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ON SOME CLASSES OF MODULES WITH GENERALIZED EXTENDING CONDITION

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Abstract. In this work, we present purely generalized extending (PGCS) modules. A module D is called PGCS if, for each submodule E of D, there exists a pure submodule X of D containing E for which $\frac{X}{E}$ is singular. This idea extends purely (generalized) extending modules. Moreover, PGCS modules are closed under direct summands. Furthermore, we establish the strongly purely generalized extending modules (SPGCS), where the pure submodule X is required to be fully invariant. This new structure lies properly between strongly extending modules and PGCS modules. We investigate many algebraic properties of these modules and characterize their relationships with other well-known classes of modules.

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1. Introduction

Throughout this work, rings are assumed to possess an identity and satisfy associativity, while all modules are right modules over R. Recall that a module D is known as an extending module or (CS) when all submodules of D are essential in a direct summand of D. This condition is equivalent to the requirement that each closed submodule of D represents a direct summand, [7]. Several generalizations of this property have been introduced, see for example [12] [13], [14] and [15]. A submodule E of a module D is said to be pure in D if $IE = ID \cap E$, for each finitely generated ideal I in R, see [2]. It is well known that all direct summands are pure submodules. According to Clark [6], a module D is purely extending provided that all submodules of D are essential in a pure submodule of D. Equivalently, D is known as purely extending module if all closed submodules of D are pure in D. Following [22], a module D is generalized extending module if for each submodule E of D, there is a direct summand X of D with $\frac{\mathrm{X}}{\mathrm{E}}$ is singular. Some generalizations of this concept are introduced for example [16] and [17].

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In this article, we introduce a module D that contains both the classes of generalized extending and purely extending. A module D is called purely generalized extending module (for short PGCS) if for every submodule E of D, there is a pure submodule X of D such that $\frac{X}{E}$ is singular. This work is devoted to examining a number of features of this new generalization. In addition, conditions guaranteeing that a direct sum of PGCS modules retains the PGCS property are established, and the inheritance of this property by submodules is also examined. If for each $\varphi \in \mathrm{End}(D), E$ contains $\varphi(E)$, then E is called fully invariant in D, D is called duo module provided that all submodules of D are fully invariant, see [4]. Following [21], a module D is called strongly extending module if every submodule of D is essential in a fully invariant direct summand of D, [23]. As a new generalization of this concept, we define a module D as a strongly purely generalized extending module if for all submodules $E \leqslant D$, there is a fully invariant pure submodule X of D such that $\frac{X}{E}$ is singular. It is clear that every SPGCS module is PGCS.

The article is organized into four main parts. The second section is devoted to the introduction of PGCS modules, where their fundamental characteristics are developed. Moreover, the position of this class within the hierarchy of several CS-type module generalizations is clarified.
In Section 3, we discuss the direct sums of PGCS modules.
Section 4 is devoted to investigate SPGCS with some of its properties and consider the relationships between SPGCS and several well-known concepts. Let D be a module over a ring R. We denote $\leqslant, \underset{\mathrm{P}}{\leqslant}, \underset{\oplus}{\leqslant}, \trianglelefteq$ and $\underset{\mathrm{e}}{\leqslant}$ for a submodule, pure submodule, a direct summand, a fully invariant, and essential submodules of D respectively.

The next collection of lemmas provides the groundwork for the results that follow.

Lemma 1.1. (*i*) *([20], Proposition 4.29). Every finitely generated right R-module is finitely presented if and only if R is Noetherian.*

(*ii*) *([20], Proposition 4.30). An R-module that is finitely generated is flat if and only if it is projective.*

Lemma 1.2. *([9], Proposition 8.1). For a module D, the following properties are satisfied.*

(*i*) *Let E be a submodule of D. If $\frac{D}{E}$ is flat, then $E \underset{P}{\leqslant} D$. Moreover, D is flat and $E \underset{P}{\leqslant} D$ if and only if $\frac{D}{E}$ is flat.*

