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NEW BOUNDS FOR GURLAND-TYPE RATIOS IN TERMS OF THE GENERALIZED NIELSEN'S BETA FUNCTION

HESHAM MOUSTAFA[1*] and MANSOUR MAHMOUD[2]

Abstract. We establish new bounds for Gurland-Type Ratios in terms of the generalized Nielsen's beta function $\beta_h(y)$. To achieve this, we investigate the complete monotonicity of certain functions involving $\beta_h(y)$ and the Gamma function $\Gamma(y)$. Furthermore, we derive some sharp bounds that are superior to some recent results.

Keywords. Nielsen's Beta function, Digamma function, Polygamma function, completely monotonic function, bounds
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1. Introduction

Euler defined the gamma function as [1]

$$\Gamma(y) = \lim_{s\to\infty} \frac{s!\ s^y}{(y+s)(y+s-1)\cdots(y+1)y}, \qquad y \in (0,\infty)$$

and the digamma function by

$$\psi(y) = \sum_{s=1}^{\infty} \frac{y}{s(y+s)} - \gamma - \frac{1}{y} = -\gamma + \int_0^\infty \left(\frac{e^{-yt}-e^{-t}}{e^{-t}-1}\right) dt, \qquad y \in \mathbb{R}^+ \quad (1.1)$$

where $\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right) \approx 0.577\ldots$ is Euler-Mascheroni's constant. The function $\psi(y)$ has the following asymptotic expansion:

$$\psi(y) \sim \ln y - \frac{1}{2y} - \sum_{s=1}^{\infty} \frac{B_{2s}}{2s\ y^{2s}}, \qquad y \to \infty \quad (1.2)$$

where B_{2i} are the Bernoulli numbers and satisfies the functional relations

$$\psi^{(s)}(1+y) - \psi^{(s)}(y) = \frac{(-1)^s\ s!}{y^{1+s}}, \qquad y \in \mathbb{R}^+, \qquad s = 0,1,2,\cdots \quad (1.3)$$

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* Corresponding author

where the higher derivatives $\psi^{(s)}(y)$; $s = 1, 2, \ldots$; are called polygamma functions.

The Nielsen's beta function [23]

$$\beta(y) = \frac{\psi\left(\frac{y+1}{2}\right) - \psi\left(\frac{y}{2}\right)}{2} = \int_0^\infty \frac{e^{(1-y)t}}{e^t+1}dt, \qquad y \in \mathbb{R}^+ \tag{1.4}$$

helps in the evaluation of certain hypergeometric functions [10]:

$${}_2F_1(1, y; 1+y; -1) = y\,\beta(y), \qquad y \neq 0, -1, -2, \cdots$$

and also in the evaluation of certain alternating series [24]:

$$\sum_{s=0}^{\infty} \frac{(-1)^s}{s\,a+d} = \frac{1}{a}\beta\left(\frac{d}{a}\right), \qquad d \neq 0, -a, -2a, \cdots .$$

Mahmoud and Agarwal [11] introduced the asymptotic formula given below for the Bateman's $G-$function [6]; where $G(y)/2 = \beta(y)$;

$$2\beta(y) \sim \sum_{s=1}^{\infty} \frac{(4^s-1)B_{2s}}{s\,y^{2s}} + 1/y, \qquad y \to \infty \tag{1.5}$$

and in Corollary 2.3, they deduced:

$$2\beta(y) > \frac{1}{y}, \qquad y \in (0, \infty). \tag{1.6}$$

The ratio $T(y,x) = \frac{\Gamma(y)\Gamma(x)}{\Gamma^2\left(\frac{y+x}{2}\right)}$ for $y, x > 0$ is known as Gurland's ratio [7] for the Gamma function. It has important connections with modified Bessel functions[32] and [33]. In particular, the special case $T(1+(k-1)/2, 1+(k+1)/2)$ arises in formulas related to the volume of the unit ball in $\mathbb{R}^k$ [2]. Another notable form, $T(1/\gamma, 3/\gamma)$ appears as the ratio between the variance and the squared absolute mean of a generalized Gamma random variable with shape parameter γ [29]. This expression is also referred to as the generalized Gaussian ratio [27] and it has useful applications in image recognition and related areas [27] and [9]. In 2019, Yang and Zheng [34] in the Corollaries 8 and 9 presented several new inequalities for the Gurland's ratio and as a special case we have

$$\sqrt{\left(1+\frac{1}{2y}\right)} < T(y, y+1) < \sqrt{\left(1+\frac{2}{4y-1}\right)}, \tag{1.7}$$

where the inequality on the left-hand side is valid for $y > 0$, while the inequality on the right-hand side holds for $y > \frac{1}{4}$ and

$$\exp\left(\beta\left(2y+\frac{1}{2}\right)\right) - \psi\left(y+\frac{1}{4}\right) < T(y, y+1) < \exp\left(\beta\left(2y\right)\right) \qquad y > 0. \tag{1.8}$$

In Corollary 11, as a special case, we have

$$\exp\left(\frac{1}{4}\psi'\left(\frac{2y+1}{2}\right)\right) < T(y, y+1) < \exp\left(\frac{1}{4}\psi'(y)\right), \qquad y \in (0, \infty). \tag{1.9}$$

In 2021, Tian and Yang [30] in the page 11, Remark 5, presented the following double inequalities:

$$\exp\left[\sum_{s=1}^{2m}\frac{(2-2^{1-2s})B_{2s}}{s(2s-1)\,y^{2s-1}}\right]\leq T(y,y+1)\leq \exp\left[\sum_{s=1}^{2m-1}\frac{(2-2^{1-2s})B_{2s}}{s(2s-1)\,y^{2s-1}}\right],\qquad y>0$$

and by letting $m=1$, we get

$$\exp\left(\frac{1}{4y}-\frac{1}{96y^3}\right)\leq T(y,y+1)\leq \exp\left(\frac{1}{4y}\right),\qquad y>0. \tag{1.10}$$

In 2018, Mahmoud and Almuashi [13] introduced the generalized $G_h(y)$, for $0<h<2$ and consequently the generalized Nielsen's beta function is defined as:

$$\beta_h(y)=\frac{\psi\left((h+y)/2\right)-\psi\left(y/2\right)}{2}=\int_0^\infty\frac{1-e^{-ht}}{1-e^{-2t}}e^{-yt}dt,\qquad 0<h<2,\quad y>0 \tag{1.11}$$

and satisfied the functional relation:

$$\beta_h(y+1)+\beta_h(y)=\psi(y+h)-\psi(y)=2\beta_{2h}\Big(2y\Big),\qquad y>0. \tag{1.12}$$

The function $\beta_h(y)$ and the hypergeometric function ${}_3F_2$ are related by:

$$\beta_h(y)=\frac{h/2}{y+h}\,{}_3F_2\left(1,1,\frac{h}{2}+1;2,\frac{y+h}{2}+1;1\right),\qquad y>0$$

and as a consequence, we have

$${}_3F_2\left(1,1,3/2;2,\frac{y+3}{2};1\right)=\frac{2y+2}{y}\,{}_2F_1(1,y;1+y;-1),\qquad y>0.$$

Also, an asymptotic expansion for $\beta_h(y)$ is deduced:

$$\beta_h(y)\sim\sum_{s=1}^{\infty}\frac{(-1)^{s+1}}{2s}\left[\sum_{r=0}^{s-1}\tbinom{s}{r}\,2^r\,B_r\,h^{s-r}\right]\frac{1}{y^s},\qquad y\to\infty. \tag{1.13}$$

For more information about this topic, please see [8], [12], [14], [17], [18], [19], [20], [21], [22], [26], [28].

