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# BERNSTEIN AND JACKSON INEQUALITIES FOR THE QUATERNIONIC DUNKL TRANSFORM

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Abstract. We study approximation theory for the quaternionic Dunkl transform (QDT) on $\mathbb{R}^2$. We establish Bernstein-type inequalities and Jackson direct theorems, relating the best approximation $Q_\nu(f)_{2,Q}$ to moduli of smoothness. We also prove inverse theorems showing that summability of $Q_n(f)_{2,Q}$ implies Sobolev regularity $f \in W^m_{\mathbf{k}}(\mathbb{R}^2, \mathbb{H})$. These results extend classical Jackson-Bernstein theory to the quaternionic Dunkl setting and provide a bridge between spectral approximation and Dunkl-Sobolev smoothness.

*Keywords.* Quaternionic Dunkl transform; Bernstein inequality; Jackson theorem; inverse theorem; modulus of smoothness.
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## 1. Introduction

Approximation theory and harmonic analysis are deeply connected through Fourier-type transforms, which allow the study of smoothness and spectral properties of functions. Classical results such as Bernstein inequalities, Jackson direct theorems, and their inverse counterparts provide quantitative relationships between the smoothness of a function and the rate of its spectral approximation [13, 15, 14].

The Dunkl transform, introduced to incorporate reflection symmetries and weighted measures, has extended classical Fourier analysis to settings with root systems and reflection groups [8, 9]. On the other hand, quaternionic analysis provides a natural framework for multidimensional and color-valued signals, leading to quaternionic integral transforms such as the quaternion Fourier transform and the quaternion linear canonical transform [6, 7, 1, 2, 3, 4, 5].

Recently, the quaternionic Dunkl transform (QDT) has been introduced as a synthesis of these two directions. The QDT combines the algebraic richness of quaternion-valued functions with the reflection-invariant structure of Dunkl operators, allowing the study of functions on $\mathbb{R}^2$ with quaternionic values under spectral and smoothness constraints. Preliminary results, such as Hardy-type

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theorems, have been established for the two-sided QDT [10], but a systematic approximation theory in this setting has not yet been developed.

The main goal of this paper is to establish a framework of approximation theory for the QDT. We define Dunkl-Sobolev spaces $W^m_{\mathbf{k}}(\mathbb{R}^2, \mathbb{H})$ associated with Dunkl operators and study the best approximation of quaternionic functions in $L^2_{\mathbf{k}}(\mathbb{R}^2, \mathbb{H})$. Using finite difference operators and Dunkl translations, we prove Bernstein-type inequalities and Jackson direct theorems that relate the best approximation $Q_\nu(f)_{2,Q}$ to moduli of smoothness. Moreover, we establish inverse theorems showing that suitable summability conditions on $Q_n(f)_{2,Q}$ imply Sobolev regularity $f \in W^m_{\mathbf{k}}(\mathbb{R}^2, \mathbb{H})$.

These results extend the classical Jackson-Bernstein theory to the quaternionic Dunkl setting, providing a bridge between spectral approximation and Dunkl-Sobolev smoothness. The framework developed here opens new directions for further studies of uncertainty principles, eigenfunction expansions, and applications to quaternionic signal and image processing [11, 12].

The paper is organized as follows. Section 2 recalls the necessary background on Dunkl operators, the two-sided quaternionic Dunkl transform (QDT), and the associated $L^2_{\mathbf{k}}(\mathbb{R}^2, \mathbb{H})$ spaces, including the definitions of Dunkl translations and the Plancherel formula. Section 3 establishes Bernstein-type inequalities for functions with bounded Dunkl spectrum and proves Jackson-type direct approximation theorems, linking the best approximation $Q_\nu(f)_{2,Q}$ to moduli of smoothness. This section also contains inverse theorems showing that summability of the best approximations $Q_n(f)_{2,Q}$ implies membership in Dunkl-Sobolev spaces. Section 4 concludes the paper and discusses potential extensions.

## 2. The Two-Sided Quaternionic Dunkl Transform

Throughout this section, we recall the necessary definitions and properties of the quaternionic Dunkl transform (QDT). The symbol $\mathbf{k}$ denotes a real parameter satisfying $\mathbf{k} > -\frac{1}{2}$.

2.1. **Dunkl Weight and Quaternion Algebra.** Let $R \subset \mathbb{R}$ be a finite root system and $R_+ \subset R$ a positive subsystem. Let $\mathbf{k} : R \to [0, \infty)$ be a multiplicity function invariant under the reflection group associated with $R$. The associated Dunkl weight function is defined by

$$w_{\mathbf{k}}(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2\mathbf{k}(\alpha)}, \quad x \in \mathbb{R}^2.$$

In the rank-one case with two coordinates, this simplifies to

$$w_{\mathbf{k}}(x_1, x_2) = |x_1|^{2k_1} |x_2|^{2k_2},$$

and the weighted measure is $d\mu_{\mathbf{k}}(x_1, x_2) = w_{\mathbf{k}}(x_1) w_{\mathbf{k}}(x_2)\, dx_1\, dx_2$.

The quaternion algebra $\mathbb{H}$ is a four-dimensional real associative algebra. Every quaternion $q \in \mathbb{H}$ can be written as

$$q = q_0 + \mathfrak{i} q_1 + \mathfrak{j} q_2 + \mathfrak{k} q_3, \quad q_0, q_1, q_2, q_3 \in \mathbb{R},$$

where the imaginary units $\mathfrak{i}, \mathfrak{j}, \mathfrak{k}$ satisfy

$$\mathfrak{i}^2 = \mathfrak{j}^2 = \mathfrak{k}^2 = \mathfrak{i}\mathfrak{j}\mathfrak{k} = -1, \quad \mathfrak{i}\mathfrak{j} = \mathfrak{k}, \quad \mathfrak{j}\mathfrak{k} = \mathfrak{i}, \quad \mathfrak{k}\mathfrak{i} = \mathfrak{j},$$
$$\mathfrak{j}\mathfrak{i} = -\mathfrak{k}, \quad \mathfrak{k}\mathfrak{j} = -\mathfrak{i}, \quad \mathfrak{i}\mathfrak{k} = -\mathfrak{j}.$$

The conjugate of $q$ is $\bar{q} = q_0 - \mathfrak{i}q_1 - \mathfrak{j}q_2 - \mathfrak{k}q_3$, and the quaternionic norm is $|q|_Q = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$.

