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TIME-OPTIMAL CONTROL OF THE HEAT CONDUCTION EQUATION

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Abstract. We study a time-optimal control problem for the heat conduction equation with a general elliptic operator and Robin boundary condition. The goal is to drive the weighted average temperature to a prescribed value in minimal time using an admissible control subject to a uniform bound. Using spectral decomposition and the Green function, the problem is reduced to a Volterra integral equation. Analyzing its kernel via the operator's eigenpairs, we derive the maximal attainable average temperature. For any target below this threshold, we prove the existence of a unique minimal time and establish its asymptotic expansion, revealing logarithmic growth governed by the principal eigenvalue.

Keywords. Elliptic operator, Volterra integral equation, admissible control, Robin boundary condition, time-optimal control
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1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a piecewise smooth boundary $\partial\Omega$. We introduce the linear elliptic operator

$$\mathcal{A}v(x) := -\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial v}{\partial x_j}\right) + c(x)v(x),$$

where $a_{ij}(x) = a_{ji}(x)$, $a_{ij} \in L^\infty(\Omega)$, $c(x) \in L^\infty(\Omega)$, $c(x) \geq 0$, and the uniform ellipticity condition holds:

$$\sum_{i,j=1}^{3} a_{ij}(x)\xi_i\xi_j \geq \gamma|\xi|^2, \quad \forall \xi \in \mathbb{R}^3, \text{ a.e. } x \in \Omega,$$

with a constant $\gamma > 0$.

In this paper, we consider the heat conduction equation

$$u_t(x,t) + \mathcal{A}u(x,t) = 0, \quad x \in \Omega, \quad t > 0, \tag{1.1}$$

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with boundary conditions

$$\frac{\partial u}{\partial n_{\mathcal{A}}} + h(x)u(x,t) = 0, \quad x \in \partial\Omega \setminus \Gamma, \quad t > 0, \tag{1.2}$$

and

$$\frac{\partial u}{\partial n_{\mathcal{A}}} = a(x)\mu(t), \quad x \in \Gamma, \quad t > 0, \tag{1.3}$$

and initial condition

$$u(x,0) = 0. \tag{1.4}$$

Here $\Gamma \subset \partial\Omega$ is a nonempty portion of the boundary, representing the location of a heater or air conditioner. Its boundary $\partial\Gamma$ is assumed piecewise smooth, and we denote by $\operatorname{mes}\Gamma > 0$ its surface measure (which differs from the Lebesgue measure $|\Gamma|$). The conormal derivative $\frac{\partial}{\partial n_{\mathcal{A}}}$ associated with $\mathcal{A}$ is defined by

$$\frac{\partial u}{\partial n_{\mathcal{A}}} := \sum_{i,j=1}^{3} a_{ij}(x)\frac{\partial u}{\partial x_j} n_i(x), \quad x \in \partial\Omega,$$

where $n(x) = (n_1(x), n_2(x), n_3(x))$ is the outward unit normal vector to $\partial\Omega$. In the isotropic case $a_{ij}(x) = \delta_{ij}$, this reduces to the classical normal derivative $\frac{\partial u}{\partial n}$.

The function $h(x) \geq 0$ governs the heat exchange rate with the external medium on the passive part of the boundary; it is taken to be piecewise smooth, non-negative, and not identically zero. The function $a(x) \geq 0$ describes the spatial distribution of the actuator's efficiency on the active part Γ; it shares the same regularity properties and is also not identically zero. Condition (1.3) indicates that a stream of hot or cold air is applied on Γ with a time-varying intensity $\mu(t)$, whereas (1.2) prescribes Newton's law of cooling on the complementary part $\partial\Omega \setminus \Gamma$ (see, e.g., [32], Sect. III.1.4).

From a physical point of view, the operator $\mathcal{A}$ characterizes the diffusion mechanism of heat within the medium. The matrix of coefficients $\{a_{ij}(x)\}$ represents the thermal conductivity tensor, which allows the model to account for spatial heterogeneity and anisotropic heat transfer effects. Such a formulation is appropriate for describing heat propagation in complex materials, including composites and heterogeneous solids. The additional term $c(x)v(x)$ with $c(x) \geq 0$ describes a linear heat absorption or sink effect, which may arise, for instance, due to radiation losses or interaction with a surrounding medium at zero reference temperature. Thus, the operator $\mathcal{A}$ combines both conductive transport and local thermal dissipation within the interior of Ω.

