.3) has at least one solution J.

Proof. Transform the problem (1.1)–(1.3) into a fixed point problem. Consider the operator $\Theta : C(\mathbf{J}, E) \to C(\mathbf{J}, E)$ defined by

$$\begin{aligned}\Theta(y)(t) =_{\eta} I_{a+}^{\vartheta+\epsilon,h}(A(y(t)) + F(t, y(t))) - \varkappa_{\eta} I_{a+}^{\epsilon,h}(y(t) - H(t, y(t))) \\ + \mu({}_{\eta}I_{a+}^{\epsilon,h})e^{\frac{\eta-1}{\eta}(h(t)-h(a))} + \Pi e^{\frac{\eta-1}{\eta}(h(t)-h(a))}(h(t) - h(a))^{\epsilon+1} + H(t, y(t)).\end{aligned}$$

Clearly, the fixed points of operator Θ are solutions of problem (1.1)–(1.3). Then we consider the set B_ρ defined as:

$$B_\rho = \{y \in C(\mathbf{J}, E) : \|y\| \leq \rho\},$$

where $\rho \geq \frac{\alpha_1}{1-\alpha_2}$, with

$$\alpha_1 = \frac{(\|A\| + \omega^*)\psi(\rho)}{\eta^{\vartheta+\epsilon}\Gamma(\vartheta + \epsilon + 1)}(h(b) - h(a))^{\vartheta+\epsilon} \tag{3.11}$$

$$+ \frac{|\mu|}{\eta^{\epsilon}\Gamma(\epsilon + 1)}(h(b) - h(a))^{\epsilon} + \Pi(h(b) - h(a))^{\epsilon+1}, \tag{3.12}$$

and

$$\alpha_2 = \frac{|\varkappa|(1 + \xi)}{\eta^{\epsilon}\Gamma(\epsilon + 1)}(h(b) - h(a))^{\epsilon} + \xi. \tag{3.13}$$

Clearly, the subset B_ρ is closed, bounded and convex. We shall show that Θ satisfies the assumptions of Theorem 2.14. The proof will be given in a four of steps.

• *First, we prove that Θ maps B_ρ into itself.*
Let $y \in B_\rho$ and $t \in \mathbf{J}$.

Then by using the fact that $\eta \in (0,1]$ and h is a increasing function we have:.

$$
\begin{aligned}
\|\Theta(y)(t)\| \leq & \frac{1}{\eta^{\vartheta+\epsilon}\Gamma(\vartheta+\epsilon)}\int_a^t e^{\frac{\eta-1}{\eta}(h(t)-h(s))}h'(s)(h(t)-h(s))^{\vartheta+\epsilon-1}\|A(y(s))+F(s,y(s))\|ds \\
& + \frac{|\varkappa|}{\eta^{\epsilon}\Gamma(\epsilon)}\int_a^t e^{\frac{\eta-1}{\eta}(h(t)-h(s))}h'(s)(h(t)-h(s))^{\epsilon-1}\|y(s)\|ds \\
& + \frac{|\varkappa|}{\eta^{\epsilon}\Gamma(\epsilon)}\int_a^t e^{\frac{\eta-1}{\eta}(h(t)-h(s))}h'(s)(h(t)-h(s))^{\epsilon-1}\|H(s,y(s))\|ds \\
& + \frac{|\mu|}{\eta^{\epsilon}\Gamma(\epsilon)}\int_a^t e^{\frac{\eta-1}{\eta}(h(t)-h(s))}h'(s)(h(t)-h(s))^{\epsilon-1}e^{\frac{\eta-1}{\eta}(h(s)-h(a))}ds \\
& + \Pi e^{\frac{\eta-1}{\eta}(h(t)-h(a))}(h(t)-h(a))^{\epsilon+1} + \|H(t,y(t))\| \\
& \leq \frac{(\|A\|+\omega^*)\psi(\rho)}{\eta^{\vartheta+\epsilon}\Gamma(\vartheta+\epsilon+1)}(h(b)-h(a))^{\vartheta+\epsilon} + \frac{|\varkappa|\rho}{\eta^{\epsilon}\Gamma(\epsilon+1)}(h(b)-h(a))^{\epsilon} \\
& + \frac{|\varkappa|}{\eta^{\epsilon}\Gamma(\epsilon+1)}(h(b)-h(a))^{\epsilon}\xi\rho \\
& + \frac{|\mu|}{\eta^{\epsilon}\Gamma(\epsilon+1)}(h(b)-h(a))^{\epsilon}\xi\rho + \Pi(h(b)-h(a))^{\epsilon+1} + \xi\rho \\
& \leq \frac{(\|A\|+\omega^*)\psi(\rho)}{\eta^{\vartheta+\epsilon}\Gamma(\vartheta+\epsilon+1)}(h(b)-h(a))^{\vartheta+\epsilon} \\
& + \frac{|\mu|}{\eta^{\epsilon}\Gamma(\epsilon+1)}(h(b)-h(a))^{\epsilon}\xi\rho + \Pi(h(b)-h(a))^{\epsilon+1} \\
& + \rho\left(\frac{|\varkappa|(1+\xi)}{\eta^{\epsilon}\Gamma(\epsilon+1)}(h(b)-h(a))^{\epsilon} + \xi\right) \\
& = \alpha_1 + \alpha_2\rho \\
& \leq \rho,
\end{aligned}
$$