The organization of this paper is as follows:
Section 1 introduces the generalized Nielsen's beta function and Gurland's ratio together with several previously known inequalities. In Section 2, we present a number of auxiliary functions and study their monotonic behavior. These results are then applied to show that our conclusions sharpen some recent published results. Section 3 is devoted to the study of complete monotonicity for certain functions involving $\beta_h(y)$ and $\Gamma(y)$. We also establish several inequalities for Gurland-type ratios $T_h\left(\frac{y}{2},\frac{y+1}{2}\right)=\frac{\Gamma\left(\frac{y}{2}\right)\Gamma\left(\frac{y+1+h}{2}\right)}{\Gamma\left(\frac{y+1}{2}\right)\Gamma\left(\frac{y+h}{2}\right)}$ that improve upon recently published results. In Section 4, we deduce some sharp bounds for $T_h\left(\frac{y}{2},\frac{y+1}{2}\right)$ for $y>0$:

$$\exp\Big(\frac{\beta_h\left(y+\frac{1}{2}\right)+\beta_{2h}(2y)}{2}-\frac{h}{48y^3}\Big)<T_h\left(\frac{y}{2},\frac{y+1}{2}\right)<\exp\Big(\frac{\beta_h\left(y+\frac{1}{2}\right)+\beta_{2h}(2y)}{2}\Big),$$

and

$$\exp\left(\frac{\beta_h\left(y+\frac{1}{4}\right)+\beta_h\left(y+\frac{3}{4}\right)}{2}\right) < T_h\left(\frac{y}{2},\frac{y+1}{2}\right) < \exp\left(\frac{\beta_h\left(y+\frac{1}{4}\right)+\beta_h\left(y+\frac{3}{4}\right)}{2}+\frac{h}{96y^3}\right).$$

All computations in this paper were performed using Mathematica 10. The result given [25] will be employed throughout this paper:

Corollary 1.1. *Suppose S is a real-valued function defined on $y > y_0$, $y_0 \in (-\infty,\infty)$, and suppose that $\lim\limits_{y\to\infty} S(y) = 0$. Then for $\alpha \in (0,\infty)$, we have $S(y) > 0$, if $S(\alpha + y) < S(y)$ for $y \in (y_0,\infty)$ and $S(y) < 0$, if $S(\alpha+y) > S(y)$ for $y \in (y_0,\infty)$.*

2. Auxiliary Results

From (1.3) and (1.12), we can conclude that

$$\beta_h(y) - \beta_h(2+y) = \frac{h}{y(h+y)}, \qquad y \in \mathbb{R}^+, \qquad 0 < h < 2. \tag{2.1}$$

Lemma 2.1.

- *For $y > 1/4$, the function $\Upsilon_1(y) = \ln\left(1+\frac{2}{4y-1}\right) - \frac{1}{2y} > 0$.*
- *For $y \geq 3/20$, the function $\Upsilon_2(y) = 2\beta\left(2y+\frac{1}{2}\right) - \ln\left(1+\frac{1}{2y}\right) > 0$.*
- *For $y \in \mathbb{R}^+$, the function $\Upsilon_3(y) = \psi'(y) - \frac{1}{y} > 0$.*
- *For $y \in \mathbb{R}^+$, the function $\Upsilon_4(y) = \beta\left(2y+\frac{1}{2}\right) - \frac{1}{4}\psi'\left(y+\frac{1}{2}\right) > 0$.*
- *For $y \in \mathbb{R}^+$ and $0 < h < 2$, the function $\Upsilon_5(y) = \frac{1}{2}\beta_{2h}(2y) - \frac{1}{2}\beta_h\left(y+\frac{1}{2}\right) > 0$.*
- *For $y \in (0,\infty)$ and $0 < h < 2$, the function*

$$\Upsilon_6(y) = \frac{1}{2}\left(\beta_h\left(y+\frac{1}{4}\right)+\beta_h\left(y+\frac{3}{4}\right)\right) - \beta_h\left(y+\frac{1}{2}\right) > 0.$$

- *For $y \geq 4$ and $1 \leq h < 2$, the function*

$$\Upsilon_7(y) = \frac{1}{2}\beta_h\left(y+\frac{1}{2}\right) + \frac{1}{2}\beta_{2h}(2y) - \frac{h}{48y^3} - \frac{1}{2}\left(\beta_h\left(y+\frac{1}{4}\right)+\beta_h\left(y+\frac{3}{4}\right)\right) > 0.$$

- *For $y \geq 1$ and $1 \leq h < 2$, the function*

$$\Upsilon_8(y) = \frac{\beta_h\left(y+\frac{1}{2}\right)}{2} + \frac{1}{2}\beta_{2h}(2y) - \frac{1}{2}\left(\beta_h(y+1/4)+\beta_h\left(y+\frac{3}{4}\right)\right) - \frac{h}{96y^3} > 0.$$

Proof.

- The function $\Upsilon_1'(y) = \frac{-1}{2y^2(-1+4y)(1+4y)} < 0$ for $y > \frac{1}{4}$. Thus $\Upsilon_1(y)$ is decreasing on $y > \frac{1}{4}$ with $\lim\limits_{y\to\infty} \Upsilon_1(y) = 0$ and then $\Upsilon_1(y) > 0$ on $y > \frac{1}{4}$.