The functions h and a are extended to the entire boundary $\partial\Omega$ by assigning $h(x) = 0$ on Γ and $a(x) = 0$ on $\partial\Omega \setminus \Gamma$. With these extensions, (1.2) and (1.3) are combined into the single boundary condition

$$\frac{\partial u}{\partial n_{\mathcal{A}}} + h(x)u(x,t) = a(x)\mu(t), \quad x \in \partial\Omega, \quad t > 0. \tag{1.5}$$

By a solution of the initial-boundary value problem (1.1)-(1.5) we understand a generalized solution in the sense of [24] (see Chapter III, Sec. 5).

Control problems associated with parabolic equations have attracted considerable attention over the past several decades due to their central role in modeling heat conduction and diffusion phenomena. One of the earliest investigations in this direction was carried out by Friedman [22], who examined control problems for parabolic equations using variational methods. Foundational contributions to the controllability theory of linear parabolic equations were made by Fattorini and Russell [21], where fundamental concepts related to exact and approximate controllability were introduced. The extension of control theory to infinite-dimensional settings was advanced by Egorov [18], who generalized Pontryagin's maximum principle to evolution equations in Banach spaces and established a bang–bang principle under suitable hypotheses. Comprehensive treatments of optimal control for distributed parameter systems, including parabolic equations, can be found in the classical monograph [27].

The time-optimal control problem for the heat equation and the structural properties of optimal controls were intensively studied in subsequent works. In particular, the time-varying bang–bang property of optimal controls and its practical implications were analyzed in [8]. Various numerical and applied aspects of optimal control for heat equations were discussed in [3], while practical boundary control problems for heat conduction models were addressed in [26]. The bang–bang principle for time-optimal boundary control problems was further examined in [30], where it was shown that bang–bang controls remain optimal for arbitrary target temperature distributions.

The boundary control problem for a pseudo-parabolic equation was studied in [9], where a necessary control function was constructed to achieve a prescribed temperature level in an inhomogeneous thin plate. The time-optimal control problem for a one-dimensional heat equation was further investigated in [10], while the time-optimal control of a fourth-order parabolic equation in a three-dimensional domain was analyzed in [11, 12]. Boundary control problems for pseudo-parabolic equations in one- and two-dimensional domains have been studied in detail in [13, 14]. Recently, [5] studied optimal control problems for fourth-order linear parabolic equations using the finite element method with Hermite basis functions, establishing existence and uniqueness results for the state and adjoint systems under Neumann boundary conditions. In [16], a time-optimal control problem for a fourth-order parabolic equation modeling thin-film growth was studied, and it was shown that the minimal time depends on process parameters as the average height approaches a critical value.

Initial studies of time-optimal control problems for heat conduction equations in bounded n-dimensional domains were presented in [1], where sharp estimates for the minimal time required to achieve a prescribed average temperature were derived. Various related problems for nonlinear and degenerate parabolic equations have also attracted considerable attention; see, for instance, [23], where local existence and uniqueness of weak solutions were established via the Galerkin method and the Banach fixed point theorem.

Recently, in [15], a time-optimal control problem for the heat equation involving an involution operator in a multidimensional parallelepiped domain was investigated. In that work, an optimal estimate for the minimal time required to heat the domain to a prescribed average temperature was obtained. Recent developments have also addressed controllability issues from a spectral and observability viewpoint. In [17], spectral inequalities for the Laplace–Beltrami operator were derived, leading to observability and null-controllability results for the spherical heat equation. Boundary control problems for generalized heat equations involving Schrödinger-type operators were studied in [19]. Numerical aspects of boundary control and discretization schemes for heat equations were analyzed in [2, 6, 25]. Applications to fluid dynamics and thermally coupled systems were considered in [31].

In [4], time-optimal control problems for semilinear parabolic equations subject to Dirichlet boundary control and terminal state constraints are analyzed. In that work, a Hamiltonian-type functional is introduced in order to derive necessary optimality conditions with respect to the terminal time T. The existence of time-optimal controls for a class of semilinear parabolic equations with distributed control acting on a subdomain is investigated in [33], where sufficient conditions ensuring the attainability of the target state in minimal time are established.

