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DIFFERENCE SCHEMES FOR CONVECTION-DIFFUSION EQUATIONS WITH GENERALIZED SOLUTIONS

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Abstract. High-accuracy difference schemes are proposed for a multidimensional convection-diffusion equation in classes of generalized solutions. The piecewise polynomial finite element method is used to approximate spatial variables and the resulting Cauchy problem for a system of ordinary differential equations. As a result, third-order accuracy schemes with respect to spatial variables and fourth-order accuracy schemes with respect to time are obtained. Stability conditions and a priori estimates for the solution of the scheme in classes of nonsmooth solutions are obtained. Based on a special technique, convergence theorems were established in a weaker functional norm. Convergence and accuracy theorems are proven.

Keywords. Convection-diffusion equation, finite element method, difference schemes, stability, convergence, accuracy
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1. Introduction

Convection-diffusion problems play a fundamental role in the mathematical modeling of various issues in continuum mechanics, including hydrodynamics, heat transfer, mass transfer, environmental concerns and many other fields [5, 6, 12, 21]. In references [7, 9, 19], the convection-diffusion equation was employed to describe water movement in soil, and the propagation of tracers in porous media [7, 9, 19]. Similar research was conducted in [4, 8, 10, 22].

Analytical methods for solving one-dimensional convection-diffusion problems are considered in [12, 21, 13]. Numerous publications are devoted to numerical methods for solving these problems. Second-order accuracy Crank-Nicolson schemes with respect to the spatial and temporal variables are constructed in [24, 25]. Explicit one-step time-monotone schemes are proposed and investigated in reference [29]. The stability of difference schemes for non-stationary convection-diffusion problems is studied in [1]. In [17], fourth-order accuracy schemes with respect to the spatial and temporal variables are proposed, where

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the method of lines is used to approximate the spatial variable, and the collocation method with cubic C^1-splines is used to approximate the temporal variable. Numerous difference schemes for convection-diffusion problems in various settings are constructed and investigated in [23]. Compact difference schemes for convection-diffusion equations with different convective terms are the subject of research conducted in [26]. In [3], schemes of high accuracy for a multidimensional convection-diffusion equation in symmetric form are considered. References [2, 16, 18, 26, 28] are devoted to the convergence of difference schemes for various equations in classes of generalized solutions.

This paper presents a high-accuracy numerical method for solving initial - boundary value problems for a multidimensional convection-diffusion equation. To address this problem, we utilize a piecewise polynomial finite element method to first approximate the spatial variables. A fourth-order finite element method is then applied through a one-parameter difference scheme to solve the resulting system of linear ordinary differential equations. Theorems on the convergence of difference schemes to the solution of the original problem in classes of generalized solutions are obtained.

2. Statement of the problem

Let us consider the following problem:

$$\frac{\partial \rho}{\partial \eta} + K_1\rho + K_2\rho = \theta(\xi,\eta), \tag{2.1}$$

$$(\xi,\eta) \in \mathbb{Q}_T = \{\xi = (\xi_1,\xi_2,...,\xi_p) \in G \subset \mathbb{R}^p,\ \ \eta \in (0,T]\},$$

$$\rho(\xi,\eta) = 0,\ \ \xi \in \Gamma = \partial G\,,\ \ \ \eta \in (0,T], \tag{2.2}$$

$$\rho(\xi,0) = \rho_0(\xi),\ \ \xi \in G, \tag{2.3}$$

where $G = \{\xi :\ 0 < \xi_m < \ell_m,\ \ m = \overline{1,p}\}\,,\ \ \bar{G} = G \cup \Gamma,\ K_1 u$ is the convective transport operator, and $K_2 u$ is the diffusion transport operator. The following operators [23] define the operators of the convection-diffusion problem in symmetric form:

$$K_1 \equiv \mathbb{C}_0 = \frac{1}{2}\left(\mathbb{C}_1 + \mathbb{C}_2\right), K_2 = -k\sum_{m=1}^{p}\frac{\partial^2}{\partial \xi_m^2},$$

$$\mathbb{C}_1 = \sum_{m=1}^{p} a_m(\xi)\frac{\partial}{\partial \xi_m},\quad \mathbb{C}_2 = \sum_{m=1}^{p}\frac{\partial a_m(\xi)}{\partial \xi_m}\,.$$