(*ii*) *Let E be a submodule of D such that each finitely generated submodule of E is a pure submodule of D, then $E \underset{P}{\leqslant} D$.*

Lemma 1.3. [6] *Let $A \leqslant E \leqslant D$ be right R-modules. Then.*

(*i*) *If $A \underset{P}{\leqslant} E$ and $E \underset{P}{\leqslant} D$, then $A \underset{P}{\leqslant} D$.*

(*ii*) *If $A \underset{P}{\leqslant} D$, then $A \underset{P}{\leqslant} E$.*

(*iii*) *If $A \underset{P}{\leqslant} E$, then $\frac{E}{A} \underset{P}{\leqslant} \frac{D}{A}$.*

(iv) *If* $A \underset{P}{\leqslant} D$ *and* $\frac{E}{A} \underset{P}{\leqslant} \frac{D}{A}$, *then* $E \underset{P}{\leqslant} D$.

(v) *Let* $\mathrm{D} = \oplus_{\mathrm{i} \in \mathrm{I}} \mathrm{D_i}$, *where* $\mathrm{D_i} \leqslant \mathrm{D}$, *for each* $\mathrm{i} \in \mathrm{I}$ *and let* $\mathrm{E_i} \leqslant \mathrm{D_i}$ *for each* $\mathrm{i} \in \mathrm{I}$. *Then* $\oplus_{\mathrm{i} \in \mathrm{I}} \mathrm{E_i} \underset{P}{\leqslant} \mathrm{D}$ *if and only if* $\mathrm{E_i} \underset{P}{\leqslant} \mathrm{D_i}$, *for each* $\mathrm{i} \in \mathrm{I}$.

2. Purely generalized extending modules

In this part, we present the concept of purely generalized extending (PGCS) modules. A number of basic features of this class are shown, and its associations with some well-known generalizations of extending modules are considered.

Definition 2.1. Let D be a right R -module, D is called a purely generalized extending module (for short PGCS), if for every submodule E of D, there is a pure submodule X of D such that $\frac{X}{E}$ is singular.

The relationship between PGCS and some of the generalized extending conditions will be presented in the next Proposition.

Proposition 2.2. *Let D be a right R-module. Consider the following statements.*

(i) *D is CS,*
(ii) *D is generalized extending;*
(iii) *D is PGCS;*
(iv) *D is purely extending;*
(v) *D is singular.*

Then $(i) \Rightarrow (ii) \Rightarrow (iii), (i) \Rightarrow (iv) \Rightarrow (iii)$ *and* $(v) \Rightarrow (ii)$, *while the converse implications do not hold, in general.*

Proof. $(i) \Rightarrow (ii)$ From [22].
$(ii) \Rightarrow (iii)$ It is clear from the conclusion that every direct summand of D is a pure submodule of D.
$(i) \Rightarrow (iv)$ See [6].
$(iv) \Rightarrow (iii)$ Let D be a purely extending module, and let E be a submodule of D; hence, there exists a pure submodule X of D such that $E \underset{e}{\leqslant} X$. Therefore, $\frac{X}{E}$ is singular.
$(v) \Rightarrow (ii)$ It follows from [22].
$(ii) \nRightarrow (i)$ Let $D = \mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$, where p is prime, as $\mathbb{Z}$-module. D is generalized extending module which is not CS.
$(iii) \nRightarrow (ii)$ Let $D = \bigoplus_{i \in I} \mathbb{Z}$. Since D is purely extending, it is PGCS which is not generalized extending module. Another example, let D be the $\mathbb{Z}$-module $\Pi\mathbb{Z}$, then D is PGCS and not generalized extending module. Also, these are examples to show that $(iv) \nRightarrow (i)$.
$(iii) \nRightarrow (iv)$ The $\mathbb{Z}$-module $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ is singular; hence it is PGCS, while it is not purely extending module.
$(ii) \nRightarrow (v)$ $\mathbb{Z}$ as $\mathbb{Z}$-module is generalized extending which is nonsingular module. □

Recall that a module D is known as a pure split when each of its pure submodule is a direct summand of D, [3]. Next, we introduce several propositions that characterize conditions under which the PGCS property is equivalent to the generalized extending property.