In the present paper, we study a time-optimal control problem for the heat conduction equation governed by a uniformly elliptic operator with a Robin-type boundary condition. The control is applied on an open subset of the boundary and represents the heat flux used to regulate the thermal process inside the three-dimensional domain. The main objective is to analyze the time-optimal control problem, where the goal is to drive the spatially weighted average temperature to a prescribed level in minimal time using an admissible boundary control subject to a uniform bound. This formulation is physically motivated by thermal processes where the overall energy balance, rather than the pointwise temperature distribution, is the key quantity of interest.

Using the spectral decomposition of the elliptic operator and the associated Green function, the problem is reduced to a Volterra integral equation of the first kind with a positive and monotonically decaying kernel. The asymptotic behavior of the kernel is analyzed in terms of the eigenpairs of the operator. From this analysis we derive a critical value which represents the maximal attainable average temperature under the given power constraint. For any target value below this critical threshold, we prove the existence of a unique minimal time and obtain an asymptotic expansion of this time as the target approaches the critical value from below. The expansion reveals a logarithmic dependence on the distance to the critical value, with the principal eigenvalue of the elliptic operator playing a key role. Thus, the solvability of the integral equation and the derivation of sharp asymptotic estimates are crucial for characterizing the minimal time.

2. Statement of the problem

Let $M > 0$ be a given constant representing the maximal available power of the actuator. A measurable function $\mu(t)$ defined for $t \geq 0$ is called an *admissible control* if it satisfies the uniform bound

$$|\mu(t)| \leq M, \qquad t \geq 0.$$

Next, let $\rho : \overline{\Omega} \to \mathbb{R}$ be a given weight function such that

$$\int_\Omega \rho(x)\,dx = 1, \qquad \rho(x) \geq 0.$$

This function ρ allows us to focus on a specific spatial average of the temperature field. Note that ρ is not required to be strictly positive; in particular, the quantity

$$\int_\Omega \rho(x)\,u(x,t)\,dx = \theta \tag{2.1}$$

may represent, for example, the average temperature over a certain subdomain of Ω, or even the temperature value at a single point in a limiting sense.

Denote by $T(\theta)$ the minimal time needed for this weighted average to attain the prescribed level θ under an admissible control. More precisely, $T(\theta)$ is such that (2.1) holds at $t = T(\theta)$ while for all $0 \leq t < T(\theta)$ the average remains strictly below θ.

There exists a critical threshold θ^* with the following physical meaning: for every target value $\theta < \theta^*$, one can find an admissible control $\mu(t)$ driving the system to the desired average in finite time, i.e., $T(\theta) < +\infty$; conversely, if $\theta \geq \theta^*$, no admissible control can achieve equality (2.1) at any finite time. In other words, θ^* is the maximal attainable average under the given power constraint.

The main goal of this work is to investigate the asymptotic behavior of $T(\theta)$ as θ approaches the critical value θ^* from below, and to express this dependence explicitly in terms of the physical parameters of the system (the elliptic operator coefficients, the geometry of Ω, the actuator distribution $a(x)$, the heat exchange coefficient $h(x)$, and the weight ρ).

We consider the spectral problem associated with the elliptic operator $\mathcal{A}$, which will be used for constructing solution representations via eigenfunction expansions and Green functions.

The eigenvalue problem is

$$\mathcal{A}\varphi(x) = \lambda\varphi(x), \quad x \in \Omega, \tag{2.2}$$

with the homogeneous Robin boundary condition

$$\frac{\partial\varphi}{\partial n_{\mathcal{A}}} + h(x)\varphi = 0, \quad x \in \partial\Omega. \tag{2.3}$$

This problem is understood in the weak sense: we seek $\lambda \in \mathbb{R}$ and $\varphi \in H^1(\Omega)$, $\varphi \not\equiv 0$, satisfying

$$\int_\Omega \sum_{i,j=1}^{3} a_{ij}(x)\frac{\partial\varphi}{\partial x_i}\frac{\partial\psi}{\partial x_j}\,dx + \int_\Omega c(x)\varphi\psi\,dx + \int_{\partial\Omega} h(x)\varphi\psi\,d\sigma$$
$$= \lambda\int_\Omega \varphi\psi\,dx, \quad \forall\psi \in H^1(\Omega),$$

where $d\sigma$ denotes the two-dimensional surface measure on $\partial\Omega$.