- Using (1.5), we get $\lim_{y\to\infty}\Upsilon_2(y)=0$ and letting $x=2y$, we have $\Upsilon_2(x)=2\beta\left(x+\frac{1}{2}\right)-\ln\left(1+\frac{1}{x}\right)$ and $\Upsilon_2'(x)=2\beta'\left(x+\frac{1}{2}\right)+\frac{1}{x(x+1)}$. Using the functional formula(2.1) at $h=1$, we get

$$\Upsilon_2'(2+x)-\Upsilon_2'(x)=\frac{2f(x-0.3)}{x(1+x)(2+x)(3+x)(1+2x)^2(3+2x)^2}>0,\quad x\geq 3/10$$

where

$$625f(x)=2313+120690x+230200x^2+146000x^3+30000x^4>0,\quad x\geq 0$$

with $\lim_{y\to\infty}\Upsilon_2'(y)=0$ and by using Corollary 1.1, we have $\Upsilon_2'(x)<0$ for $x\geq 3/10$ and then $\Upsilon_2(x)$ is decreasing on $[3/10,\infty)$ with $\lim_{x\to\infty}\Upsilon_2(x)=0$. Hence $\Upsilon_2(x)>0$ for all $x\in[3/10,\infty)$ and consequently $\Upsilon_2(y)>0$ for all $y\in[3/20,\infty)$
- By using (1.2), we get $\lim_{y\to\infty}\Upsilon_3(y)=0$ and using the functional formula(1.3), we get $\Upsilon_3(y+1)-\Upsilon_3(y)=\frac{-1}{y^2(1+y)}<0$ for $y>0$ and by using Corollary 1.1, we have $\Upsilon_3(y)>0$ for $y>0$
- Using (1.2)and (1.5), we get $\lim_{y\to\infty}\Upsilon_4(y)=0$ and using the functional formulas (1.3) and (2.1) at $h=1$, we get

$$\Upsilon_4(y+2)-\Upsilon_4(y)=\frac{-2(159+440y+476y^2+256y^3+64y^4)}{(1+2y)^2(3+2y)^2(1+4y)(3+4y)(5+4y)(7+4y)}<0,\quad y>0$$

and by the same method, we have $\Upsilon_4(y)>0$ for $y>0$.
- Using (1.2)and (1.13), we get $\lim_{y\to\infty}\Upsilon_5(y)=0$ and using the functional formulas (1.3) and (2.1), we get

$$\Upsilon_5(y+2)-\Upsilon_5(y)=\frac{-h(1+3h+2h^2+6y(1+h)+6y^2)}{4y(1+y)(h+y)(1+h+y)(1+2y)(1+2h+2y)}<0,\quad y>0,\ \ 0<h<2$$

and then $\Upsilon_5(y)>0$ for $y>0$ and $0<h<2$.
- Similarly, we have

$$\Upsilon_6(y+2)-\Upsilon_6(y)=\frac{-2h(1+2(y+h))^{-1}(3+4y)^{-1}\,\xi_h(y)}{(1+2y)(1+4y)(1+4(h+y))(3+4(h+y))}<0,\quad y\in(0,\infty),\ \ 0<h<2$$

where

$$\xi_h(y)=31+96h+80h^2+(192+448h+256h^2)y+(448+768h+256h^2)y^2+(512+512h)y^3+256y^4$$

and then $\Upsilon_6(y)>0$ for $y>0$ and $0<h<2$.
- By the same way, we have

$$\Upsilon_7(2+y)-\Upsilon_7(y)=\frac{-h\Big((y+1)(3+4y)(1+4h+4y)(3+4h+4y)\Big)^{-1}\chi_h(y-4)}{24y^3(y+2)^3(h+y)(1+h+y)(1+2y)(1+2h+2y)(1+4y)}<0,$$

where

$$\begin{aligned}
\chi_h(y) &= -36\Big(-1907793684-462573025h+283176765h^2+130932590h^3+19088784h^4\\
&\quad+981920h^5\Big)-2\Big(-78457524462-15906984535h+10333641555h^2\\
&\quad+4078684730h^3+512584560h^4+22319072h^5\Big)y+\Big(162276970142\\
&\quad+26630058835h-18643538055h^2-6181685570h^3-655677872h^4\\
&\quad-23483744h^5\Big)y^2+\Big(100235826517+12728899009h-9817525605h^2\\
&\quad-2678968410h^3-233061904h^4-6588896h^5\Big)y^3-2\Big(-20535164184\\
&\quad-1888451677h+1662241940h^2+363060150h^3+24861632h^4+519968h^5\Big)y^4\\
&\quad+\Big(11714700519+699445432h-750458600h^2-126032960h^3-6367232h^4\\
&\quad-87552h^5\Big)y^5-32\Big(-74126934-2304916h+3527875h^2+427460h^3\\
&\quad+14160h^4+96h^5\Big)y^6-8\Big(-42587115-308884h+1363700h^2+106080h^3\\
&\quad+1728h^4\Big)y^7-64\Big(-530974+5477h+9600h^2+360h^3\Big)y^8-32\Big(-69967\\
&\quad+1368h+480h^2\Big)y^9-1536(-57+h)y^{10}+1536y^{11}>0,\quad y\geq 0,\ \ 1\leq h<2
\end{aligned}$$

and then $\Upsilon_7(y)>0$ for $y\geq 4$ and $1\leq h<2$.

- By the same method, we have

$$\Upsilon_8(2+y)-\Upsilon_8(y)=\frac{-h\Big((4y+3)(1+4h+4y)(3+4h+4y)\Big)^{-1}\varpi_h(y-1)}{48y^3(y+1)(y+2)^3(y+h)(y+1+h)(1+2y)(1+2h+2y)(1+4y)}<0,$$

where

$$\begin{aligned}
\varpi_h(y) &= -6(-394608-633125h-257535h^2+76810h^3+76368h^4+14560h^5)\\
&\quad+(16464444+24044297h+8948385h^2-2046070h^3-1864752h^4\\
&\quad-302368h^5)y+(51308507+67111606h+22052670h^2-4221920h^3\\
&\quad-3278240h^4-433664h^5)y^2+(94557715+108700789h+30316755h^2\\
&\quad-5331210h^3-3232912h^4-330464h^5)y^3+(114456873+113012108h\\
&\quad+25490900h^2-4462860h^3-1931968h^4-141376h^5)y^4+(95484345\\
&\quad+78663274h+13444420h^2-2483360h^3-698816h^4-32256h^5)y^5\\
&\quad+4(13991784+9256793h+1086880h^2-219280h^3-35328h^4-768h^5)y^6\\
&\quad+4(5754639+2898320h+197120h^2-44160h^3-3072h^4)y^7\\
&\quad-64(-101545-35902h-960h^2+240h^3)y^8+64(18701+4032h)y^9\\
&\quad+6144(21+2h)y^{10}+6144y^{11}\\
&> 0,\quad y\geq 0,\ \ 1\leq h<2
\end{aligned}$$

and then $\Upsilon_8(y)>0$ for $y\geq 1$ and $1\leq h<2$.

□

3. Main Results

Recall that a function $\omega(y)$ defined on $\mathbb{R}^+$ is completely monotonic (CM) if

$$(-1)^s \omega^{(s)}(y) \geq 0, \qquad y \in \mathbb{R}^+;\ s \geq 0.$$

A condition that is simultaneously necessary and sufficient [31] for the function $\omega(y)$ to be CM on $(0,\infty)$ is the convergence of the integral:

$$\omega(y) = \int_0^\infty e^{-yt} dv(t), \qquad 0 < y < \infty$$

where $v(t)$ is non-negative measure on $[0,\infty)$. For more information about this topic, see [15] and [16].