In this paper, we consider quaternion-valued functions $f : \mathbb{R}^2 \to \mathbb{H}$ expressed as $f(x) = f_0(x) + \mathfrak{i}f_1(x) + \mathfrak{j}f_2(x) + \mathfrak{k}f_3(x)$ with real-valued components $f_n : \mathbb{R}^2 \to \mathbb{R}$ for $n = 0, 1, 2, 3$.

The inner product of two quaternion-valued functions $f, g : \mathbb{R}^2 \to \mathbb{H}$ is

$$\langle f, g\rangle = \int_{\mathbb{R}^2} f(x)\, \overline{g(x)}\, d\mu_{\mathbf{k}}(x),$$

and the corresponding norms are

$$\|f\|_{p,Q} = \left(\int_{\mathbb{R}^2} |f(x)|_Q^p \, d\mu_{\mathbf{k}}(x)\right)^{\frac{1}{p}}, \quad 1 \le p < \infty, \qquad \|f\|_{\infty,Q} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^2} |f(x)|_Q.$$

We denote by $L^p(\mathbb{R}^2; \mathbb{H})$ the space of such functions with finite $\|f\|_{p,Q}$ norm. For real-valued functions, $L^p(\mathbb{R}^2; d\mu_{\mathbf{k}})$ is defined analogously with the usual absolute value. The constant $c_{\mathbf{k}} = \frac{\Gamma(\mathbf{k}+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\mathbf{k})}$ appears throughout.

### 2.2. **Quaternionic Dunkl Transform.**

**Definition 2.1.** Let $f \in L^1_{\mathbf{k}}(\mathbb{R}^2; \mathbb{H})$. The two-sided quaternionic Dunkl transform (QDT) of $f$ is defined by

$$\mathcal{F}_{q,D}\{f\}(y) = \int_{\mathbb{R}^2} E_{\mathbf{k}}(x_1, -\mathfrak{i}y_1)\, f(x_1, x_2)\, E_{\mathbf{k}}(x_2, -\mathfrak{j}y_2)\, d\mu_{\mathbf{k}}(x), \qquad x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2,$$

where $E_{\mathbf{k}}$ is the Dunkl kernel associated with the root system of rank one.

For a quaternion-valued function $f = f_0 + \mathfrak{i}f_1 + \mathfrak{j}f_2 + \mathfrak{k}f_3$, the QDT acts componentwise, and we define the pointwise modulus

$$\|\mathcal{F}_{q,D}\{f\}(y)\|_Q = \sqrt{\sum_{n=0}^{3} |\mathcal{F}_{q,D}\{f_n\}(y)|_Q^2}.$$

The corresponding $L^2$-norm is

$$\|\mathcal{F}_{q,D}\{f\}\|_{2,Q} = \left(\int_{\mathbb{R}^2} \|\mathcal{F}_{q,D}\{f\}(y)\|_Q^2 \, d\mu_{\mathbf{k}}(y)\right)^{\frac{1}{2}}.$$

**Definition 2.2** (Inversion formula)**.** Let $f \in L^1_{\mathbf{k}}(\mathbb{R}^2; \mathbb{H}) \cap L^2_{\mathbf{k}}(\mathbb{R}^2; \mathbb{H})$ and assume $\mathcal{F}_{q,D}\{f\} \in L^1_{\mathbf{k}}(\mathbb{R}^2; \mathbb{H})$. Then $f$ can be recovered almost everywhere via

$$f(x_1, x_2) = c_{\mathbf{k}}^2 \int_{\mathbb{R}^2} E_{\mathbf{k}}(x_1, \mathfrak{i}y_1)\, \mathcal{F}_{q,D}\{f\}(y)\, E_{\mathbf{k}}(x_2, \mathfrak{j}y_2)\, d\mu_{\mathbf{k}}(y), \quad x, y \in \mathbb{R}^2.$$

**Definition 2.3** (Plancherel formula)**.** For any $f \in L^2_{\mathbf{k}}(\mathbb{R}^2; \mathbb{H})$, the QDT satisfies the Plancherel identity

$$\|\mathcal{F}_{q,D}\{f\}\|_{2,Q} = \|f\|_{2,Q}.$$

2.3. **Dunkl Operators and Translations.** The Dunkl operators are differential-difference operators defined by

$$\mathrm{D}_{\mathfrak{i},\mathbf{k}} f(x_1,x_2) = \frac{\partial}{\partial x_1} f(x_1,x_2) + \mathbf{k}\,\frac{f(x_1,x_2)-f(-x_1,x_2)}{x_1},$$
$$\mathrm{D}_{\mathfrak{j},\mathbf{k}} f(x_1,x_2) = \frac{\partial}{\partial x_2} f(x_1,x_2) + \mathbf{k}\,\frac{f(x_1,x_2)-f(x_1,-x_2)}{x_2}.$$

These operators satisfy the eigenfunction relations

$$\mathrm{D}_{\mathfrak{i},\mathbf{k}}\big(E_{\mathbf{k}}(x_1,\mathfrak{i}\lambda_1)\big) = \mathfrak{i}\lambda_1\, E_{\mathbf{k}}(x_1,\mathfrak{i}\lambda_1), \qquad \mathrm{D}_{\mathfrak{j},\mathbf{k}}\big(E_{\mathbf{k}}(x_2,\mathfrak{j}\lambda_2)\big) = \mathfrak{j}\lambda_2\, E_{\mathbf{k}}(x_2,\mathfrak{j}\lambda_2).$$

We define the QDT differential operator $D_{\mathbf{k}}$ and its powers by

$$D_{\mathbf{k}} = \mathrm{D}_{\mathfrak{i},\mathbf{k}}\,\mathrm{D}_{\mathfrak{j},\mathbf{k}}, \qquad D_{\mathbf{k}}^r = \mathrm{D}_{\mathfrak{i},\mathbf{k}}^r\,\mathrm{D}_{\mathfrak{j},\mathbf{k}}^r, \quad r\in\mathbb{N}.$$