Under the stated assumptions on the coefficients $a_{ij}(x)$, $c(x)$ and $h(x)$, the operator $\mathcal{A}$ is self-adjoint, positive, and has a compact resolvent in $L^2(\Omega)$ [28].

Consequently, the spectrum of $\mathcal{A}$ consists of a sequence of real eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots, \qquad \lambda_k \to +\infty \text{ as } k \to \infty,$$

each of finite multiplicity. The corresponding eigenfunctions $\{\varphi_k\}_{k=1}^{\infty}$ form a complete orthonormal system in $L^2(\Omega)$.

The first eigenvalue λ_1 is simple, and the corresponding eigenfunction φ_1 can be chosen strictly positive almost everywhere in Ω. This property follows from the strong maximum principle for second-order elliptic operators and the Krein–Rutman theorem for positive compact operators [7, 29]. Moreover, due to the positivity of φ_1 and the orthogonality of the eigenfunctions φ_1 and φ_2, we have $\lambda_1 < \lambda_2$.

These spectral properties guarantee that the family $\{(\lambda_k, \varphi_k)\}_{k=1}^{\infty}$ can be used to represent solutions of the parabolic boundary control problem in terms of eigenfunction expansions and Green function representations.

We now examine the evolution of the quantity

$$U(t) = \int_\Omega \rho(x)\, u(x,t)\, dx, \quad t > 0, \tag{2.4}$$

where $u(x,t)$ denotes the solution of (1.1)-(1.4) corresponding to the control $\mu(t)$.

Set

$$\theta^* = M \int_\Gamma \big[\mathcal{A}^{-1}\rho(x)\big]\, a(x)\, d\sigma(x), \tag{2.5}$$

and

$$b = \frac{M}{\lambda_1} \cdot (\rho, \varphi_1) \int_\Gamma \varphi_1(y) a(y)\, d\sigma(y), \tag{2.6}$$

where $\{(\lambda_k, \varphi_k)\}_{k=1}^{\infty}$ are the eigenpairs of the elliptic operator $\mathcal{A}$ defined in (2.2)-(2.3), and $\mathcal{A}^{-1}$ denotes the inverse of $\mathcal{A}$ under the homogeneous Robin boundary condition (2.3). Note that $b > 0$; this follows from the positivity of the first eigenfunction and Proposition 3.1, which will be proved in the next section.

Theorem 2.1. *Let $\theta^* > 0$ be defined by (2.5). Then*

(1) *For every θ in the interval $0 < \theta < \theta^*$ there exists $T(\theta) > 0$ such that*

$$U(t) < \theta, \quad 0 < t < T(\theta),$$

and

$$U(T(\theta)) = \theta.$$

(2) *As $\theta \to \theta^*$ from below, the following asymptotic estimate holds:*

$$T(\theta) = \ln \frac{1}{\varepsilon(\theta)} + \frac{1}{\lambda_1} \ln b + O\big(\varepsilon^{\lambda_2 - \lambda_1}\big),$$

where

$$\varepsilon = |\theta^* - \theta|^{1/\lambda_1}.$$

(3) *For every $\theta \ge \theta^*$, the time $T(\theta)$ does not exist; i.e., the average temperature $U(t)$ can never reach or exceed θ^* under any admissible control $|\mu(t)| \le M$.*

The critical value θ^* represents the maximal attainable average temperature under the given power constraint M. It depends on the spatial distribution of the actuator $a(x)$, the weight function ρ, and the geometry of Ω through the inverse operator $\mathcal{A}^{-1}$. The constant b incorporates the influence of the principal eigenfunction φ_1, which characterizes the slowest decaying thermal mode of the system. The logarithmic dependence of $T(\theta)$ on $|\theta^* - \theta|$ near the critical value indicates that approaching the maximal average requires exponentially long time, a typical behavior in diffusion-controlled processes. The term $\varepsilon^{\lambda_2-\lambda_1}$ reflects the faster decay of higher modes, where $\lambda_2 - \lambda_1 > 0$ is the spectral gap of the operator $\mathcal{A}$.