Theorem 3.1. *For $y \in (0,\infty)$ and $0 < h < 2$, we have*

$$\exp\left(\beta_h\left(y+\frac{1}{2}\right)\right) \leq T_h\left(\frac{y}{2}, \frac{y+1}{2}\right) \leq \exp\Big(\beta_{2h}(2y)\Big). \tag{3.1}$$

Proof. From the definition (1.11), we deduce that $\beta_h(y)$ is CM on $(0,\infty)$ and hence it is convex on $(0,\infty)$. Applying the Hermite-Hadamard's inequality [5] for $0 < \alpha < \delta < \infty$, we get

$$\beta_h\left(\frac{\alpha+\delta}{2}\right) \leq \frac{1}{\delta-\alpha}\int_\alpha^\delta \beta_h(u)du \leq \frac{\beta_h(\alpha)+\beta_h(\delta)}{2}$$

and by letting $\alpha = y$ and $\delta = y+1$, we get

$$\beta_h(y+1/2) \leq \ln\left[T_h\left(\frac{y}{2}, \frac{y+1}{2}\right)\right] \leq \frac{\beta_h(y)+\beta_h(y+1)}{2}.$$

and by using (1.12), we get (3.1) □

Letting $h = 1$ in (3.1) and using the relation (1.3), we get

$$\exp\left(\beta\left(2y+\frac{1}{2}\right)\right) \leq T(y, y+1) \leq \exp\left(\frac{1}{4y}\right), \quad y > 0. \tag{3.2}$$

Remark 3.2.
- As $\Upsilon_1(y) > 0$ for $y > \frac{1}{4}$ and then the upper bound given in (3.2) improves the upper bound given in (1.7) for all $y > \frac{1}{4}$.
- As $\Upsilon_2(y) > 0$ for $y \geq \frac{3}{20}$ and then the lower bound given in (3.2) improves the lower bound given in (1.7) for all $y \geq \frac{3}{20}$.
- As $\psi(y)$ is increasing on $y > 0$ and we have $\psi(y+1/4) > \psi(31/20) \sim 0.0822226$. Then $\exp\left(\beta\left(2y+\frac{1}{2}\right)\right) > \exp\left(\beta\left(2y+\frac{1}{2}\right)\right) - \psi\left(y+\frac{1}{4}\right)$ and this proves that the lower bound given in (3.2) improves the lower bound given in (1.8) for all $y > 1.3$.
- The inequality (1.6) proves that the upper bound given in (3.2) improves the upper bound given in (1.8) for all $y > 0$.
- As $\Upsilon_3(y) > 0$ for $y > 0$ and this proves that the upper bound given in (3.2) improves the upper bound given in (1.9) for $y > 0$.
- As $\Upsilon_4(y) > 0$ for all $y > 0$ and this proves that the lower bound given in (3.2) improves the lower bound given in (1.9) for all $y > 0$.

Theorem 3.3. *For $0 < h < 2$, the functions*

$$H_h(y) = \beta_h(y+1/2) + \frac{\beta_h(y)+\beta_h(y+1)}{2} - 2\ln\left[T_h\left(\frac{y}{2},\frac{y+1}{2}\right)\right] \tag{3.3}$$

and

$$S_h(y) = 2\left[T_h\left(\frac{y}{2},\frac{y+1}{2}\right)\right] - \beta_h(y+1/4) - \beta_h\left(y+\frac{3}{4}\right) \tag{3.4}$$

are CM on $(0,\infty)$.

Proof. By using (1.1) and (1.11), we have

$$\beta_h(y+1/2)+\frac{\beta_h(y)+\beta_h(y+1)}{2} = \int_0^\infty \frac{e^{-\frac{yt}{2}}}{4(1-e^{-t})}\left(-2e^{-\frac{(2h+1)t}{4}}+2e^{-\frac{t}{4}}-e^{-\frac{ht}{2}}+1-e^{-\frac{(h+1)t}{2}}+e^{-\frac{t}{2}}\right)dt.$$

By using Malmstén's formula [4], page 6, equation(19):

$$\ln\Gamma(z) = \int_0^\infty \left(e^{-t}(z-1)+\frac{e^{-zt}-e^{-t}}{1-e^{-t}}\right)\frac{dt}{t}, \qquad \Re(z)>0 \tag{3.5}$$

we have

$$\ln\left[T_h\left(\frac{y}{2},\frac{y+1}{2}\right)\right] = \int_0^\infty \frac{e^{-\frac{yt}{2}}}{t(1-e^{-t})}\left[e^{-\frac{(h+1)t}{2}}-e^{-\frac{t}{2}}+1-e^{-\frac{ht}{2}}\right]dt.$$

And then

$$H_h(y) = \int_0^\infty \frac{e^{-\frac{yt}{2}}\, e^{-\frac{(h+1)t}{2}}\, \varphi_h(t)}{4t(1-e^{-t})}$$

where

$$\begin{aligned}
\varphi_h(t) &= t\Big[-2e^{\frac{t}{4}}+2e^{\frac{(2h+1)t}{4}}-e^{\frac{t}{2}}+e^{\frac{(h+1)t}{2}}-1+e^{\frac{ht}{2}}\Big]-8\left[1-e^{\frac{ht}{2}}+e^{\frac{(h+1)t}{2}}-e^{\frac{t}{2}}\right]\\
&= \frac{h\,t^4}{96}+\sum_{m=4}^\infty \frac{C_h(m)\,2^{-m}\,t^{m+1}}{(m+1)!},
\end{aligned}$$

where

$$\begin{aligned}
C_h(m) &= (m+1)\Big[-2\left(\frac{1}{2}\right)^m+2\left(h+\frac{1}{2}\right)^m-1+(h+1)^m+h^m\Big]+4\Big[h^{m+1}-(h+1)^{m+1}+1\Big]\\
&= (m+1)\sum_{s=1}^{m-3}\tbinom{m}{s}h^s\left(2\left(\frac{1}{2}\right)^{m-s}+1\right)-4\sum_{s=1}^{m-3}\tbinom{m+1}{s}h^s+\frac{(m+1)m(m-1)\,h^{m-2}}{12}\\
&= (m+1)\sum_{s=1}^{m-3}\tbinom{m}{s}h^s\left(2\left(\frac{1}{2}\right)^{m-s}+1-\frac{4}{m+1-s}\right)+\frac{(m+1)m(m-1)\,h^{m-2}}{12}\\
&= (m+1)\sum_{s=1}^{m-3}\tbinom{m}{s}h^s\left(2\left(\frac{1}{2}\right)^{m-s}+\frac{m-3-s}{m+1-s}\right)+\frac{(m+1)m(m-1)\,h^{m-2}}{12}\\
&> 0, \quad 0<h<2, \quad m\ge 4.
\end{aligned}$$