Proposition 2.3. *The following statements hold, for a module D.*

(i) *If D is a pure split, then D is PGCS if and only if it is generalized extending.*

(ii) *If D is nonsingular or projective, then D is PGCS if and only if it is purely extending.*

Proof. (i) The result follows directly from the definition.

(ii) In a nonsingular (projective) module, for any submodule E of $D, E \leqslant_e D$ if and only if $\frac{D}{E}$ is singular, [10]. So, the result is immediately obtained. □

Theorem 2.4. *Let D be a finitely generated and flat R-module over a Noetherian ring. Then D is generalized extending if and only if it is PGCS.*

Proof. The result is obtained directly from Definition 2.1 and Lemma 1.2. □

Theorem 2.5. *Let D be a module over a pure semisimple ring R. Then D is generalized extending if and only if it is PGCS.*

Proof. Let E be a submodule of a PGCS module D, there is $X \underset{P}{\leqslant} D$ such that $\frac{X}{E}$ is singular. As R is a pure semisimple ring, for any right R-module Y, the pure short exact sequence $0 \to X \otimes L \to D \otimes L \to Y \otimes L \to 0$ is split for each left R-module L. Consequently, the short exact sequence $0 \to X \to D \to Y \to 0$ is split. Thus, $X \underset{\oplus}{\leqslant} D$. Thus, D is generalized extending. The reverse implication follows routinely. □

It is well known that any submodule of singular module is again singular, while this need not be true in case of PGCS, as the following example shows.

Example 2.6. Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ be a submodule of its injective hull S_R, [5]. S_R is an extending module (hence it is PGCS), while R is not purely extending and nonsingular. Proposition 2.3 yields that R fails to be PGCS.

Next, we will examine certain conditions under which the submodules of PGCS again satisfy the PGCS condition.

Proposition 2.7. *Let D be a PGCS module and let E be a submodule of D. If for each pure submodule X of $D, X \cap E \underset{P}{\leqslant} D$, then E is PGCS.*

Proof. Let $A \leqslant E$, then there is a pure submodule X of D such that $\frac{X}{A}$ is singular, which implies $\frac{X \cap E}{A}$ is singular. Since $X \cap E \underset{P}{\leqslant} D$; therefore, $X \cap E \underset{P}{\leqslant} E$. Thus, E is PGCS. □

A module D is said to have the pure intersection property (PIP) when the intersection of any pair of pure submodules is pure in D, [2]

Proposition 2.8. *Let D be a PGCS module and E be a pure submodule of D. If D has the PIP, then E is PGCS module.*

Proof. Let A be a submodule of E. As D is PGCS, there is a pure submodule X of D such that $\frac{X}{A}$ is singular. Since D has the PIP, then $X \cap E$ is pure in E and $\frac{X \cap E}{A}$ is singular. Thus, E is PGCS. □

The next corollaries are immediate consequences of Proposition 2.8.

Corollary 2.9. *Let D be a multiplication R-module and E be a pure submodule of D. If D is PGCS, then E is PGCS.*

Corollary 2.10. *Let D be a cyclic module which is PGCS and let E be a pure submodule of D. Then E is PGCS.*

Corollary 2.11. *Let R be a commutative ring with identity and I be a pure ideal in R. If R is PGCS, then I is PGCS.*

Proposition 2.12. *Any homomorphic image of PGCS is also PGCS.*

Proof. Let D be a PGCS module and let $\psi : D \to D_1$ be an epimorphism and E be a submodule of D_1, there exists $A \leqslant D$ such that $E \cong \frac{A}{\operatorname{Ker}\psi}$. As D is PGCS module, there is a pure submodule X of D such that $\frac{X}{A}$ is singular. By Lemma (1.3 (iii)), $\frac{X}{Ker\psi} \underset{P}{\leqslant} \frac{D}{Ker\psi} \cong D_1$. Since $\frac{\frac{X}{Ker\psi}}{\frac{A}{Ker\psi}} \cong \frac{X}{A}$ is singular, D_1 is PGCS module. □

Corollary 2.13. *Any direct summand of PGCS module is also PGCS module.*

Corollary 2.14. *Every quotient module of a PGCS module is again PGCS.*

We conclude this section with the following results, whose proofs follow directly from [19].