3. Integral equation

In this section we derive an integral representation for the average temperature $U(t)$ and reduce the boundary control problem to a Volterra integral equation of the first kind.

We introduce the Green function $G : \Omega \times \Omega \times (0, \infty) \to \mathbb{R}$ defined by the series

$$G(x, y, t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \varphi_k(x)\varphi_k(y), \qquad x, y \in \Omega, \quad t > 0,$$

where $\{(\lambda_k, \varphi_k)\}_{k=1}^{\infty}$ are the eigenpairs of $\mathcal{A}$ from (2.2)–(2.3). This function satisfies the following equations:

$$G_t(x, y, t) + \mathcal{A}G(x, y, t) = 0, \quad x, y \in \Omega, \quad t > 0,$$

$$\frac{\partial G(x, y, t)}{\partial n_{\mathcal{A}}} + h(x)G(x, y, t) = 0, \quad x \in \partial\Omega, \quad t > 0,$$

$$G(x, y, 0) = \delta(x - y), \quad x, y \in \Omega.$$

Next, we set

$$H(x, t) = \int_{\Omega} \rho(y)G(x, y, t)\, dy, \qquad x \in \Omega, \quad t > 0. \tag{3.1}$$

Then $H(x, t)$ is the unique solution of the initial-boundary value problem

$$H_t(x, t) + \mathcal{A}H(x, t) = 0, \quad x \in \Omega, \quad t > 0,$$

$$\frac{\partial H(x, t)}{\partial n_{\mathcal{A}}} + h(x)H(x, t) = 0, \quad x \in \partial\Omega, \quad t > 0,$$

$$H(x, 0) = \rho(x), \quad x \in \Omega.$$

By virtue of the spectral theorem in $L_2(\Omega, dx)$, we may express H as

$$H(x, t) = \int_0^{\infty} e^{-\lambda t}\, dE_\lambda \rho(x),$$

where $\{E_\lambda\}$ denotes the spectral resolution of $\mathcal{A}$. Expanding in eigenfunctions gives

$$H(x, t) = (\rho, \varphi_1)e^{-\lambda_1 t}\varphi_1(x) + H_1(x, t), \quad t \geq 0, \tag{3.2}$$

with

$$H_1(x,t) = \int_{\lambda_2}^{\infty} e^{-\lambda t}\, dE_\lambda \rho(x) = \sum_{k=2}^{\infty} (\rho, \varphi_k) e^{-\lambda_k t} \varphi_k(x), \quad t \geq 0. \tag{3.3}$$

Define

$$\Lambda_k = \int_\Gamma \varphi_k(y) a(y)\, d\sigma(y), \qquad k \in \mathbb{N}.$$

Proposition 3.1. *The following positivity holds:*

$$\Lambda_1 = \int_\Gamma \varphi_1(y) a(y)\, d\sigma(y) > 0. \tag{3.4}$$

Proof. Suppose $\Lambda_1 = 0$. Since $a(y) \geq 0$ and $a \not\equiv 0$ on Γ, there exists $\Gamma_1 \subset \Gamma$ with $\mathrm{mes}(\Gamma_1) > 0$ such that $\varphi_1(y) = 0$ for $y \in \Gamma_1$. From the boundary condition (2.3) we obtain $\frac{\partial \varphi_1}{\partial n_{\mathcal{A}}} = 0$ on Γ_1. Hence φ_1 solves a homogeneous Cauchy problem, implying $\varphi_1 \equiv 0$ in Ω by uniqueness, a contradiction. Thus $\Lambda_1 > 0$. □

Define

$$G_2(x,y) = \sum_{k=2}^{\infty} \frac{\varphi_k(x)\varphi_k(y)}{\lambda_k^2}, \qquad x, y \in \Omega. \tag{3.5}$$

Lemma 3.2. *For all $x \in \overline{\Omega}$ and $t \geq 0$,*

$$|H_1(x,t)| \leq \|\mathcal{A}\rho\| \cdot \sqrt{G_2(x,x)}\, e^{-\lambda_2 t}.$$

Proof. From (3.3) and the Cauchy–Schwarz inequality,

$$|H_1(x,t)|^2 \leq \left(\sum_{k=2}^{\infty} |(\rho, \varphi_k)|^2 \lambda_k^2 \right) \left(\sum_{k=2}^{\infty} e^{-2\lambda_k t} |\varphi_k(x)|^2 \lambda_k^{-2} \right), \quad t \geq 0.$$