Following [8], the generalized translation operators for $f \in L^2_{\mathbf{k}}(\mathbb{R}^2, d\mu_{\mathbf{k}})$ are given by

$$\mathrm{T}_{\mathfrak{i},h_1} f(x_1,x_2) = c_{\mathbf{k}}\Bigg(\int_0^\pi f_{e_1}\big(G(x_1,h_1,\varphi_1),x_2\big)\, h^e(x_1,h_1,\varphi_1)\,\sin^{2\mathbf{k}-1}\varphi_1\,d\varphi_1 + \int_0^\pi f_{o_1}\big(G(x_1,h_1,\varphi_1),x_2\big)\, h^o(x_1,h_1,\varphi_1)\,\sin^{2\mathbf{k}-1}\varphi_1\,d\varphi_1\Bigg),$$

and analogously for $\mathrm{T}_{\mathfrak{j},h_2}$, where

$$G(x_i,h_i,\varphi_i) = \sqrt{x_i^2+h_i^2-2|x_ih_i|\cos\varphi_i},$$
$$h^e(x_i,h_i,\varphi_i) = 1-\operatorname{sgn}(x_ih_i)\cos\varphi_i,$$

and

$$h^o(x_i,h_i,\varphi_i) = \begin{cases} \dfrac{(x_i+h_i)\,h^e(x_i,h_i,\varphi_i)}{G(x_i,h_i,\varphi_i)} & \text{if } (x_i,h_i)\neq(0,0),\\ 0 & \text{if } (x_i,h_i)=(0,0). \end{cases}$$

The even and odd parts of $f$ with respect to each variable are

$$f_{e_1}(x_1,x_2) = \tfrac12(f(x_1,x_2)+f(-x_1,x_2)), \quad f_{o_1}(x_1,x_2) = \tfrac12(f(x_1,x_2)-f(-x_1,x_2)),$$
$$f_{e_2}(x_1,x_2) = \tfrac12(f(x_1,x_2)+f(x_1,-x_2)), \quad f_{o_2}(x_1,x_2) = \tfrac12(f(x_1,x_2)-f(x_1,-x_2)).$$

From Lemma 2.5 in [8], we obtain the following fundamental identities:

$$\mathcal{F}_{q,D}\big(\mathrm{T}_{\mathfrak{i},h_1} f\big)(\lambda_1,\lambda_2) = E_{\mathbf{k}}(h_1,\mathfrak{i}\lambda_1)\,\mathcal{F}_{q,D}\{f\}(\lambda_1,\lambda_2), \tag{2.1}$$

$$\mathcal{F}_{q,D}\big(\mathrm{T}_{\mathfrak{j},h_2} f\big)(\lambda_1,\lambda_2) = \mathcal{F}_{q,D}\{f\}(\lambda_1,\lambda_2)\,E_{\mathbf{k}}(h_2,\mathfrak{j}\lambda_2). \tag{2.2}$$

For a quaternion-valued function $f \in L^2_{\mathbf{k}}(\mathbb{R}^2,\mathbb{H})$, we define the shifted (translated) function

$$\tau_{h_1,h_2} f(x_1,x_2) = \mathrm{T}_{\mathfrak{i},h_1}\circ\mathrm{T}_{\mathfrak{j},h_2}\big(f(x_1,x_2)\big).$$

The following norm inequalities hold (see [8] for the real-valued case, extended componentwise to $\mathbb{H}$):

$$\|\mathrm{T}_{\mathfrak{i},h_1} f\|_{2,Q} \le 8\sqrt2\|f\|_{2,Q}, \qquad \|\mathrm{T}_{\mathfrak{j},h_2} f\|_{2,Q} \le 8\sqrt2\|f\|_{2,Q}, \qquad \|\tau_{h_1,h_2} f\|_{2,Q} \le 128\|f\|_{2,Q}.$$

**Lemma 2.4.** *Let $f \in L^2_{\mathbf{k}}(\mathbb{R}^2,\mathbb{H})$. Then for any $h=(h_1,h_2)\in\mathbb{R}^2$,*

$$\mathcal{F}_{q,D}\big(\tau_{h_1,h_2}f\big)(\lambda) = E_{\mathbf{k}}(h_1,\mathfrak{i}\lambda_1)\,\mathcal{F}_{q,D}(f)(\lambda_1,\lambda_2)\,E_{\mathbf{k}}(h_2,\mathfrak{j}\lambda_2).$$

*Proof.* Write $f=f_0+\mathfrak{i}f_1+\mathfrak{j}f_2+\mathfrak{k}f_3$ with real-valued components $f_n$. By linearity of $\mathrm{T}_{\mathfrak{i},h_1}$ and $\mathrm{T}_{\mathfrak{j},h_2}$, and using identities (2.1) and (2.2) together with the quaternionic multiplication rules, the result follows by direct computation. □

2.4. **Sobolev Spaces and K-Functionals.** For $r\in\mathbb{N}$, the Sobolev space associated with $D_{\mathbf{k}}$ is

$$\mathrm{W}^r_{2,\mathbf{k}} = \left\{ f\in L^2_{\mathbf{k}}(\mathbb{R}^2,\mathbb{H}) \;\middle|\; D^m_{\mathbf{k}}f \in L^2_{\mathbf{k}}(\mathbb{R}^2,\mathbb{H}),\quad m=1,2,\dots,r \right\}.$$

For $t>0$, the corresponding $K$-functional is defined by

$$K_m(f,t)_2 = \inf_{g\in \mathrm{W}^m_{2,\mathbf{k}}} \left\{ \|f-g\|_{2,Q} + t\|D^m_{\mathbf{k}}g\|_{2,Q} \right\}.$$

**Lemma 2.5.** *If $f\in\mathrm{W}^r_{2,\mathbf{k}}$, then*

$$\|\mathcal{F}_{q,D}(D^r_{\mathbf{k}}f)(\lambda)\|_{2,Q} = |\lambda_1\lambda_2|^r\,\|\mathcal{F}_{q,D}(f)(\lambda_1,\lambda_2)\|_{2,Q}.$$