Consequently, $H_h(y)$ is CM on $(0,\infty)$. By the same way, we have

$$S_h(y) = \int_0^\infty \frac{e^{-\frac{yt}{2}}\, e^{-\frac{(h+1)t}{2}}\, \phi_h(t)}{2t(1-e^{-t})}$$

where

$$\begin{aligned}\phi_h(t) &= 4\Big[1 - e^{\frac{ht}{2}} + e^{\frac{(h+1)t}{2}} - e^{\frac{t}{2}}\Big] - t\Big[e^{\frac{(4h+3)t}{8}} - e^{\frac{3t}{8}} + e^{\frac{(4h+1)t}{8}} - e^{\frac{t}{8}}\Big]\\ &= \frac{ht^4(4+t+ht)}{1536} + \sum_{m=5}^{\infty} \frac{A_h(m)\, 2^{-m}\, t^{m+1}}{(m+1)!},\end{aligned}$$

where

$$\begin{aligned}A_h(m) &= 2\Big[(h+1)^{m+1} - h^{m+1} - 1\Big] - (m+1)\Big[(h+3/4)^m - (3/4)^m + (h+1/4)^m - (1/4)^m\Big]\\ &= 2\sum_{s=1}^{m-4} \tbinom{m+1}{s} h^s - (m+1)\sum_{s=1}^{m-4} \tbinom{m}{s} h^s \left(\left(\frac{3}{4}\right)^{m-s} + \left(\frac{1}{4}\right)^{m-s}\right)\\ &\quad + \frac{(m+1)m(m-1)\Big(2h+(m-2)\Big)\, h^{m-3}}{96}, \qquad m \geq 5, \quad 0<h<2\\ &> (m+1)\sum_{s=1}^{m-4} \tbinom{m}{s} h^s \left(\frac{2}{m+1-s} - \left(\frac{3}{4}\right)^{m-s} - \left(\frac{1}{4}\right)^{m-s}\right)\\ &= (m+1)\sum_{s=1}^{m-4} \tbinom{m}{s} \frac{h^s}{(m+1-s)} \left[2 - (m+1-s)\left(\left(\frac{3}{4}\right)^{m-s} + \left(\frac{1}{4}\right)^{m-s}\right)\right]\\ &> 0, \quad 0<h<2, \quad m \geq 5\end{aligned}$$

as the sequence $\theta_1(m-s) = (m+1-s)\left(\frac{3}{4}\right)^{m-s}, \quad 4 \leq m-s \leq m-1$ is decreasing for $4 \leq m-s \leq m-1$ because

$$\theta_1(m-s+1) - \theta_1(m-s) = \left(\frac{3}{4}\right)^{m-s} \left(\frac{2-(m-s)}{4}\right) < 0, \quad 4 \leq m-s \leq m-1$$

and so also $\theta_2(m-s) = (m+1-s)\left(\frac{1}{4}\right)^{m-s}$ and this implies that

$$2 - (m+1-s)\left[\left(\frac{3}{4}\right)^{m-s} + \left(\frac{1}{4}\right)^{m-s}\right] > 2 - \frac{205}{128} = \frac{51}{128} > 0, \quad 1 \leq s \leq m-4$$

and consequently, $S_h(y)$ is CM on $(0,\infty)$. □

A direct result from Theorem 3.3 and by using the relation (1.12), we have

Corollary 3.4. *For $y \in (0,\infty)$ and $0<h<2$, we have*

$$\exp\left(\frac{\beta_h(y+1/4) + \beta_h\left(y+\frac{3}{4}\right)}{2}\right) < T_h\left(\frac{y}{2}, \frac{y+1}{2}\right) < \exp\left(\frac{\beta_h\left(y+\frac{1}{2}\right) + \beta_{2h}\Big(2y\Big)}{2}\right). \tag{3.6}$$

Remark 3.5.
- As $\Upsilon_5(y) > 0$ for $y>0$ and $0<h<2$, and then
$$\exp\left(\frac{1}{2}\beta_h\left(y+\frac{1}{2}\right) + \frac{1}{2}\beta_{2h}\Big(2y\Big)\right) < \exp\Big(\beta_{2h}\Big(2y\Big)\Big), \tag{3.7}$$
which shows that the upper bound in (3.6) improves upon the upper bound in (3.1) for $y>0$.

- Letting $h=1$ in (3.6) and (3.7) yields

$$\exp\left(\frac{\beta\left(2y+\frac{1}{4}\right)+\beta\left(2y+\frac{3}{4}\right)}{2}\right)<T(y,y+1)<\exp\left(\frac{1}{2}\beta\left(2y+\frac{1}{2}\right)+\frac{1}{8y}\right),\quad y>0. \tag{3.8}$$

And

$$\exp\left(\frac{1}{2}\beta\left(2y+\frac{1}{2}\right)+\frac{1}{8y}\right)<\exp\left(\frac{\psi(2y+1)-\psi(2y)}{2}\right)=\exp\left(\frac{1}{4y}\right)$$

which shows that the upper bound in (3.8) improves upon the upper bound in (1.10) for $y>0$.
- As $\Upsilon_6(y)>0$ for $y>0$ and $0<h<2$, and then the lower bound given in (3.6) refines the lower bound given in (3.1) for $y>0$ and $0<h<2$.

4. Sharp bounds for $T_h\left(\frac{y}{2},\frac{y+1}{2}\right)$

Lemma 4.1. *For $1\le h<2$, the function*

$$\lambda_h(y)=y^3\ H_h(y),\qquad y>0 \tag{4.1}$$

is strictly increasing on $(0,\infty)$ with sharp bounds $0<\lambda_h(y)<\frac{h}{24}$ and consequently, for $y>0$, we have

$$\exp\left(\frac{\beta_h\left(y+\frac{1}{2}\right)+\beta_{2h}(2y)}{2}-\frac{h}{48y^3}\right)<T_h\left(\frac{y}{2},\frac{y+1}{2}\right)<\exp\left(\frac{\beta_h\left(y+\frac{1}{2}\right)+\beta_{2h}(2y)}{2}\right). \tag{4.2}$$

Proof. Differentiating (4.1) with respect to y, we have

$$\begin{aligned}\frac{\lambda_h'(y)}{y^2} &= y\ H_h'(y)+3H_h(y)\\ &= y\left(\beta_h'\left(y+\frac{1}{2}\right)+\frac{1}{2}\beta_h'(y)+\frac{1}{2}\beta_h'(y+1)-2\beta_h(y+1)+2\beta_h(y)\right)\\ &\quad+3\left(\beta_h\left(y+\frac{1}{2}\right)+\frac{\beta_h(y)+\beta_h(y+1)}{2}-2\ln T_h\left(\frac{y}{2},\frac{y+1}{2}\right)\right)\triangleq F_h(y).\end{aligned}$$