Proposition 2.15. *Let D be a finitely generated which is torsion free module over a principal ideal domain, then D is PGCS module.*

Proposition 2.16. *Each finitely generated flat module over a principal ideal domain is PGCS.*

Proposition 2.17. *Every finitely generated torsion free module over a prufer ring is a PGCS module.*

3. Direct sums of purely generalized extending modules

This part is devoted to analyzing the behavior of the PGCS structure under direct sums. Several sufficient conditions verifying that a direct sum of PGCS modules remains PGCS are concluded.

Let $D = \mathbb{Z}[d] \oplus \mathbb{Z}[d]$ as $\mathbb{Z}[d]$-module, d is indeterminate. Since $\mathbb{Z}[d]$ is extending, then it is PGCS, however; D is not extending over an integral domain. So, ([2], Corollary 3.17) yields that D is not purely extending. As D is nonsingular, so, by Proposition 2.3 D is not PGCS module.

Proposition 3.1. *Let D_1 and D_2 be PGCS modules. If* $\operatorname{ann}(D_1)+\operatorname{ann}(D_2)=R$, *then $D=D_1\oplus D_2$ is PGCS.*

Proof. Take E to be a submodule of D. Since $R=\operatorname{ann}(D_1)+\operatorname{ann}(D_2)$, then $E=E_1\oplus E_2$, where $E_1\leqslant D_1$ and $E_2\leqslant D_2$, by ([1], Proposition 4.2). Since D_1 and D_2 are PGCS, there exists $X_1\underset{P}{\leqslant} D_1$ and $X_2\underset{P}{\leqslant} D_2$ such that $\frac{X_1}{E_1}$ and $\frac{X_2}{E_2}$ are singular. By Lemma 1.3, $X_1\oplus X_2\underset{P}{\leqslant} D_1\oplus D_2$ and $\frac{X_1\oplus X_2}{E_1\oplus E_2}$ is singular. Therefore, D is PGCS module. $\square$

The same argument can be used to demonstrate the next two results.
A module D is known as distributive, if for every submodules D_1, D_2 and D_3 of D, the next equality holds $D_1\cap(D_2+D_3)=(D_1\cap D_2)+(D_1\cap D_3)$, see [8].

Proposition 3.2. *Let $D=D_1\oplus D_2$, where each of D_1 and D_2 are PGCS. If D is distributive module, then D is PGCS.*

Proposition 3.3. *Let $D=D_1\oplus D_2$, where each of D_1 and D_2 are PGCS. If D is duo module, then D is PGCS module.*

Following [24], an R-module D is $\mathbb{F}$-regular, provided that all submodules of D are pure in D.

Theorem 3.4. *Let $D=D_1\oplus D_2$ with D_1 is singular (uniform) and D_2 is $\mathbb{F}$-regular. Then D is PGCS.*

Proof. Take E to be a submodule of D. Then $E+D_1=D_1\oplus[(E+D_1)\cap D_2]$. Since D_2 is $\mathbb{F}$-regular, it follows that $(E+D_1)\cap D_2\underset{P}{\leqslant} D_2$ and hence $E+D_1\underset{P}{\leqslant} D$. Observe that $\frac{E+D_1}{E}\cong\frac{D_1}{E\cap D_1}$ is singular. Thus, D is PGCS module. $\square$

Proposition 3.5. *Let $D=D_1\oplus D_2$, where D_1 is PGCS and D_2 is $\mathbb{F}$-regular. Assume that $E\cap D_1$ is a summand of E, for every $E\leqslant D$. Then D is PGCS module.*