*Proof.* For each real-valued component, using the eigenfunction relations and the fact that $\mathcal{F}_{q,D}$ maps $D^r_{\mathbf{k}}$ to multiplication by $(-\mathfrak{i}\lambda_1)^r$ on the left and $(-\mathfrak{j}\lambda_2)^r$ on the right, we have

$$\mathcal{F}_{q,D}(D^r_{\mathbf{k}}f)(\lambda) = (-\mathfrak{i}\lambda_1)^r\,\mathcal{F}_{q,D}\{f\}(\lambda)\,(-\mathfrak{j}\lambda_2)^r.$$

Taking the $\|\cdot\|_{2,Q}$-norm and noting $|\mathfrak{i}|=|\mathfrak{j}|=1$ together with multiplicativity of the quaternionic norm yields the result. □

**Lemma 2.6.** *For $x=(x_1,x_2)\in\mathbb{R}^2$, the following inequalities hold:*

(1) $|E_{\mathbf{k}}(x_1,\mathfrak{i}y_1)|_Q \le 1$ *and* $|E_{\mathbf{k}}(x_2,\mathfrak{j}y_2)|_Q\le 1$, *with equality iff* $x_1=0$ *and* $x_2=0$, *respectively.*
(2) $|1-E_{\mathbf{k}}(x_1,\mathfrak{i}y_1)|_Q \le 2|x_1|$ *and* $|1-E_{\mathbf{k}}(x_2,\mathfrak{j}y_2)|_Q\le 2|x_2|$.
(3) *There exists a constant $c>0$, depending only on $\mathbf{k}$, such that* $|1-E_{\mathbf{k}}(x_1,\mathfrak{i}y_1)|_Q \ge c$ *for all* $|x_1|\ge 1$, *and* $|1-E_{\mathbf{k}}(x_2,\mathfrak{j}y_2)|_Q\ge c$ *for all* $|x_2|\ge 1$.

*Proof.* See Lemma 2.9 in [8]. □

2.5. **Finite Difference Operators.** We define the finite difference operators of first and higher orders as follows:

$$\Delta_{h_1,h_2}f(x) = \tau_{h_1,h_2}f(x) - \mathrm{T}_{\mathfrak{i},h_1}f(x) - \mathrm{T}_{\mathfrak{j},h_2}f(x) + f(x) = (\mathrm{T}_{\mathfrak{i},h_1}-I)(\mathrm{T}_{\mathfrak{j},h_2}-I)f(x),$$

$$\Delta^{m,n}_{h_1,h_2}f(x) = \sum_{k_2=0}^{m}\sum_{k_1=0}^{n}(-1)^{n-k_1}(-1)^{m-k_2}\binom{n}{k_1}\binom{m}{k_2}\mathrm{T}^{k_1}_{\mathfrak{i},h_1}\circ\mathrm{T}^{k_2}_{\mathfrak{j},h_2}f(x),$$

where $\mathrm{T}^0_{\cdot,h_p}f(x) = f(x)$ and $\mathrm{T}^i_{\cdot,h_p}f(x) = \mathrm{T}_{\cdot,h_p}\big(\mathrm{T}^{i-1}_{\cdot,h_p}f(x)\big)$, and $I$ denotes the identity operator on $L^2_{\mathbf{k}}(\mathbb{R}^2,\mathbb{H})$.

For $f\in L^2(\mathbb{R}^2,\mathbb{H})$ and a scalar step $h>0$, we will use the notation $\Delta_h f(x) = (\mathrm{T}_{\mathfrak{i},h}-I)(\mathrm{T}_{\mathfrak{j},h}-I)f(x)$, where the same step $h$ is taken in both directions. When we write $\Delta_{1/\nu}$ with a scalar argument, it is understood that the step vector is $(1/\nu,1/\nu)$.

The following norm inequality holds (see [8]):

$$\|\Delta_h^m f\|_2 \leq 12^{2m}\|f\|_2. \tag{2.3}$$

## 3. Bernstein and Jackson-Type Theorems for the Quaternionic Dunkl Transform

In this section, we establish Bernstein-type inequalities and Jackson direct and inverse theorems in the framework of the two-sided quaternionic Dunkl transform (QDT). These results provide quantitative estimates for approximating quaternion-valued functions in $L^2_{\mathbf{k}}(\mathbb{R}^2, \mathbb{H})$ by functions with bounded spectral support.

### 3.1. **Sobolev Spaces and Modulus of Smoothness.**

Let $m \in \mathbb{N}$. The Sobolev space associated with the quaternionic Dunkl differential operator $D_{\mathbf{k}}$ is

$$W^m_{2,\mathbf{k}}(\mathbb{R}^2, \mathbb{H}) = \Big\{ f \in L^2_{\mathbf{k}}(\mathbb{R}^2, \mathbb{H}) \ : \ D^r_{\mathbf{k}} f \in L^2_{\mathbf{k}}(\mathbb{R}^2, \mathbb{H}), \quad r = 1, \ldots, m \Big\}.$$

For $f \in L^2_{\mathbf{k}}(\mathbb{R}^2, \mathbb{H})$, the modulus of smoothness of order $m \in \mathbb{N}$ is defined by

$$\omega_m(f, \delta)_{2,Q} = \sup_{|h| \leq \delta} \|\Delta_h^m f\|_{2,Q}\,,$$

where $\Delta_h^m$ denotes the $m$-th order finite difference operator associated with the two-sided quaternionic Dunkl translations (see Section 2), and the norm $\|\cdot\|_{2,Q}$ is taken with respect to the weighted measure $d\mu_{\mathbf{k}}$ on $\mathbb{R}^2$.