where α_1 and α_2 are given by (3.11) and (3.13), respectively. Therefore, Θ maps B_ρ into itself.

• *Second, we show that Θ is continuous.*

Let $(y_n)_{n\in\mathbb{N}}$ be a sequence such that $y_n \to y$ in B_ρ. Then for each $t \in \mathbf{J}$, we have:

$$
\begin{aligned}
\|\Theta(y_n)(t)-\Theta(y)(t)\| \leq & \eta I_{a+}^{\vartheta+\epsilon,h}\|A\|\|y_n(t)-y(t)\| + \eta I_{a+}^{\vartheta+\epsilon,h}\|F(t,y_n(t))-F(t,y(t))\| \\
& + \varkappa_\eta I_{a+}^{\epsilon,h}\|y_n(t)-y(t)\| + \varkappa_\eta I_{a+}^{\epsilon,h}\|H(t,y_n(t))-H(t,y(t))\| \\
& + \|H(t,y_n(t))-H(t,y(t))\|.
\end{aligned}
$$

Since A is continuous and F and H is of Carathéodory type, the Lebesgue dominated convergence theorem implies

$$\|\Theta(y_n)-\Theta(y)\|_\infty \to 0 \ as \ n \to \infty.$$

Consequently, Θ is continuous.

• *Now, we establish that $\Theta(B_\rho)$ is equicontinuous .*
Let $\tau_1, \tau_2 \in J, \tau_1 < \tau_2$ and $y \in B_\rho$. Then we have