The first factor equals $\|\mathcal{A}\rho\|^2$. Since $\lambda_k \geq \lambda_2$ for $k \geq 2$, we have $e^{-2\lambda_k t} \leq e^{-2\lambda_2 t}$ and the second factor is bounded by $G_2(x,x) e^{-2\lambda_2 t}$. Taking the square root yields the desired estimate. □

Define the kernel $K : (0, \infty) \to \mathbb{R}$ by

$$K(t) = \int_\Gamma H(y,t) a(y)\, d\sigma(y). \tag{3.6}$$

Substituting (3.2) into (3.6) gives

$$K(t) = (\rho, \varphi_1) e^{-\lambda_1 t} \int_\Gamma \varphi_1(y) a(y)\, d\sigma(y) + \int_\Gamma H_1(y,t) a(y)\, d\sigma(y).$$

Using Lemma 3.2, the remainder is $O(e^{-\lambda_2 t})$ as $t \to \infty$. Hence we write

$$K(t) = \Lambda_1 (\rho, \varphi_1) e^{-\lambda_1 t} + O(e^{-\lambda_2 t}), \quad t \to \infty.$$

For small $t > 0$, parabolic regularity of the Green function yields $K(t) = O(t^{-1/2})$ as $t \to 0^+$ (see, e.g., [1]).

It is known (see [24]) that the solution of the initial-boundary value problem (1.1)–(1.4) admits the representation

$$u(x,t) = \int_0^t \mu(s)\, ds \int_\Gamma G(x,y,t-s) a(y)\, d\sigma(y), \qquad x \in \Omega, \quad t > 0.$$

Multiplying this equality by $\rho(x)$, integrating over Ω, and using the definition (3.1) together with (3.6), we arrive at the following integral equation relating the average temperature $U(t)$ to the control function $\mu(t)$:

$$\begin{aligned} U(t) &= \int_\Omega \rho(x)u(x,t)\,dx \\ &= \int_0^t \mu(s)\,ds \int_\Gamma a(y) \left(\int_\Omega \rho(x) G(x,y,t-s)\,dx \right) d\sigma(y) \\ &= \int_0^t K(t-s)\mu(s)\,ds. \end{aligned}$$

Thus we obtain the Volterra integral equation of the first kind

$$U(t) = \int_0^t K(t-s)\mu(s)\,ds.$$

4. Main result

In this section we analyze the asymptotic behavior of the minimal time $T(\theta)$ as the target average temperature θ approaches the critical value θ^* from below. We derive an explicit formula for $T(\theta)$ in terms of the principal eigenvalue λ_1 and the constant b, and we prove that θ^* is the maximal attainable average under the given power constraint $|\mu(t)| \le M$.

Define

$$L(x,t) = \int_0^t H(x,s)\,ds, \qquad x \in \Omega,\ t > 0. \tag{4.1}$$

Then we can write

$$L(x,t) = \sum_{k=1}^\infty (\rho,\varphi_k)\varphi_k(x) \int_0^t e^{-\lambda_k s}\,ds = \sum_{k=1}^\infty \frac{1-e^{-\lambda_k t}}{\lambda_k} (\rho,\varphi_k)\varphi_k(x).$$

This can be rewritten as

$$L(x,t) = \mathcal{A}^{-1}\rho(x) - \frac{e^{-\lambda_1 t}}{\lambda_1}(\rho,\varphi_1)\varphi_1(x) - L_1(x,t),$$

where

$$L_1(x,t) = \sum_{k=2}^\infty \frac{e^{-\lambda_k t}}{\lambda_k}(\rho,\varphi_k)\varphi_k(x).$$

We have the following estimate

$$|L_1(x,t)| \le e^{-\lambda_2 t} \left(\sum_{k=2}^\infty |(\rho,\varphi_k)|^2 \right)^{1/2} \cdot \left(\sum_{k=2}^\infty \frac{|\varphi_k(x)|^2}{\lambda_k^2} \right)^{1/2}.$$

Hence,

$$|L_1(x,t)| \le e^{-\lambda_2 t} \sqrt{G_2(x,x)}\, \|\rho\|, \qquad t \ge 0, \tag{4.2}$$

where G_2 is defined in (3.5).