### 3.2. **Spectral Truncation and Functions with Bounded Spectrum.**

For any $\nu > 0$, define the spectral projection operator $P_\nu$ by

$$P_\nu(f)(x) = \mathcal{F}^{-1}_{q,D}\big\{\mathcal{F}_{q,D}\{f\}(\lambda)\, \chi_\nu(\lambda)\big\}(x),$$

where $\chi_\nu$ is the characteristic function of the square $[-\nu, \nu] \times [-\nu, \nu] \subset \mathbb{R}^2$. Since $P_\nu(f)$ has compact spectral support, it follows that $P_\nu(f)$ is infinitely differentiable in the sense of the Dunkl operator and belongs to $W^m_{2,\mathbf{k}}(\mathbb{R}^2, \mathbb{H})$ for all $m \in \mathbb{N}$.

A function $f \in L^2_{\mathbf{k}}(\mathbb{R}^2, \mathbb{H})$ is said to have bounded spectrum of order $\nu > 0$ if

$$\mathcal{F}_{q,D}\{f\}(\lambda) = 0 \quad \text{for all } |\lambda| > \nu.$$

We denote the set of such functions by $\mathcal{I}_\nu$. The best approximation of $f \in L^2_{\mathbf{k}}(\mathbb{R}^2, \mathbb{H})$ by functions in $\mathcal{I}_\nu$ is

$$Q_\nu(f)_{2,Q} = \inf_{g \in \mathcal{I}_\nu} \|f - g\|_{2,Q},$$

where $\|\cdot\|_{2,Q}$ denotes the $L^2_{\mathbf{k}}$-norm associated with the quaternionic Dunkl measure. The operator $P_\nu$ is the orthogonal projection onto $\mathcal{I}_\nu$.

### 3.3. **Bernstein-Type Inequalities.**

**Theorem 3.1.** *Let $f \in \mathcal{I}_\nu$, i.e., $f$ has bounded spectrum of order $\nu > 0$. Then $D_{\mathbf{k}} f \in \mathcal{I}_\nu$ and*

$$\|D_{\mathbf{k}} f\|_{2,Q} \le \nu^2 \, \|f\|_{2,Q}. \tag{3.1}$$

*More generally, for any integer $m \ge 1$,*

$$\|D_{\mathbf{k}}^m f\|_{2,Q} \le \nu^{2m} \, \|f\|_{2,Q}. \tag{3.2}$$

*Proof.* Since $f \in \mathcal{I}_\nu$, its quaternionic Dunkl transform $\mathcal{F}_{q,D}\{f\}(\lambda)$ is supported in the square $[-\nu,\nu]^2$. By definition of $D_{\mathbf{k}}$ and properties of the QDT,

$$\mathcal{F}_{q,D}\{D_{\mathbf{k}}^m f\}(\lambda) = (-\mathfrak{i}\lambda_1)^m \, \mathcal{F}_{q,D}\{f\}(\lambda) \, (-\mathfrak{j}\lambda_2)^m,$$

so $D_{\mathbf{k}}^m f$ also has spectrum contained in $[-\nu,\nu]^2$, i.e., $D_{\mathbf{k}}^m f \in \mathcal{I}_\nu$. Applying the Plancherel formula for the QDT,

$$\begin{aligned}
\|D_{\mathbf{k}}^m f\|_{2,Q}^2 &= \int_{\mathbb{R}^2} |\mathcal{F}_{q,D}\{D_{\mathbf{k}}^m f\}(\lambda)|_Q^2 \, d\mu_{\mathbf{k}}(\lambda) \\
&= \int_{[-\nu,\nu]^2} |(\lambda_1\lambda_2)^m \, \mathcal{F}_{q,D}\{f\}(\lambda)|_Q^2 \, d\mu_{\mathbf{k}}(\lambda) \\
&\le \nu^{4m} \int_{[-\nu,\nu]^2} |\mathcal{F}_{q,D}\{f\}(\lambda)|_Q^2 \, d\mu_{\mathbf{k}}(\lambda) \\
&= \nu^{4m} \, \|f\|_{2,Q}^2.
\end{aligned}$$

Taking the square root yields (3.2) (and (3.1) for $m = 1$). $\square$

### 3.4. **Jackson-Type Direct Theorem.**

**Theorem 3.2.** *Let $f \in W_{2,\mathbf{k}}^m(\mathbb{R}^2, \mathbb{H})$, where $m$ is a positive integer. Then, for any $\nu > 0$, the best approximation by functions with spectrum bounded by $\nu$ satisfies*

$$Q_\nu(f)_{2,Q} = \inf_{g \in \mathcal{I}_\nu} \|f - g\|_{2,Q} \le c\, \nu^{-2m} \, \omega_k(D_{\mathbf{k}}^m f, 1/\nu)_{2,Q}, \tag{3.3}$$

*where $c > 0$ is a constant independent of $f$ and $\nu$, and $\omega_k$ is the modulus of smoothness of order $k$ defined via the QDT.*

*Proof.* Consider the spectral projection $P_\nu(f)(x) = \mathcal{F}_{q,D}^{-1}\{\mathcal{F}_{q,D}\{f\}(\lambda)\, \chi_\nu(\lambda)\}(x)$, where $\chi_\nu$ is the characteristic function of $[-\nu,\nu]^2$. By the Plancherel formula for QDT,

$$\|f - P_\nu(f)\|_{2,Q}^2 = \int_{\mathbb{R}^2} |(1 - \chi_\nu(\lambda))\mathcal{F}_{q,D}\{f\}(\lambda)|_Q^2 \, d\mu_{\mathbf{k}}(\lambda).$$

**Key observation:** Since $\chi_\nu$ is the characteristic function of $[-\nu,\nu]^2$, we have $\lambda \notin [-\nu,\nu]^2$ if and only if $\max(|\lambda_1|, |\lambda_2|) > \nu$. Equivalently, $|\lambda_1| > \nu$ or $|\lambda_2| > \nu$. Using the inequality $|\varphi_{\mathfrak{i},\lambda_1}(1/\nu)| \le |\lambda_1/\nu|$, we obtain

$$\|f - P_\nu(f)\|_{2,Q} \le c_1 \, \|\Delta_{1/\nu}^{k+m} f\|_{2,Q},$$

where $\Delta_{1/\nu}^{k+m}$ denotes the finite difference operator with uniform step vector $(1/\nu, 1/\nu)$. Applying the QDT analogue of the inequality $\|\Delta_h f\|_{2,Q} \le h^2 \|D_{\mathbf{k}} f\|_{2,Q}$ repeatedly $m$ times gives

$$\|\Delta_{1/\nu}^{k+m} f\|_{2,Q} \le (\nu^{-2})^m \, \|\Delta_{1/\nu}^k D_{\mathbf{k}}^m f\|_{2,Q} \le \nu^{-2m} \, \omega_k(D_{\mathbf{k}}^m f, 1/\nu)_{2,Q}.$$