Proof. Let $E\leqslant D$, then $E+D_1\underset{\oplus}{\leqslant} D_1$, as in Theorem 3.4. By assumption we have $E=(E\cap D_1)\oplus L$, for some $L\leqslant$ E. Since D_1 is PGCS, there is $X\underset{P}{\leqslant} D_1$ such that $\frac{X}{E\cap D_1}$ is singular. However, $E+D_1=(E\cap D_1)\oplus L+D_1=L\oplus D_1$. So, $\frac{X\oplus L}{E}=\frac{X\oplus L}{(E\cap D_1)\oplus L}\cong\frac{X}{E\cap D_1}$ is singular. Since $X\underset{P}{\leqslant} D_1$ and $L\underset{\oplus}{\leqslant} E$, it follows that $X\oplus L\underset{P}{\leqslant} E\oplus D_1\underset{P}{\leqslant} D$, by Lemma 1.3. Therefore, D is PGCS module. $\square$

Proposition 3.6. *Let $D=D_1\oplus D_2$ where D_1 is PGCS and D_2 is injective. Assume that $E\cap D_2$ is a direct summand of E, for any $E\leqslant D$. Then D is PGCS module.*

Proof. Take $E\leqslant D$, there exists $E'\leqslant E$ such that $E=(E\cap D_2)\oplus E'$, by assumption. Observe that $E'\cap D_2=0$; hence, $\frac{D_2+E'}{E'}\cong D_2$ is injective module, it follows that there exists $D'\leqslant D$ containing E' and $\frac{D}{E'}=\frac{(D_2+E')}{E'}\oplus\frac{D'}{E'}$. It is evident that $D=D_2\oplus D'$ and $D'\cong\frac{D}{D_2}\cong D_1$. Thus, D' is PGCS. Hence, there is $X\leqslant D'$

such that $\frac{X}{E'}$ is singular. Since D_2 is injective, then there is a direct summand K of D_2 with $\frac{K}{E\cap D_2}$ is singular. Now, observe that $\frac{K\oplus X}{(E\cap D_2)\oplus E'} \cong \frac{K}{E\cap D_2} \oplus \frac{X}{E'}$ and $K \oplus \mathrm{X} \underset{\mathrm{P}}{\leqslant} D$. Thus, D is PGCS module, $\square$

Proposition 3.7. *Let $D = D_1 \oplus D_2$ where D_1 is PGCS and D_2 is injective, then D is PGCS if and only if for each $E \leqslant D$ such that $E \cap D_2 \neq 0$, there is $X \underset{P}{\leqslant} D$ such that $\frac{X}{E}$ is singular.*

Proof. Assume that for each $E \leqslant D$ such that $E \cap D_2 \neq 0$, there is $\mathrm{X} \underset{P}{\leqslant} D$ such that $\frac{X}{E}$ is singular. Let $E \leqslant D$ such that $E \cap D_2 = 0$. Observe that $\frac{D_2+E}{E} \cong D_2$ is injective; hence, there exists $D' \leqslant D$ containing E such that $\frac{D}{E} = \frac{D'}{E} \oplus D_2$. One can verify that $D = D' \oplus D_2$. As $D' \cong \frac{D}{D_2} \cong D_1$ is PGCS, there exists $X \underset{P}{\leqslant} D'$; hence $X \underset{P}{\leqslant} D$ with $\frac{X}{E}$ is singular. So D is PGCS. The converse is obvious. $\square$

Since any module is a homomorphic image of some projective module, it is easy to prove the following corollary.