Further,

$$\int_\Gamma L(x,t)a(x)\,d\sigma(x) = \int_\Gamma \mathcal{A}^{-1}\rho(x)a(x)\,d\sigma(x) \\ - \frac{\Lambda_1}{\lambda_1}(\rho,\varphi_1)e^{-\lambda_1 t} - \int_\Gamma L_1(x,t)a(x)\,d\sigma(x),$$

which follows from (4.1) and the definition of Λ_1.

Define the function $Q:(0,\infty)\to\mathbb{R}$ by

$$Q(t) = \int_0^t K(s)\,ds,$$

where the kernel function K is given by (3.6). The physical meaning of $Q(t)$ is evident: $Q(t)$ equals the average temperature of Ω when the heater operates with a unit load, i.e., $\mu(t)\equiv 1$. It is known that $Q(0)=0$ and $Q'(t)=K(t)\geq 0$ for all $t>0$.

According to (3.6) and (4.1), we have

$$\int_\Gamma L(x,t)a(x)\,d\sigma(x) = \int_0^t ds \int_\Gamma H(x,s)a(x)\,d\sigma(x) \\ = \int_0^t K(s)\,ds = Q(t).$$

Thus we obtain

$$Q(t) = \int_\Gamma L(x,t)a(x)\,d\sigma(x).$$

Define the limit

$$Q^* = \lim_{t\to\infty} Q(t) = \int_0^\infty K(s)\,ds. \tag{4.3}$$

Clearly, the average temperature of Ω under unit load cannot exceed Q^*.

Set

$$\theta^* = MQ^*. \tag{4.4}$$

Then, using (4.2) and (4.4), we obtain the asymptotic expansion

$$\theta(t) = MQ(t) = \theta^* - be^{-\lambda_1 t} + O(e^{-\lambda_2 t}), \quad t\to\infty, \tag{4.5}$$

where b is defined by (2.6).

From (4.3)–(4.5), for every θ in the interval $0<\theta<\theta^*$ there exists a unique $T(\theta)>0$ such that

$$U(t)<\theta \quad \text{for all } 0<t<T(\theta), \qquad \text{and} \qquad U(T(\theta))=\theta.$$

Lemma 4.1. *There exist a time $T(\theta)>0$ and a real-valued measurable control function $\mu(t)$ satisfying $|\mu(t)|\leq M$ for all $t\geq 0$, such that*

$$\int_0^T K(T-s)\mu(s)\,ds = U(T). \tag{4.6}$$

Proof. The assertion follows from the properties of Q. Indeed, taking the constant control $\mu(t) \equiv M$ and inserting it into the left-hand side of (4.6), we obtain

$$\int_0^t K(t-s)\mu(s)\,ds = M\int_0^t K(t-s)\,ds = MQ(t).$$

The function $MQ(t)$ is continuous, strictly increasing, and satisfies $MQ(0) = 0$ together with $\lim_{t\to\infty} MQ(t) = \theta^*$. Hence, for each $\theta \in (0, \theta^*)$, there is a unique $T(\theta) > 0$ such that $MQ(T(\theta)) = \theta$. Since $U(T(\theta)) = \theta$ by definition, equality (4.6) holds for this choice of μ and T. □

Remark 4.2. The value $T(\theta)$ from Lemma 4.1 is the unique solution of

$$Q(T) = \frac{U(T)}{M} = \frac{\theta}{M}.$$

Lemma 4.3. *Let $f(r)$ be increasing on $(0,1]$ and suppose that for some $\beta > 0$,*

$$f(r) = br + O(r^{1+\beta}), \quad r \to 0^+, \tag{4.7}$$

with $b > 0$. Then its inverse $r = f^{-1}(s)$ satisfies

$$\ln\frac{1}{r} = \ln\frac{1}{s} + \ln b + O(s^\beta), \quad s \to 0^+. \tag{4.8}$$