For $K_1 = \mathbb{C}_0$, we obtain the convection-diffusion problem in symmetric form, for $K_1 = \mathbb{C}_1$ in non-divergent form, and for $K_1 = \mathbb{C}_2$ in divergent form. Here, $a_m(\xi),\ \ m = 1,2,...,p$ are the velocity components that determine stationary convective transport, and $k = const$ is the diffusion coefficient.

We present some properties of operators $\mathbb{C}_l,\ l = 0,1,2$ and K_2. Let $\Pi = L_2(G)$ be a Hilbert space with inner product $(\rho,v) = \int\limits_G \rho(\xi)v(\xi)d\xi$ for arbitrary functions $\rho(\xi)$ and $v(\xi)$, vanishing on ∂G, and norm $\|\rho\| = (\rho,\rho)^{1/2} = \left(\int\limits_G \rho^2(\xi)d\xi\right)^{1/2}$.

The diffusion transport operator is self-adjoint and positive definite in Π, i.e. $K_2 = K_2^* \geq k\lambda_0 E$, where $\lambda_0 > 0$ is the minimal eigenvalue of the Laplace operator with Dirichlet boundary conditions. The convective transport operator has the following properties [23]:

1. The convective transport operators in non-divergent and divergent forms are conjugate to each other up to a sign $\mathbb{C}_1^* = -\mathbb{C}_2$;
2. The convective transport operators in symmetric form are skew-symmetric $\mathbb{C}_0 = -\mathbb{C}_0^*$;
3. The convective transport operators in non-divergent and divergent forms are bounded by:

$$|(\mathbb{C}_m\rho, \rho)| \leq M_1\|\rho\|^2, \quad m = 1, 2, \quad M_1 = \frac{1}{2}\|div\vec{v}\|_{C(G)};$$

4. The convective transport operators are subject to the diffusion operator, i.e. $\|\mathbb{C}_m\rho\|^2 \leq M_2(K_2\rho, \rho), \quad m = 1, 2$ with constant M_2, depending on the velocity.

In many areas of modern research, problems arise that lead to the solution of problem (2.1)–(2.3). For example, three-dimensional mathematical models of the location of pollution sources in reservoirs, coastal seas and a linearized three-dimensional mathematical model of the dynamics of sea and ocean currents [15], and the problem for a parabolic equation are written [1] in the form (2.1).

We call a function $\rho(\xi, \eta)$ a generalized solution to problem (2.1)–(2.3), that belongs to $\Pi = \overset{\circ}{H}{}_2^1(G)$ almost for each $\eta \in (0, T)$, has derivative $\partial\rho/\partial\eta \in L_2(\mathbb{Q}_T)$, and satisfies the following relations almost everywhere on (0,T):

$$\left(\frac{d}{d\eta}\rho(\eta), v\right) + \sum_{\alpha=1}^{2} q_\alpha(\rho(\eta), v) = (\theta(\eta), v), \quad \forall\ v \in \Pi, \ \ \rho(0) = \rho_0, \tag{2.4}$$

where $q_1(\rho(\eta), v) = \int\limits_G \mathbb{C}_\alpha v\, d\xi, \quad \alpha = 0, 1, 2,$ $q_2(\rho(\eta), v) = \int\limits_G \sum\limits_{m=1}^{p} k\frac{\partial\rho}{\partial\xi_m} \cdot \frac{\partial v}{\partial\xi_m}\, d\xi,$ $\rho(\eta)$ is the function of an abstract argument $\eta \in (0, T]$ with values in Π. The issues of existence, smoothness, and uniqueness of a solution to this problem are studied in [11].