Combining the inequalities, we obtain

$$\|f - P_\nu(f)\|_{2,Q} \le c\,\nu^{-2m}\,\omega_k(D^m_{\mathbf{k}} f, 1/\nu)_{2,Q}.$$

Since $P_\nu(f) \in \mathcal{I}_\nu$, we conclude

$$Q_\nu(f)_{2,Q} \le \|f - P_\nu(f)\|_{2,Q} \le c\,\nu^{-2m}\,\omega_k(D^m_{\mathbf{k}} f, 1/\nu)_{2,Q}.$$

□

3.5. **Auxiliary Lemmas for Inverse Theorems.**

**Lemma 3.3.** *For $j, k \in \mathbb{N}^*$, the following inequality holds for the QDT best approximation:*

$$2^{2k(j-1)} Q_{2^j}(f)_{2,Q} \le \sum_{\eta=2^{j-1}+1}^{2^j} \eta^{2k-1} Q_\eta(f)_{2,Q}.$$

*Proof.* Observe that

$$\sum_{\eta=2^{j-1}+1}^{2^j} \eta^{2k-1} \ge (2^{j-1})^{2k-1} \cdot 2^{j-1} = 2^{2k(j-1)}.$$

Since $Q_\eta(f)_{2,Q}$ is monotonically decreasing with respect to $\eta$, we have

$$2^{2k(j-1)} Q_{2^j}(f)_{2,Q} \le \sum_{\eta=2^{j-1}+1}^{2^j} \eta^{2k-1} Q_\eta(f)_{2,Q}.$$

□

**Lemma 3.4.** *For $n, k \in \mathbb{N}^*$,*

$$2^k Q_n(f)_{2,Q} \le \frac{c_4}{n^{2k}} \sum_{j=0}^{n} (j+1)^{2k-1} Q_j(f)_{2,Q}.$$

*Proof.* A simple calculation shows

$$\sum_{j=0}^{n} (j+1)^{2k-1} \ge \sum_{j \ge n/2-1}^{n} (j+1)^{2k-1} \ge \left(\frac{n}{2}\right)^{2k-1} \frac{n}{2} = 2^{-2k} n^{2k}.$$

Taking into account this inequality and the fact that $Q_j(f)_{2,Q}$ is decreasing with respect to $j$, we obtain the result. □

**Lemma 3.5.** *Let $\Phi_\nu \in \mathcal{I}_\nu$ satisfy $\|f - \Phi_\nu\|_{2,Q} = Q_\nu(f)_{2,Q}$ for each $\nu \in \mathbb{N}$. Then*

$$\|D^k_{\mathbf{k}} \Phi_{2^{j+1}} - D^k_{\mathbf{k}} \Phi_{2^j}\|_{2,Q} \le 2^{2k(j+1)+1} Q_{2^j}(f)_{2,Q}.$$

*In particular,*

$$\|D^k_{\mathbf{k}} \Phi_1\|_{2,Q} = \|D^k_{\mathbf{k}} \Phi_1 - D^k_{\mathbf{k}} \Phi_0\|_{2,Q} \le 2^{4k+1} Q_0(f)_{2,Q}.$$

*Proof.* By the QDT analogue of Bernstein's inequality and the monotonicity of $Q_\nu(f)_{2,Q}$:

$$\begin{aligned}\|D^k_{\mathbf{k}}\Phi_{2^{j+1}} - D^k_{\mathbf{k}}\Phi_{2^j}\|_{2,Q} &\le 2^{2k(j+1)}\|\Phi_{2^{j+1}} - \Phi_{2^j}\|_{2,Q} \\ &= 2^{2k(j+1)}\|(f-\Phi_{2^j}) - (f-\Phi_{2^{j+1}})\|_{2,Q} \\ &\le 2^{2k(j+1)}\big(Q_{2^j}(f)_{2,Q} + Q_{2^{j+1}}(f)_{2,Q}\big) \\ &\le 2^{2k(j+1)+1}Q_{2^j}(f)_{2,Q}.\end{aligned}$$

The second inequality follows similarly. □

### 3.6. **Inverse Theorem.**

**Theorem 3.6.** *For each function $f \in L^2_{\mathbf{k}}(\mathbb{R}^2, \mathbb{H})$ and every $n \in \mathbb{N}^*$,*

$$\omega_k\Big(f, \frac{1}{n}\Big)_{2,Q} \le \frac{c}{n^{2k}}\sum_{j=0}^{n}(j+1)^{2k-1}Q_j(f)_{2,Q},$$

*where $c = c(k,\alpha) > 0$ is a constant.*

*Proof.* Let $2^m \le n < 2^{m+1}$ for some $m \in \mathbb{N}$. For each $\nu \ge 0$, let $\Phi_\nu \in \mathcal{I}_\nu$ be the best approximation of $f$, i.e., $\|f - \Phi_\nu\|_{2,Q} = Q_\nu(f)_{2,Q}$.