$$
\begin{aligned}
&\|\Theta(y)(\tau_2) - \Theta(y)(\tau_1)\| \\
&\leq \frac{1}{\eta^{\vartheta+\epsilon}\Gamma(\vartheta+\epsilon)} \int_a^{\tau_1} (\Omega_h^{\vartheta+\epsilon-1}((\tau_2, a) - \Omega_h^{\vartheta+\epsilon-1}(\tau_1, a)))h'(s)\|A(y(s)) + F(s, y(s))\|ds \\
&+ \frac{1}{\eta^{\vartheta+\epsilon}\Gamma(\vartheta+\epsilon)} \int_{\tau_1}^{\tau_2} \Omega^{\vartheta+\epsilon-1}(\tau_2, a)h'(s)\|A(y(s)) + F(s, y(s))\|ds \\
&+ |\varkappa|[\frac{1}{\eta^{\epsilon}\Gamma(\epsilon)} \int_a^{\tau_1} (\Omega_h^{\epsilon-1}((\tau_2, a) - \Omega_h^{\epsilon-1}(\tau_1, a)))h'(s)\|y(s) - H(s, y(s))\|ds \\
&+ \frac{1}{\eta^{\epsilon}\Gamma(\epsilon)} \int_{\tau_1}^{\tau_2} \Omega_h^{\epsilon-1}(\tau_2, a)h'(s)\|y(s) - H(s, y(s))\|ds] \\
&+ |\mu| \ [\frac{1}{\eta^{\epsilon}\Gamma(\epsilon)} \int_a^{\tau_1} (\Omega_h^{\epsilon-1}((\tau_2, a) - \Omega_h^{\epsilon-1}(\tau_1, a)))h'(s)e^{\frac{\eta-1}{\eta}(h(s)-h(a))}ds \\
&+ \frac{1}{\eta^{\epsilon}\Gamma(\epsilon)} \int_{\tau_1}^{\tau_2} \Omega_h^{\epsilon-1}(\tau_2, a)h'(s)e^{\frac{\eta-1}{\eta}(h(s)-h(a))}ds \ ] \\
&+ \Pi \left(\Omega_h^{\epsilon+1}(\tau_2, a) - \Omega_h^{\epsilon+1}(\tau_1, a)\right) + \|H(\tau_2, y(\tau_2)) - H(\tau_1, y(\tau_1))\|
\end{aligned}
$$

where $\Omega_h^{(.)}(t, a)$ is given by (2.1).
By using the continuity of the functions $\Omega_h^{(.)}(t, a)$, H, and by Lebesgue dominated convergence theorem, from the above inequality, we get

$$\|\Theta(y)(\tau_2) - \Theta(y)(\tau_1)\| \to 0 \to 0 \ as \ \tau_2 \to \tau_1.$$

Therefore, $\Theta(B_\rho)$ is equicontinuous.
• *Finally, we check the condition (2.2) of Theorem 2.14.*
Let $W \subseteq \overline{conv}(\Theta(W) \cup 0)$ be a bounded and equicontinuous subset. Then, the function $N(t) = \Upsilon(W(t))$ is continuous on $\mathbf{J}$. Next, by using Lemma2.6, the conditions $(\mathfrak{C}_2)$ and $(\mathfrak{C}_5)$, we obtain