Then

$$\begin{aligned}F_h'(y) &= y\left(\beta_h''\left(y+\frac{1}{2}\right)+\frac{1}{2}\beta_h''(y)+\frac{1}{2}\beta_h''(y+1)-2\beta_h'(y+1)+2\beta_h'(y)\right)\\ &\quad+4\left(\beta_h'\left(y+\frac{1}{2}\right)+\frac{1}{2}\beta_h'(y)+\frac{1}{2}\beta_h'(y+1)-2\beta_h(y+1)+2\beta_h(y)\right)\end{aligned}$$

and by using (2.1) and its derivatives, we get

$$\begin{aligned}F_h'(y+2)-F_h'(y) &= 2\beta_h''\left(y+\frac{5}{2}\right)+\beta_h''(y+2)+\beta_h''(y+3)-4\beta_h'(y+3)+4\beta_h'(y+2)\\ &\quad+\frac{h\ U_h(y)}{y^2(1+y)^3(h+y)^3(1+h+y)^3(1+2y)^3(1+2h+2y)^3}\triangleq T_h(y).\end{aligned}$$

where

$$\begin{aligned} U_h(y) &= h^2(1+h)^3(1+2h)^3+3h(1+h)^2(1+2h)^2(1+7h+9h^2+2h^3)y \\ &+(1+h)(1+2h)(1+50h+271h^2+552h^3+479h^4+168h^5+20h^6)y^2 \\ &+2(1+h)(7+179h+916h^2+1981h^3+2081h^4+1048h^5+236h^6+16h^7)y^3 \\ &+3(25+470h+2200h^2+4629h^3+5078h^4+2968h^5+864h^6+96h^7)y^4 \\ &+2(1+h)(104+1542h+4839h^2+5952h^3+3120h^4+576h^5)y^5 \\ &+2(164+2526h+8187h^2+10764h^3+6288h^4+1344h^5)y^6 \\ &+12(1+h)(23+402h+712h^2+312h^3)y^7-8(1+h)(37-88h+32h^2)y^9 \\ &-2(-7-1490h-2896h^2-1360h^3+32h^4)y^8-24(17+24h+16h^2)y^{10} \\ &-256(1+h)y^{11}-64y^{12}. \end{aligned}$$

And by using again(2.1) and its derivatives, we get

$$T_h(y+2)-T_h(y)=\frac{-2h\ s_h(y)\ y^{-2}(1+y)^{-3}(2+y)^{-3}(3+y)^{-3}(h+y)^{-3}(1+h+y)^{-3}}{(2+h+y)^3(3+h+y)^3(1+2y)^3(5+2y)^3(1+2h+2y)^3(5+2h+2y)^3},$$

where $s_h(y)$ is a polynomial of degree 27th with coefficients of h and for $1\le h<2$, we have

$$\begin{aligned} s_h(y) &> 12\Big(6+11y+6y^2+y^3\Big)^4\Big(111132000+941711400y+3907385064y^2 \\ &+10311122157y^3+18897523384y^4+25049039258y^5+24575186800y^6 \\ &+18095674295y^7+10070680672y^8+4237252664y^9+1338257408y^{10}+12544y^{15} \\ &+311985664y^{11}+52043264y^{12}+5872768y^{13}+401408y^{14}\Big)>0 \end{aligned}$$

and hence $T_h(2+y)-T_h(y)<0$ for $y>0$ and $1\le h<2$. Using the asymptotic, see [3], page 141, 5.11.12:

$$\frac{\Gamma(z+a)}{\Gamma(z+b)}\sim z^{a-b},\qquad a,b\in\mathbb{R},\quad z\longrightarrow\infty$$

and the asymptotic expansion (1.13) and its derivatives, we get

$$\begin{aligned} \lim_{y\to\infty}T_h(y) &= \lim_{y\to\infty}\left(\frac{h(5+2y)^{-3}}{(2+y)^3(3+y)^3}\Big(-9169+4750h+5(-3677+1890h)y\right. \\ &+3(-4919+2510h)y^2+4(-1482+751h)y^3+8(-149+75h)y^4 \\ &\left.+48(-2+h)y^5\Big)+O(y^{-5})\right)=0, \end{aligned}$$

$$\begin{aligned} \lim_{y\to\infty}F'_h(y) &= \lim_{y\to\infty}\left(\frac{h}{6y^3(1+y)^3(1+2y)^3}\Big(2(-2+h)(-1+4h)+3(15-56h+24h^2)y\right. \\ &+3(77-216h+88h^2)y^2+(389-1194h+496h^2)y^3+12(9-85h+40h^2)y^4 \\ &\left.+24(-2+h)(7+8h)y^5+144(-2+h)y^6\Big)+O(y^{-4})\right)=0, \end{aligned}$$

$$\lim_{y\to\infty} F_h(y) = \lim_{y\to\infty}\left(\frac{-h\Big(-3+2h+2(-11+6h)y+2(-23+12h)y^2+16(-2+h)y^3\Big)}{4y(1+y)^2(1+2y)^2} +O(y^{-3})\right)=0,$$

$$\lim_{y\to\infty} H_h(y) = \lim_{y\to\infty}\left[\frac{h(1+8y+8y^2)}{4y(1+y)(1+2y)}+O(y^{-2})\right]-2\lim_{y\to\infty}\ln\left[T_h\left(\frac{y}{2},\frac{y+1}{2}\right)\right]=0$$

And by using L'Hôpital's rule, we get

$$\begin{aligned}
\lim_{y\to\infty}\lambda_h(y) &= \lim_{y\to\infty} y^3\, H_h(y)=\lim_{y\to\infty}\frac{H_h(y)}{y^{-3}}=\frac{-1}{3}\lim_{y\to\infty} y^4\, H_h'(y)\\
&= \frac{-1}{3}\lim_{y\to\infty} y^4\left(\beta_h'\left(y+\frac{1}{2}\right)+\frac{1}{2}\beta_h'(y)+\frac{1}{2}\beta_h'(y+1)-2\beta_h(y+1)+2\beta_h(y)\right)\\
&= \frac{-1}{3}\lim_{y\to\infty} y^4\Bigg(\frac{-h}{12y^4(1+y)^4(1+2y)^4}\Big(3(-2+h)(-1-h+h^2)\\
&\quad +(-2+h)(-35-40h+36h^2)y+3(113+100h-202h^2+62h^3)y^2\\
&\quad +2(418+585h-934h^2+270h^3)y^3+(1223+2577h-3494h^2+960h^3)y^4\\
&\quad +6(191+541h-664h^2+176h^3)y^5+2(329+1140h-1304h^2+336h^3)y^6\\
&\quad +48(4+15h-16h^2+4h^3)y^7+24y^8\Big)+O(y^{-5})\Bigg)=\frac{h}{24}.
\end{aligned}$$

Now $T_h(y+2)-T_h(y)<0$ for all $y>0$ and $1\le h<2$ and $\lim_{y\to\infty} T_h(y)=0$ then by using Corollary 1.1, we have $T_h(y)>0$ for all $y>0$ and $1\le h<2$, and then $F_h'(y+2)-F_h'(y)>0$ with $\lim_{y\to\infty} F_h'(y)=0$ then $F_h'(y)<0$ which implies that $F_h(y)$ is decreasing for all $y>0$ and $1\le h<2$ with $\lim_{y\to\infty} F_h(y)=0$ then $F_h(y)>0$ and then $\lambda(y)$ is strictly increasing on $\mathbb{R}^+$. By using (1.1), we have $\lim_{x\to 0}\lambda_h(y)=0$. Then

$$0<\beta_h\left(y+\frac{1}{2}\right)+\frac{\beta_h(y)+\beta_h(y+1)}{2}-2\ln\left[\ln T_h\left(\frac{y}{2},\frac{y+1}{2}\right)\right]<\frac{h}{24y^3},\qquad y>0,\qquad 1\le h<2$$

and by using (1.12), we have (4.2) $\square$

Remark 4.2. As $\Upsilon_7(y)>0$ for $y\ge 4$ and $1\le h<2$, and this proves than the lower bound in(4.2) improves the lower bound in (3.6) for all $y\ge 4$ and $1\le h<2$
.