Using the triangle inequality and the modulus of smoothness properties:

$$\begin{aligned}\omega_k\Big(f, \frac{1}{n}\Big)_{2,Q} &\le \omega_k\Big(f - \Phi_{2^{m+1}}, \frac{1}{n}\Big)_{2,Q} + \omega_k\Big(\Phi_{2^{m+1}}, \frac{1}{n}\Big)_{2,Q} \\ &\le 2^k\|f - \Phi_{2^{m+1}}\|_{2,Q} + \omega_k\Big(\Phi_{2^{m+1}}, \frac{1}{n}\Big)_{2,Q} \\ &\le 2^k Q_{2^{m+1}}(f)_{2,Q} + \omega_k\Big(\Phi_{2^{m+1}}, \frac{1}{n}\Big)_{2,Q} \\ &\le 2^k Q_n(f)_{2,Q} + \omega_k\Big(\Phi_{2^{m+1}}, \frac{1}{n}\Big)_{2,Q}.\end{aligned}$$

By the QDT analogues of Lemmas 2.5 and 2.6:

$$\begin{aligned}\omega_k\Big(\Phi_{2^{m+1}}, \frac{1}{n}\Big)_{2,Q} &\le \frac{c_3}{n^{2k}}\|D^k_{\mathbf{k}}\Phi_{2^{m+1}}\|_{2,Q} \\ &\le \frac{c_3}{n^{2k}}\Big\{\|D^k_{\mathbf{k}}\Phi_1 - D^k_{\mathbf{k}}\Phi_0\|_{2,Q} + \sum_{j=0}^{m}\|D^k_{\mathbf{k}}\Phi_{2^{j+1}} - D^k_{\mathbf{k}}\Phi_{2^j}\|_{2,Q}\Big\} \\ &\le \frac{c_3}{n^{2k}}\Big\{2^{4k+1}Q_0(f)_{2,Q} + \sum_{j=0}^{m}2^{2k(j+1)+1}Q_{2^j}(f)_{2,Q}\Big\} \\ &\le \frac{c_3 2^{4k+1}}{n^{2k}}\Big\{Q_0(f)_{2,Q} + \sum_{j=0}^{m}2^{2k(j-1)}Q_{2^j}(f)_{2,Q}\Big\}.\end{aligned}$$

Using the discrete summation lemma (QDT analogue of Lemma 2.4 in [8]):

$$\omega_k\Big(\Phi_{2^{m+1}}, \frac{1}{n}\Big)_{2,Q} \le \frac{c_5}{n^{2k}}\sum_{j=0}^{2^m}(j+1)^{2k-1}Q_j(f)_{2,Q}.$$

Therefore, combining the estimates:

$$\omega_k\left(f, \frac{1}{n}\right)_{2,Q} \le 2^k Q_n(f)_{2,Q} + \frac{c_5}{n^{2k}} \sum_{j=0}^{n} (j+1)^{2k-1} Q_j(f)_{2,Q}.$$

Finally, by the monotonicity of $Q_\nu(f)_{2,Q}$ and the summation lemma,

$$\omega_k\left(f, \frac{1}{n}\right)_{2,Q} \le \frac{c}{n^{2k}} \sum_{j=0}^{n} (j+1)^{2k-1} Q_j(f)_{2,Q}.$$

□

3.7. **Numerical Illustration.** We briefly summarize the numerical procedure used to illustrate the Jackson-type theorems. (Full implementation details are provided in the supplementary material.) For a given quaternion-valued signal $f$ with components $f_0, f_1, f_2, f_3$ defined on a spatial grid $x \in [-5, 5]$, we compute the spectral best approximation error $Q_\nu(f)_{2,Q}$ and the modulus of smoothness $\omega_1(f, 1/\nu)_{2,Q}$ for spectral radii $\nu = 1, 2, \ldots, 20$ using an FFT-based approximation of the QDT. The spectral mask $M_\nu$ is the characteristic function of $[-\nu, \nu]^2$.

The numerical results are displayed in Figure 1, showing the decay of the best spectral approximation error and the modulus of smoothness with respect to $\nu$ on a log-log scale. The observed asymptotic equivalence $Q_\nu(f)_{2,Q} \asymp \omega_1(f, 1/\nu)_{2,Q}$ confirms the theoretical predictions of Theorems 3.2 and 3.6.

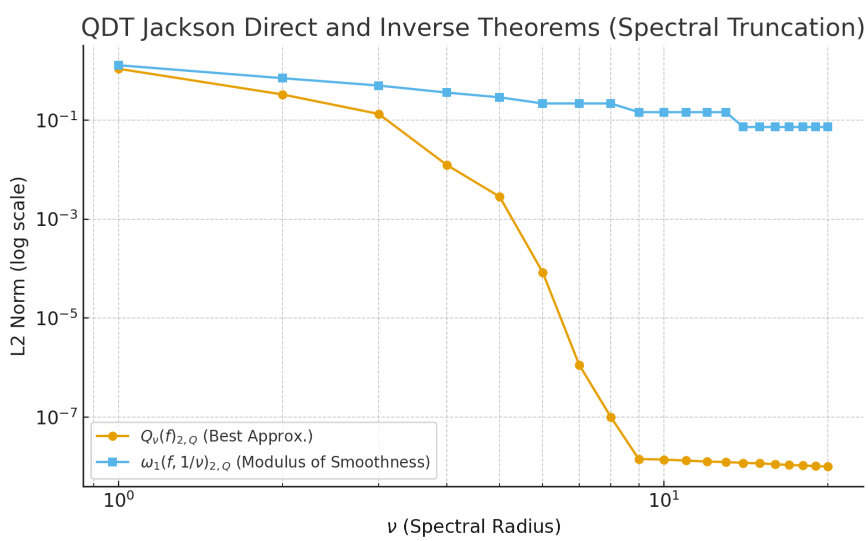


FIGURE 1. Best spectral approximation error $Q_\nu(f)_{2,Q}$ (solid line) and modulus of smoothness $\omega_1(f, 1/\nu)_{2,Q}$ (dashed line) versus spectral radius $\nu$ on a log-log scale. The parallel decay confirms the Jackson-type direct and inverse theorems.

3.8. **Inverse Theorem for Higher Derivatives.**

**Theorem 3.7.** *Suppose* $f \in L^2_{\mathbf{k}}(\mathbb{R}^2, \mathbb{H})$ *and*

$$\sum_{j=1}^{\infty} j^{2m-1} Q_j(f)_{2,Q} < \infty, \quad m \in \mathbb{N}^*.$$

*Then* $f \in W^m_{\mathbf{k}}(\mathbb{R}^2, \mathbb{H})$ *and, for every positive integer* $n$,

$$\omega_k\Big(D^m_{\mathbf{k}} f, \frac{1}{n}\Big)_{2,Q} \leq C\Bigg\{\frac{1}{n^{2k}} \sum_{j=0}^{n} (j+1)^{2(k+m)-1} Q_j(f)_{2,Q} + \sum_{j=n+1}^{\infty} j^{2m-1} Q_j(f)_{2,Q}\Bigg\},$$

*where* $C = C(k, m, \alpha) > 0$.