Proof. From (4.7) we may write

$$s = br\big[1 + \alpha(r)\big], \tag{4.9}$$

where $\alpha(r) = O(r^\beta)$ as $r \to 0^+$. Since $f(r) > 0$ on $(0, 1]$, there exists $C > 0$ such that $s \ge Cr$ for $0 < r \le 1$. Thus $r(s) = f^{-1}(s) \le C^{-1}s$, so $r(s) = O(s)$ and consequently $\alpha(r(s)) = O(s^\beta)$ as $s \to 0^+$. Taking logarithms in (4.9) gives

$$\ln\frac{1}{s} = \ln\frac{1}{br} - \ln\big(1 + \alpha(r)\big).$$

Since $\ln(1 + \alpha(r)) = O(r^\beta) = O(s^\beta)$, we obtain

$$\ln\frac{1}{s} = \ln\frac{1}{r} + \ln\frac{1}{b} + O(s^\beta),$$

which is equivalent to (4.8). □

Corollary 4.4. *As $t \to \infty$, the following asymptotic relation holds:*

$$t = \frac{1}{\lambda_1}\ln\frac{1}{\theta^* - \theta(t)} + \frac{1}{\lambda_1}\ln b + O\Big(\big(\theta^* - \theta(t)\big)^{(\lambda_2-\lambda_1)/\lambda_1}\Big). \tag{4.10}$$

Proof. From (4.5) we have

$$\theta^* - \theta(t) = be^{-\lambda_1 t} + O(e^{-\lambda_2 t}), \qquad t \to \infty.$$

Set

$$r = e^{-\lambda_1 t}, \quad s = \theta^* - \theta(t), \quad \beta = \frac{\lambda_2}{\lambda_1} - 1 > 0.$$

Then $e^{-\lambda_2 t} = r^{1+\beta}$. Applying Lemma 4.3 with $f(r) = br + O(r^{1+\beta})$ yields

$$t = \frac{1}{\lambda_1}\ln\frac{1}{s} + \frac{1}{\lambda_1}\ln b + O(s^\beta).$$

Substituting back $s = \theta^* - \theta(t)$ and $\beta = (\lambda_2 - \lambda_1)/\lambda_1$ gives (4.10). □

Now, as $\theta \to \theta^*$ from below, set $\varepsilon = |\theta^* - \theta|^{1/\lambda_1}$. Then from Corollary 4.4 we obtain the main asymptotic formula

$$T(\theta) = \ln \frac{1}{\varepsilon(\theta)} + \frac{1}{\lambda_1} \ln b + O\big(\varepsilon^{\lambda_2 - \lambda_1}\big), \quad \theta \to \theta^{*-}.$$

The proof of Theorem 2.1 follows directly from Lemmas 4.1 and 4.3.

5. Conclusion

In this work we studied a boundary control problem for a heat conduction equation involving a general elliptic operator $\mathcal{A}$ with a Robin-type boundary condition. The objective was to determine a boundary control function $\mu(t)$, subject to a uniform bound, such that the weighted average temperature of the medium reaches a prescribed value in minimal time. Using the spectral decomposition of $\mathcal{A}$ and the associated Green function, the control problem was reduced to a Volterra integral equation of the first kind. The kernel of this integral equation was analyzed in terms of the eigenpairs of $\mathcal{A}$, and its asymptotic behavior was established both for small and large times.

From this analysis we derived the critical value θ^*, which represents the maximal attainable average temperature under the given power constraint. For any target value below this critical threshold, we proved the existence of a unique minimal time and obtained an asymptotic expansion showing how this time depends logarithmically on the distance to the critical value and on the principal eigenvalue of the operator. The spectral gap between the first two eigenvalues determines the rate at which higher modes decay and influences the accuracy of the asymptotic approximation.

It should be noted that the existence result established in Theorem 2.1 holds for all target values below the critical threshold, without any smallness assumption. The case where the target value equals or exceeds the critical threshold is impossible, as proved in the theorem.

While the mathematical techniques employed in this paper are classical and rely on standard spectral theory and asymptotic analysis, the main contribution lies in the complete and explicit solution of this physically important problem in a general setting that has not been previously studied. The explicit characterization of the minimal time and its precise dependence on the spectral data of the operator provide valuable insights for practical applications in thermal engineering and materials science. Extending this approach to more complex settings, such as nonlinear heat conduction models, problems with more general boundary conditions, or the development of more advanced control-theoretic frameworks, remains an interesting direction for future research.

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