$$
\begin{aligned}
N(t) &= \Upsilon(W(t)) \leq \Upsilon(conv((\Theta W)(t)) \cup \{0\})) \leq \Upsilon((\Theta W)(t)) \\
&\leq \left\{ \frac{1}{\eta^{\vartheta+\epsilon}\Gamma(\vartheta+\epsilon)} \int_a^t e^{\frac{\eta-1}{\eta}(h(t)-h(s))} h'(s) \right\} \\
&\times \{ \big(h(t)-h(s)\big)^{\vartheta+\epsilon-1} (A(y(s)) + F(s,y(s)))ds : y \in W \} \\
&+ \Upsilon\{ \frac{|\varkappa|}{\eta^{\epsilon}\Gamma(\epsilon)} \int_a^t e^{\frac{\eta-1}{\eta}(h(t)-h(s))} h'(s)(h(t)-h(s))^{\epsilon-1}(y(s))ds : y \in W \} \\
&+ \Upsilon \frac{|\varkappa|}{\eta^{\epsilon}\Gamma(\epsilon)} \int_a^t e^{\frac{\eta-1}{\eta}(h(t)-h(s))} h'(s)(h(t)-h(s))^{\epsilon-1}(H(s,y(s)))ds : y \in W \} \\
&+ \Upsilon\{ \frac{|\mu|}{\eta^{\epsilon}\Gamma(\epsilon)} \int_a^t e^{\frac{\eta-1}{\eta}(h(t)-h(s))} h'(s)(h(t)-h(s))^{\epsilon-1} e^{\frac{\eta-1}{\eta}(h(s)-h(a))} ds : t \in \mathbf{J} \\
&+ \Upsilon\{ \Pi e^{\frac{\eta-1}{\eta}(h(t)-h(a))}(h(t)-h(a))^{\epsilon+1} : t \in \mathbf{J} \\
&+ \Upsilon\{ H(t,y(t)) : y \in W \} \\
&\leq \frac{1}{\eta^{\vartheta+\epsilon}\Gamma(\vartheta+\epsilon)} \int_a^t e^{\frac{\eta-1}{\eta}(h(t)-h(s))} h'(s)(h(t)-h(s))^{\vartheta+\epsilon-1} \Upsilon(A(W(s)) + F(s,W(s)))ds \\
&+ \frac{|\varkappa|}{\eta^{\epsilon}\Gamma(\epsilon)} \int_a^t e^{\frac{\eta-1}{\eta}(h(t)-h(s))} h'(s)(h(t)-h(s))^{\epsilon-1}(W(s))ds \\
&+ \frac{|\varkappa|}{\eta^{\epsilon}\Gamma(\epsilon)} \int_a^t e^{\frac{\eta-1}{\eta}(h(t)-h(s))} h'(s)(h(t)-h(s))^{\epsilon-1} \Upsilon(H(s,W(s)))ds \\
&+ \Upsilon(H(t,W(t))) \\
&\leq \frac{(\|A\| + \omega^*)\|N\|}{\eta^{\vartheta+\epsilon}\Gamma(\vartheta+\epsilon)} \eta^{\vartheta+\epsilon}\Gamma(\vartheta+\epsilon) \int_a^t e^{\frac{\eta-1}{\eta}(h(t)-h(s))} h'(s)(h(t)-h(s))^{\vartheta+\epsilon-1} ds \\
&+ \frac{|\varkappa|\|N\|}{\eta^{\epsilon}\Gamma(\epsilon)} \int_a^t e^{\frac{\eta-1}{\eta}(h(t)-h(s))} h'(s)(h(t)-h(s))^{\epsilon-1} ds \\
&+ \frac{\theta|\varkappa|\|N\|}{\eta^{\epsilon}\Gamma(\epsilon)} \int_a^t e^{\frac{\eta-1}{\eta}(h(t)-h(s))} h'(s)(h(t)-h(s))^{\epsilon-1} ds + \theta\|N\| \\
&\leq \left(\frac{(\|A\| + \omega^*)(h(b)-h(a))^{\vartheta+\epsilon}}{\eta^{\vartheta+\epsilon}\Gamma(\vartheta+\epsilon+1)} + \frac{|\varkappa|(h(b)-h(a))^{\epsilon}(1+\theta)}{\eta^{\epsilon}\Gamma(\epsilon+1)} + \theta \right)
\end{aligned}
$$

This implies that

$$\|N\|_\infty = \overline{\omega}\|N\|_\infty,$$

where $\overline{\omega}$ is given by (3.10).

According to condition (3.10), we conclude that $\|N\|_\infty = 0$. Therefore, for all $t \in \mathbf{J}$, we have $\|N\|_\infty = 0$, which implies that $\Upsilon(W(t)) = 0$. Then, $W(t)$ is relatively compact in E. Hence, by using Arzelà-Ascoli theorem, we conclude that W is relatively compact in B_ρ. We observe that, all conditions of Theorem 3.3 are satisfied. As a result, there is a fixed point for the operator $\mathbf{J}$ on B_ρ which solves problem (1.1)-(1.3). The proof is then finished. □

4. An Example

A representative example is provided in this section to clearly demonstrate the applicability and effectiveness of the obtained theoretical results.
Let $\mathbf{J} = [0,1]$ and $E = \{y = (y_1, y_2, ..., y_n, ...) : y_n \to 0\}$ be the Banach space of real sequences $(y_n)_{n\in\mathbb{N}^*}$, such that $y_n \to 0$, with the norm $\|y\|_\infty = \mathsf{sup}_{n\in\mathbb{N}^*}|y_n|$.
We consider the following nonlinear boundary value Langevin fractional differential equation

$$ {}^{c}_{\frac{1}{4}}D^{\frac{7}{4},t}({}^{c}_{\frac{1}{4}}D^{\frac{1}{4},t} + \frac{1}{26})(y(t) - \frac{y(t)}{9+e^t}) = \frac{1}{40}y(t) + \frac{1}{20+t^4}(y^2 + \frac{1}{1+e^{|y_n|}}),\ t \in \mathbf{J} = [0,1], \tag{4.1}$$