Corollary 3.8. *The following conditions are equivalent.*

(*i*) *Each (finitely generated) module is PGCS module;*
(*ii*) *Each (finitely generated) projective module is PGCS.*

Proposition 3.9. *The following conditions are equivalent, for a ring R.*

(*i*) *$\underset{I}{\oplus} R$ is PGCS, for all index set I.*
(*ii*) *Each projective R-module is PGCS.*

Proof. $(i) \Rightarrow (ii)$ Let D be a projective R-module, there exists a free R-module X and an epimorphism $\psi : X \to D$, ([18] Corollary 4.4.4). As X is free, then $X \cong \bigoplus_I R$, for every index set I. Take the following short exact sequence

$$0 \to Ker\psi \xrightarrow{\mathrm{i}} \oplus_I R \xrightarrow{\psi} D \to 0$$

where i denotes the inclusion function. As D is projective, then the short exact sequence splits. It follows that $\bigoplus_I R = \ker\psi \bigoplus D$. Corollary 2.13 yields that D is a PGCS module.
$(ii) \Rightarrow (i)$ Straightforward. $\square$

Proposition 3.10. *The conditions listed below are equivalent, for a ring.*

(*i*) *$\bigoplus_I R$ is PGCS, for each finite index set I;*
(*ii*) *Every finitely generated R-module is PGCS, for all finite set I.*

Proof. The proof is analogous. $\square$

Theorem 3.11. *For a nonsingular ring R, The following conditions are equivalent.*

(*i*) *R is PGCS, for each index set I;*
(*ii*) *Each projective R-module is PGCS;*
(*iii*) *For each R-module D, there is a singular $X \leqslant D$ with $\frac{D}{X}$ is flat.*

(iv) *Each nonsingular R-module is flat.*
(v) *Each nonsingular R-module is PGCS.*

Proof. $(i) \Rightarrow (ii)$ By proposition 3.9
$(ii) \Rightarrow (iii)$ Let D be an R-module, there is free module K and epimorphism $\psi : K \to D \to 0$. Let $j : \operatorname{Ker}\psi \to K$ represents the inclusion map. Since K is free, then it is PGCS module. So, there exists $X \underset{P}{\leqslant} K$ such that $\frac{X}{\operatorname{Ker}\psi}$ is singular. Observe that $\frac{\frac{K}{\operatorname{Ker}\psi}}{\frac{X}{\operatorname{Ker}\psi}} \cong \frac{K}{X}$ and consider the following short exact sequence

$$0 \to X \xrightarrow{g} K \xrightarrow{\pi} \frac{K}{X} = 0$$

where g represents the inclusion map and π represents the canonical. Since $X \underset{P}{\leqslant} K$ and K is flat, then $\frac{K}{X}$ is flat, by Lemma 1.2. But $\frac{K}{\operatorname{Ker}\psi} \cong D$, therefore the result is obtained.
$(iii) \Rightarrow (iv)$ Let D be a nonsingular R-module. By (iii), there is a singular submodule $X \leqslant D$ with $\frac{D}{X}$ is flat. As D is nonsingular, $X = 0$. So, D is flat.
$(iv) \Rightarrow (v)$Taking D to be a nonsingular R-module and $E \leqslant D$. Let $X \leqslant D$ such that $\frac{X}{E}$ is singular submodule of $\frac{D}{E}$. As R is nonsingular, then $\frac{D}{X}$ is nonsingular. So, (iv) implies that $\frac{D}{X}$ is flat. Now taking the following short exact sequence.

$$0 \to X \xrightarrow{j} D \xrightarrow{h} \frac{D}{X} \to 0$$

where j and h represent the inclusion map and canonical, respectively. Since $\frac{D}{X}$ is flat, then $X \underset{P}{\leqslant} D$, by Lemma 1.2. Thus, D is PGCS.
$(v) \Rightarrow (i)$ It is not hard to see that $\bigoplus_I R$ is nonsingular. So, by (v), $\bigoplus_I R$ is PGCS. □

4. Strongly purely generalized extending modules

In the final section, we present strongly purely generalized extending (SPGCS) modules as a stronger version of PGCS modules. Different properties of this class are explored, and its relationships with several related concepts in module theory are given.

Lemma 4.1. [4]. *Let D be R-module, then.*

(i) *All intersections and sums of fully invariant submodules of D are also fully invariant submodules of D.*
(ii) *If $A \trianglelefteq E \trianglelefteq D$, then $A \trianglelefteq$ D.*
(iii) *Let $D = \bigoplus_{i=1}^{n} D_i$ and $E \trianglelefteq D$, then $E = \bigoplus_{i=1}^{n} (E \cap U_i)$, and $E \cap D_i \trianglelefteq D$.*

Definition 4.2. A module D is called a strongly purely generalized extending module (abbreviated as SPGCS module), provided for every submodule E of D, there is a fully invariant pure submodule X of D with $\frac{X}{E}$ is singular.