Lemma 4.3. *For $1\le h<2$, the function $\sigma_h(y)=y^3\, S_h(y)$ is strictly increasing on $(0,\infty)$ with sharp bounds $0<\sigma_h(y)<\frac{h}{48}$ and consequently, we have for $y>0$,*

$$\exp\left(\frac{\beta_h\left(y+\frac{1}{4}\right)+\beta_h\left(y+\frac{3}{4}\right)}{2}\right)<T_h\left(\frac{y}{2},\frac{y+1}{2}\right)<\exp\left(\frac{\beta_h\left(y+\frac{1}{4}\right)+\beta_h\left(y+\frac{3}{4}\right)}{2}+\frac{h}{96y^3}\right).\tag{4.3}$$

Proof. By the same method, we have

$$\frac{\sigma'(y)}{y^2} = y\ S_h'(y) + 3S_h(y) \triangleq f_h(y), \quad y > 0$$

and

$$\begin{aligned} f_h'(y) &= y\ S_h''(y) + 4S_h'(y) \\ &= y\left(-\beta_h''\left(y+\frac{1}{4}\right) - \beta_h''\left(y+\frac{3}{4}\right) + 2\beta_h'\left(y+1\right) - 2\beta_h'\left(y\right)\right) \\ &\quad +4\left(-\beta_h'\left(y+\frac{1}{4}\right) - \beta_h'\left(y+\frac{3}{4}\right) + 2\beta_h\left(y+1\right) - 2\beta_h\left(y\right)\right) \end{aligned}$$

and by using (2.1) and its derivatives, we get

$$\begin{aligned} f_h'(2+y) - f_h'(y) &= -2\beta_h''\left(y+\frac{9}{4}\right) - 2\beta_h''\left(y+\frac{11}{4}\right) + 4\beta_h'\left(y+3\right) - 4\beta_h'\left(y+2\right) \\ &\quad + \frac{2h\ u_h(y)\ (3+4y)^{-3}(1+4h+4y)^{-3}(3+4h+4y)^{-3}}{y(1+y)^2(h+y)^2(1+h+y)^2(1+4y)^3} \triangleq t_h(y). \end{aligned}$$

where

$$\begin{aligned} u_h(y) &= 81h(1+h)^2(1+4h)^3(3+4h)^3 + 6(1+h)(1+4h)(3+4h)\Big(81 + 5319h + 37425h^2 \\ &\quad +107312h^3 + 142560h^4 + 84480h^5 + 16640h^6\Big)y + 2\Big(26244 + 883683h \\ &\quad +8137908h^2 + 35602752h^3 + 87992800h^4 + 130363520h^5 + 116217856h^6 \\ &\quad +59674624h^7 + 15687680h^8 + 1540096h^9\Big)y^2 + 8(1+h)\Big(83457 + 1794528h \\ &\quad +11795064h^2 + 36120128h^3 + 59360064h^4 + 53694464h^5 + 25579520h^6 + 5750784h^7 \\ &\quad +475136h^8\Big)y^3 + 2\Big(2198745 + 37723248h + 221989520h^2 + 642086272h^3 \\ &\quad +1041863680h^4 + 990167040h^5 + 546504704h^6 + 165904384h^7 + 24248320h^8 \\ &\quad +1048576h^9\Big)y^4 + 64(1+h)\Big(270549 + 3536608h + 15159416h^2 \\ &\quad +29749504h^3 + 29859328h^4 + 15415296h^5 + 3819520h^6 + 327680h^7\Big)y^5 \\ &\quad +64\Big(680705 + 8320544h + 34800008h^2 + 69735680h^3 + 74983168h^4 + 43939840h^5 \\ &\quad +13025280h^6 + 1474560h^7\Big)y^6 - 20971520(1+h)y^{13} - 4194304y^{14} \\ &\quad +2048(1+h)\Big(35039 + 360720h + 1065920h^2 + 1288320h^3 + 675328h^4 \\ &\quad +122880h^5\Big)y^7 + 1024\Big(73583 + 839840h + 2596280h^2 + 3365120h^3 \\ &\quad +1947904h^4 + 409600h^5\Big)y^8 - 131072\Big(1+h)(-311 - 4257h - 7502h^2 \\ &\quad -3168h^3 + 32h^4\Big)y^9 - 65536\Big(181 - 3122h - 6462h^2 - 2720h^3 \\ &\quad +320h^4\Big)y^{10} - 4194304(1+h)(11+10h^2)y^{11} - 655360\Big(67 + 112h + 64h^2\Big)y^{12}. \end{aligned}$$

And by using again(2.1) and its derivatives, we get

$$\frac{t_h(y+2)-t_h(y)}{\Big((1+y)(2+y)(3+y)(h+y)(1+h+y)\Big)^{-2}} = \frac{\frac{-8h\ L_h(y)\ (2+h+y)^{-2}(3+h+y)^{-2}}{y(11+4y)^3(1+4y)^3(3+4y)^3(9+4y)^3(11+4h+4y)^3}}{(1+4h+4y)^3(3+4h+4y)^3(9+4h+4y)^3)}$$

where $L_h(y)$ is a polynomial of degree 32th with coefficients of h and for $1 \leq h < 2$, we have

$$\begin{aligned} L_h(y) \ > \ & 6(y+1)^2(y+2)^2(y+3)^2(y+4)\Big(1.49917\times10^{20}+2.2446\times10^{21}y+1.60411\times10^{22}y^2 \\ & +7.31331\times10^{22}y^3+2.3899\times10^{23}y^4+5.94913\times10^{23}y^5+1.16914\times10^{24}y^6 \\ & +1.85529\times10^{24}y^7+2.41309\times10^{24}y^8+2.59877\times10^{24}y^9+2.33352\times10^{24}y^{10} \\ & +1.755\times10^{24}y^{11}+1.10833\times10^{24}y^{12}+5.88161\times10^{23}y^{13}+2.6198\times10^{23}y^{14} \\ & +9.76422\times10^{22}y^{15}+3.0287\times10^{22}y^{16}+7.75433\times10^{21}y^{17}+1.61922\times10^{21}y^{18} \\ & +2.71067\times10^{20}y^{19}+3.54804\times10^{19}y^{20}+3.49589\times10^{18}y^{21}+2.43728\times10^{17}y^{22} \\ & +1.07136\times10^{16}y^{23}+2.23201\times10^{14}y^{24}\Big) > 0, \quad y>0 \end{aligned}$$