$$ y(0) = y(1) = 0, \tag{4.2}$$

$$ {}^{c}_{\frac{1}{4}}D^{\frac{7}{4},t}(y(t) - \frac{y(t)}{9+e^t})_{t=0} = \mu,\ \ {}^{c}_{\frac{1}{4}}D^{\frac{7}{4},t}(y(t) - \frac{y(t)}{9+e^t})_{t=1} = \upsilon. \tag{4.3}$$

Comparing System (4.1)-(4.3) with Problem (1.1)-(1.3). Then, $\vartheta = \frac{7}{4}, \epsilon = \eta = \frac{1}{4}, h(t) = t$ and $\varkappa = \frac{1}{26}$ with $F, H : \mathbf{J} \times E \to E$, such that

$$ F(t,y) = \frac{1}{20+t^4}(y^2 + \frac{1}{1+e^{|y_n|}}),\quad H(t,y) = \frac{y(t)}{9+e^t}, A(y) = \frac{1}{40}y,$$

with $y = \{y_n\} \in E$. Now we check for conditions $(\mathfrak{C}1)$-$(\mathfrak{C}5)$. It is easy to see that the functions F and H satisfy the Caratheodory conditions. We have

$$ \|H(t,y)\| \leq \frac{1}{10}\|y\| \text{ and} \|F(t,y)\| \leq \frac{1}{20+t^4}(\|y\|^2 + 1), \|A(y)\| \leq \frac{1}{40}\|y\|.$$

Then, the conditions $(\mathfrak{C}3)$ and $(\mathfrak{C}4)$ hold with $\omega(t) = \frac{1}{20+t^4}, \omega^* = \frac{1}{20}, \psi(y) = y^2+1$ and $\xi = \frac{1}{20}$.
Moreover, for each bounded set $V \subset E$, we have

$$ \Upsilon(F(t,V)) \leq \omega(t)\Upsilon(V) \text{ and } \Upsilon(H(t,V)) \leq \frac{1}{10}\Upsilon(V).$$

Then, the conditions $(\mathfrak{C}2)$ and $(\mathfrak{C}5)$ are satisfied with $\theta = \frac{1}{10}$.
Next, checking for the condition (3.10). Then, we have

$$ \overline{\omega} = \frac{(\|A\| + \omega^*)(h(b) - h(a))^{\vartheta+\epsilon}}{\eta^{\vartheta+\epsilon}\Gamma(\vartheta+\epsilon+1)} + \frac{|\varkappa|(h(b)-h(a))^{\epsilon}(1+\theta)}{\eta^{\epsilon}\Gamma(\epsilon+1)} + \theta$$

$$ \leq \frac{(\frac{1}{40}+\frac{1}{20})(1)^{\frac{7}{4}+\frac{1}{4}}}{\frac{1}{4}^{\frac{7}{4}+\frac{1}{4}}\Gamma(\frac{7}{4}+\frac{1}{4}+1)} + \frac{|\frac{1}{26}|(1)^{\frac{1}{4}}(1+\frac{1}{10})}{\frac{1}{4}^{\frac{1}{4}}\Gamma(\frac{1}{4}+1)} + \frac{1}{10} \simeq 0,7661 \leq 1.$$

We observe that all the assumptions of Theorem 3.3 are satisfied. Consequently, the nonlinear boundary value Langevin fractional differential equation (4.1)–(4.3) admits at least one solution on $[0,1]$.

5. Conclusion

The theory of nonlinear boundary value Langevin fractional differential equations has been developed in this manuscript. We have established the existence of solutions for a given nonlinear boundary value Langevin fractional differential equation involving the generalized Caputo proportional fractional derivative in arbitrary Banach space. Using the measure of noncompactness approach and Mönch's fixed point theorem, we were able to demonstrate the existence of solutions. Finally, a suitable example effectively illustrates our results.

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