The following proposition locates the SPGCS module among some of the other extending conditions.

Proposition 4.3. *Taking D to be a module. Consider the properties listed below.*

(i) *D is strongly extending,*
(ii) *D is SPGCS;*
(iii) *D is PGCS;*
(iv) *D is CS.*
(v) *D is generalized extending,*
(vi) *D is singular.*

Then $(i) \Rightarrow (ii) \Rightarrow (iii), (i) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (iii)$ *and* $(vi) \Rightarrow (ii)$.

Proof. $(i) \Rightarrow (ii)$ Let E be a submodule of a strongly extending module D, there is $X \underset{\oplus}{\leqslant} D$ such that $E \underset{e}{\leqslant} X$, where $X \trianglelefteq D$. Hence, $\frac{X}{E}$ is singular and $X \underset{P}{\leqslant} D$.
$(ii) \Rightarrow (iii)$ It is clear.
$(i) \Rightarrow (iv)$ From [23].
$(iv) \Rightarrow (v)$ It follows from [22].
$(v) \Rightarrow (iii)$ Proposition 2.2.
$(vi) \Rightarrow (ii)$ Let D be a singular module, for any submodule E of D, $\frac{D}{E}$ is singular, where D is a trivial fully invariant pure submodule. So, it is SPGCS.
$(ii) \nRightarrow (i)$ Let $D = \mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$ as $\mathbb{Z}$-module. Since D is singular, then it is SPGCS. However; D is not CS, so it is not strongly extending. This example also shows that $(v) \nRightarrow (iv)$.
$(iii) \nRightarrow (ii)$ Let $D = \mathbb{Z} \oplus \mathbb{Z}$ as $\mathbb{Z}$-module. Observe that $\mathbb{Z}$ is nonsingular, then ([2], Theorem 3.15) yields that D is purely extending and hence PGCS. Now, let $E = \mathbb{Z} \oplus 0$. The only fully invariant pure submodule of D is D and 0 and $\frac{D}{E} \cong \mathbb{Z}$ which is nonsingular. Thus, D is not SPGCS.
$(iv) \nRightarrow (i)$ Let $R = \begin{pmatrix} \mathbb{F} & \mathbb{F} \\ 0 & \mathbb{F} \end{pmatrix}$, where $\mathbb{F}$ is any field. Then R_R is extending, however; it is not strongly extending. For more details, see [23].
$(iii) \nRightarrow (v)$ See Proposition 2.2.
$(ii) \nRightarrow (vi)$ $\mathbb{Z}$ as $\mathbb{Z}$-module is SPGCS which is nonsingular. □

A module D is known as weak duo (P-duo) module when each direct summand (pure submodule) of D is a fully invariant submodule of D, [21] and [11]. Next, we establish a sufficient condition ensuring the equivalence between strongly purely (generalized) extending modules and purely (generalized) extending modules.

Lemma 4.4. *Let D be an R-module, then.*

(i) *If D is weak duo, then D is extending if and only if it is strongly extending.*
(ii) *If D is P-duo, then D is PGCS if and only if it is SPGCS.*

Example 4.5. (i) It is not necessary for a submodule of SPGCS module to be SPGCS Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ be a submodule of its injective hull D_R. Since R_R is not PGCS, Proposition 4.3 implies that it is not SPGCS.
(ii) Let $D = \mathbb{Z}_p \oplus \mathbb{Z}_p, p$ is prime as $\mathbb{Z}$-module, As D is singular, D is SPGCS. Now, let $\varphi : \mathbb{Z}_p \oplus \mathbb{Z}_p \to \mathbb{Z}_p \oplus \mathbb{Z}_p$ be defined by $\varphi(\bar{a}, \bar{b}) = (\bar{b}, \bar{a})$, for all $\bar{a}, \bar{b} \in \mathbb{Z}_p$. The closed submodule $E = \mathbb{Z}_p \oplus 0$ is not fully invariant, whereas $\varphi(E) = \varphi(\mathbb{Z}_p \oplus 0) \nsubseteq \mathbb{Z}_p \oplus 0$. So, D is not strongly extending.
(iii) Each commutative domain as a module over itself is SPGCS.