and hence $t_h(y+2)-t_h(y) < 0$ for $y \in (0,\infty)$ and $1 \leq h < 2$. Using (1.13), we have

$$\begin{aligned} \lim_{y\to\infty} t_h(y) \ = \ & \lim_{y\to\infty}\Bigg[\frac{-2h(2+y)^{-3}(3+y)^{-3}}{(9+4y)^3(11+4y)^3}\Big(27(-1388996+682803h)+27(-4454051 \\ & +2194335h)y+9(-18734447+9249801h)y^2+2(-67502427+33399760h)y^3 \\ & +16(-4220069+2092497h)y^4+96(-225001+111800h)y^5+1280(-3372 \\ & +1679h)y^6+1024(-481+240h)y^7+12288(-2+h)y^8\Big)+O(y^{-5})\Bigg] = 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{y\to\infty} f_h'(y) \ = \ & \lim_{y\to\infty}\Bigg(\frac{hy^{-3}(y+1)^{-3}}{3(4y+1)^3(4y+3)^3}\Big(-108(-2+h)(-1+h)-27(-2+h)(-79+76h)y \\ & -9(4025-5655h+1828h^2)y^2+(-149927+219921h-72832h^2)y^3 \\ & -6(57003-93440h+32608h^2)y^4-48(9025-18171h+6848h^2)y^5 \\ & -64(3497-12372h+5296h^2)y^6-96(-1301-3560h+2048h^2)y^7 \\ & -384(-599+128h^2)y^8-4608(-21+8h)y^9+6144y^{10}\Big)+O(y^{-3})\Bigg) = 0 \end{aligned}$$

and

$$\begin{aligned}\lim_{y\to\infty} f_h(y) &= \lim_{y\to\infty}\Bigg(\frac{-hy^{-2}(1+y)^{-3}}{6(1+4y)^3(3+4y)^3}\Big(54(-2+h)(-1+h)+27(-2+h)(-41+38h)y \\ &\quad +9(2188-2913h+914h^2)y^2+2(44036-58065h+18208h^2)y^3 \\ &\quad +6(38187-51272h+16304h^2)y^4+96(3866-5307h+1712h^2)y^5 \\ &\quad +128(2897-4056h+1324h^2)y^6+3072(68-97h+32h^2)y^7 \\ &\quad +1536(33-48h+16h^2)y^8\Big)+O(y^{-4})\Bigg)=0\end{aligned}$$

and

$$\begin{aligned}\lim_{y\to\infty}\sigma_h(y) &= \lim_{y\to\infty} y^3\, S_h(y)=\lim_{y\to\infty}\frac{S_h(y)}{y^{-3}}=\frac{-1}{3}\lim_{y\to\infty} y^4\, S_h'(y) \\ &= \frac{-1}{3}\lim_{y\to\infty} y^4\Bigg(2\beta_h\left(y+1\right)-2\beta_h\left(y\right)-\beta_h'\left(y+1/4\right)-\beta_h'\left(y+3/4\right)\Bigg) \\ &= \frac{-1}{3}\lim_{y\to\infty} y^4\Bigg[\frac{h}{12y^4(1+y)^4(1+4y)^4(3+4y)^4}\Bigg(243(-2+h)^2h \\ &\quad +324(-2+h)(1-39h+19h^2)y+54(-2+h)(161-2702h+1275h^2)y^2 \\ &\quad +36(-5811+57892h-52450h^2+12475h^3)y^3+6\Big(-194996+1522235h \\ &\quad -1347234h^2+317120h^3\Big)y^4+4\Big(-917183+6635091h-5854144h^2 \\ &\quad +1379904h^3\Big)y^5+4\Big(-1808659+13264032h-11825216h^2+2811648h^3\Big)y^6 \\ &\quad +256\Big(-37511+290337h-263056h^2+63240h^3\Big)y^7+64\Big(-137837 \\ &\quad +1131816h-1043056h^2+253440h^3\Big)y^8+1024\Big(-5383+45939h \\ &\quad -43040h^2+10560h^3\Big)y^9+1024\Big(-2141+17904h-17056h^2 \\ &\quad +4224h^3\Big)y^{10}+98304\Big(-5+33h-32h^2+8h^3\Big)y^{11}-49152y^{12}\Bigg)+O(y^{-5})\Bigg] \\ &= \frac{h}{48}.\end{aligned}$$

Now $t_h(y+2)-t_h(y)<0$ for all $y>0$ and $1\le h<2$ and $\lim_{y\to\infty} t_h(y)=0$ then by the same way, we have $t_h(y)>0$ for all $y>0$ and $1\le h<2$, and then $f_h'(y+2)-f_h'(y)>0$ with $\lim_{y\to\infty} f_h'(y)=0$ then $f_h'(y)<0$ and hence $f_h(y)$ is decreasing for all $y>0$ and $1\le h<2$ with $\lim_{y\to\infty} f_h(y)=0$ then $f_h(y)>0$ and then $\sigma_h(y)$ is strictly increasing on $\mathbb{R}^+$ with $\lim_{x\to\infty}\sigma_h(y)=\frac{h}{48}$ and $\lim_{x\to 0}\sigma_h(y)=0$

and hence

$$0 < 2\ln\left[T_h\left(\frac{y}{2}, \frac{y+1}{2}\right)\right] - \beta_h\left(y+\frac{1}{4}\right) - \beta_h\left(y+\frac{3}{4}\right) < \frac{h}{48y^3}, \quad y > 0, \quad 1 \leq h < 2$$

and this gives (4.3). ☐

Remark 4.4. As $\Upsilon_8(y) > 0$ for $y \geq 1$ and $1 \leq h < 2$, and this proves that the upper bound given in (4.3) improves the upper bound given in (3.6) for $y \geq 1$ and $1 \leq h < 2$.

5. Conclusion

The main contributions of this paper are formulated and presented in Theorems 3.1 and 3.3. In particular, we derive several approximations for the Gurland's Ratio $T_h\left(\frac{y}{2}, \frac{y+1}{2}\right)$. Consequently, the obtained inequalities improve upon a number of recently established results. Moreover, these results lead to more accurate bounds for modified Bessel functions and other related functions.

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[1] Department of Mathematics and Statistics, Fahad Bin Sultan University, College of Sciences and Humanities, P.O. Box 15700 Tabuk, 71454, Saudi Arabia.
Email address: hmoustafa@fbsu.edu.sa

[2] Mathematics Department, King Abdulaziz University, Faculty of Science, P. O. Box 80203, Jeddah 21589, Saudi Arabia.
Email address: mmabadr@kau.edu.sa