*Proof.* Let $2^s \leq n < 2^{s+1}$ for some $s \in \mathbb{N}$. For each $r \leq m$, consider the series

$$D^r_{\mathbf{k}} \Phi_1 + \sum_{j=0}^{\infty} \big(D^r_{\mathbf{k}} \Phi_{2^{j+1}} - D^r_{\mathbf{k}} \Phi_{2^j}\big).$$

Using the QDT analogues of Lemmas 2.5 and 2.6, this series converges in $L^2_{\mathbf{k}}(\mathbb{R}^2, \mathbb{H})$ because

$$\sum_{j=0}^{\infty} \|D^r_{\mathbf{k}} \Phi_{2^{j+1}} - D^r_{\mathbf{k}} \Phi_{2^j}\|_{2,Q} \leq \sum_{j=1}^{\infty} j^{2r-1} Q_j(f)_{2,Q} < \infty.$$

Moreover, we have $f = \Phi_1 + \sum_{j=0}^{\infty} (\Phi_{2^{j+1}} - \Phi_{2^j})$, where the series converges in $L^2_{\mathbf{k}}$ and hence in $\mathcal{D}'(\mathbb{R}^2, \mathbb{H})$. Since $D_{\mathbf{k}}$ is continuous on $\mathcal{D}'$, we obtain

$$D^r_{\mathbf{k}} f = D^r_{\mathbf{k}} \Phi_1 + \sum_{j=0}^{\infty} \big(D^r_{\mathbf{k}} \Phi_{2^{j+1}} - D^r_{\mathbf{k}} \Phi_{2^j}\big) \in L^2_{\mathbf{k}},$$

so $f \in W^m_{\mathbf{k}}(\mathbb{R}^2, \mathbb{H})$.

For the modulus of smoothness, using the triangle inequality:

$$\omega_k\Big(D^m_{\mathbf{k}} f, \frac{1}{n}\Big)_{2,Q} \leq \omega_k\Big(D^m_{\mathbf{k}} f - D^m_{\mathbf{k}} \Phi_{2^{s+1}}, \frac{1}{n}\Big)_{2,Q} + \omega_k\Big(D^m_{\mathbf{k}} \Phi_{2^{s+1}}, \frac{1}{n}\Big)_{2,Q}.$$

By Lemmas 2.5 and 2.6 for QDT, the first term is bounded as

$$\omega_k\Big(D^m_{\mathbf{k}} f - D^m_{\mathbf{k}} \Phi_{2^{s+1}}, \frac{1}{n}\Big)_{2,Q} \leq c_6 \sum_{j=2^s+1}^{\infty} j^{2m-1} Q_j(f)_{2,Q}.$$

The second term is estimated by

$$\omega_k\Big(D^m_{\mathbf{k}} \Phi_{2^{s+1}}, \frac{1}{n}\Big)_{2,Q} \leq \frac{c_7}{n^{2k}} \sum_{j=0}^{2^s} (j+1)^{2(k+m)-1} Q_j(f)_{2,Q}.$$

Combining these estimates and noting that $2^s \leq n < 2^{s+1}$ yields the desired inequality. □

## 4. Conclusion

In this work we have extended classical results of approximation theory to the setting of the Quaternionic Dunkl Transform (QDT). Specifically, we established three main families of results.

First, we proved Bernstein-type inequalities showing that functions with bounded QDT spectrum have their derivatives controlled in the $L^2_{\mathbf{k}}$ norm. More precisely, for any $f \in \mathcal{I}_\nu$ we obtained $\|D^m_{\mathbf{k}} f\|_{2,Q} \leq \nu^{2m}\|f\|_{2,Q}$, which generalizes the classical Bernstein inequality to the quaternionic Dunkl setting.

Second, we established Jackson-type direct theorems providing quantitative estimates for the best approximation of functions in Sobolev-type spaces $W^m_{\mathbf{k}}(\mathbb{R}^2, \mathbb{H})$ in terms of their modulus of smoothness. The inequality $Q_\nu(f)_{2,Q} \leq c\,\nu^{-2m}\,\omega_k(D^m_{\mathbf{k}} f, 1/\nu)_{2,Q}$ captures the precise rate at which functions can be approximated by bandlimited quaternionic signals.

Third, we proved inverse theorems linking the decay of best approximation errors $Q_n(f)_{2,Q}$ to membership in Sobolev spaces. Under suitable summability conditions on $Q_n(f)_{2,Q}$, we showed that $f$ belongs to $W^m_{\mathbf{k}}(\mathbb{R}^2, \mathbb{H})$ and obtained explicit estimates for higher-order QDT derivatives. These inverse results complete the approximation-theoretic characterization of smoothness in the QDT framework.

These results together form a comprehensive framework for approximation and smoothness analysis in the QDT setting, generalizing well-known theorems from classical Fourier and octonion Fourier analysis to the quaternionic Dunkl context. The theoretical foundation laid here opens the door for further investigations.

Promising future directions include the development of numerical algorithms for signal reconstruction using QDT and its associated prolate spheroidal wave function analogues; extensions to multi-dimensional and weighted Dunkl-type settings with more general root systems; and applications in quaternion-valued signal processing, color image analysis, and harmonic analysis on reflection groups. Another interesting direction would be to investigate uncertainty principles and eigenfunction expansions within the QDT framework, as well as the connection between the $K$-functionals defined here and other smoothness measures.

Overall, this work provides a robust bridge between classical approximation theory and the emerging field of quaternionic Dunkl analysis, offering both theoretical insights and potential computational tools for processing quaternion-valued signals with reflection symmetries.

## Conflicts of Interest

No data sets were generated or analysed during the current study. All authors declare that they have no conflict of interest. This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors. All authors contributed equally to the conceptualization, formal analysis, and writing of this manuscript. All authors have read and approved the final version of the manuscript.

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