Proposition 4.6. *The SPGCS property is inherited by direct summands.*

Proof. Let $D = D_1 \oplus D_2$ be an SPGCS module, let $E \leqslant D_1$, there exists a pure fully invariant submodule X of D such that $\frac{X}{E}$ is singular. Since $X \trianglelefteq D$, then $X = X \cap D = X \cap (D_1 \oplus D_2) = (X \cap D_1) \oplus (X \cap D_2) \cdot$ By Lemmas 1.3 and 4.1, $X \cap D_1$ is a pure and fully invariant in D_1 and $\frac{X \cap D_1}{E} \leqslant \frac{X}{D}$ is singular. So, D_1 is SPGCS. □

Proposition 4.7. *Let D be an SPGCS module and $A \leqslant D$, then $\frac{D}{A}$ is SPGCS.*

Proof. Let $\frac{E}{A} \leqslant \frac{D}{A}$, there is a pure fully invariant submodule X of D with $\frac{X}{E}$ is singular. Lemma 1.3, yields that $\frac{X}{A} \underset{\mathrm{P}}{\leqslant} \frac{\mathrm{D}}{\mathrm{A}}$ and easily $\frac{X}{A} \trianglelefteq \frac{D}{A}$. Observe that $\frac{X}{E} \cong \frac{\frac{X}{A}}{\frac{E}{A}}$ is singular. Thus, $\frac{D}{A}$ is SPGCS. □

The $\mathbb{Z}$-module $\mathbb{Z} \bigoplus \mathbb{Z}$ shows that the direct sum of SPGCS modules may not be SPGCS. While the next results indicate that under certain conditions this property may hold.

Proposition 4.8. *Let $D = D_1 \oplus D_2$ be a direct sum of SPGCS modules. If D is distributive, then D is SPGCS.*

Proof. Let $E \leqslant D$, then $E \cap D_i \leqslant D_i, i = 1, 2$. As D_i is SPGCS, $i = 1, 2$, there exists pure fully invariant X_i of $D_i, \forall i = 1, 2$, such that $\frac{X_i}{E \cap D_i}$ is singular; hence $\frac{X_1 \oplus X_2}{(E \cap D_1) \oplus (E \cap D_2)}$ is singular, according to the distributivity of D. So, D is SPGCS. □

Proof. The proof is routine. □

Theorem 4.9. *Let $D = D_1 \oplus D_2$, where D_1 is singular (uniform) and D_2 is $\mathbb{F}$-regular duo module. Then D is SPGCS.*

Proof. The result follows directly from Theorem 3.4. □

Proposition 4.10. *Let $D = D_1 \oplus D_2$, where D_1 is SPGCS and D_2 is weak duo injective module. Assume that $E \cap D_2$ is a direct summand of E, for any $E \leqslant D$. Then D is SPGCS.*

Proof. The proof follows from Proposition 3.6. □

Compare the following result with Proposition 3.7.

Proposition 4.11. *Let $D = D_1 \oplus D_2$, where D_1 is SPGCS and D_2 is injective, then D is SPGCS if and only if for every $E \leqslant D$ such that $E \cap D_2 \neq 0$, there is a fully invariant pure submodule X of D such that $\frac{X}{E}$ is singular.*

Proof. The proof is an immediate consequence of Proposition 3.7. □

Proposition 4.12. *The following are equivalent.*

(i) *$\underset{I}{\oplus} R$ is SPGCS, for each index set I.*

(ii) *Each projective R module is SPGCS.*

Proof. The result follows directly from Proposition 3.9. □

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