We denote the energy seminorms in Π corresponding to bilinear forms $q_\alpha(\rho, v)$ by $|\rho|_\alpha = \sqrt{q_\alpha(\rho, \rho)}, \quad \alpha = 1, 2.$ The energy space Π_{A_1} generated by seminorm $|\rho|_1$ is equivalent to space $\Pi = \overset{\circ}{H}{}_2^1(G)$, so the following estimate holds $s_1|\rho|_1 \leq |\rho|_1 \leq s_2|\rho|_1$, $s_1 > 0$, where $|\rho|_1$ is the norm in Π. For another energy seminorm, estimate $0 \leq |\rho|_2 \leq \sqrt{k}s_2|\rho|_1$ holds.

For bilinear forms $q_\alpha(\rho, v)$, the following estimates hold:

$$q_\alpha(v, v) \geq k_\alpha\, \|v\|_\alpha^2, \quad \forall\ v \in \Pi,$$

where $k_\alpha > 0$ is a constant and $\alpha = 1, 2$.

Operator $\mathbb{C}_0$ is skew-symmetric, so $q_0(\rho, \rho) = 0$.

3. Space discretization

Let $\Pi_h \subset \Pi$ be the set of elements $v_h = \sum_{i=1}^{N} a_i \varphi_i(\xi)$, where $\{\varphi_i = \varphi_i(\xi)\}_{i=1}^{N}$ is the basis of piecewise polynomial functions that are polynomials of degree k on each finite element.

Let us set the problem in (2.4):

$$\left(\frac{d\rho_h}{d\eta}, v_h\right) + \sum_{\alpha=1}^{2} q_\alpha(\rho_h, v_h) = (\theta, v_h), \;\; \forall\, v_h \in \Pi_h, \;\; \eta \in (0, T], \;\; \rho_h(0) = \rho_{0,h}. \quad (3.1)$$

Problem (3.1) corresponds to the Cauchy problem for the coefficients of the approximate solution $\rho_h(\eta) = \sum_{i=1}^{N} a_i(\eta)\varphi_i$ from Π_h:

$$M\frac{d\vec{a}(\eta)}{d\eta} + \Im\vec{a}(\eta) = \vec{\mathbb{F}}(\eta), \quad \vec{a}(0) = \vec{a}^0. \quad (3.2)$$

Here, $\vec{a}(\eta) = \{a_i(\eta)\}_{i=1}^{N}, \;\; \vec{a}(0) = \{a_i(0)\}_{i=1}^{N}$ are the vectors of dimension N, $M = ((\varphi_i, \varphi_j))_{i,j=1}^{N}$ is the mass matrix, $\Im = (a(\varphi_i, \varphi_j))_{i,j=1}^{N}$ is the stiffness matrix, and $\vec{\mathbb{F}}(\eta) = \{\mathbb{F}(\eta)\}_{i=1}^{N}$ is the right-hand side.

Problem (3.2) can be written in the operator form:

$$\mathrm{D}\frac{d\rho_h(\eta)}{d\eta} + A\rho_h(\eta) = \theta_h(\eta), \;\; \rho_h(0) = \rho_{0,h}, \quad (3.3)$$

where $\rho_h(\eta), \;\; \theta_h(\eta)$ are the elements of the finite-dimensional space Π_h, $\forall\eta$; $\mathrm{D}, \; A$ are operators from Π_h into Π_h (they correspond to matrices $M, \; \Im$). Problem (3.3) approximates the differential problem (2.1)–(2.3) with order $O(h^3)$.

4. Time discretization

Now let us consider the discretization of the Cauchy problem (3.3). Let $\bar{x}_\tau = \{\eta_n = n\tau, \; n = 0, 1, 2, ...\}$ be a uniform grid on the interval $\eta \in [0, T]$. For any interval $(\eta_n, \eta_{n+1}) \in [0, T], \;\; \eta_{n+1} = \eta(t_n + \tau)$ from (3.3), the following identity holds:

$$\int_{\eta_n}^{\eta_{n+1}} (-\mathrm{D}\rho_h\dot{v} + \rho_h v)d\eta + \mathrm{D}\rho_h v\Big|_{\eta_n}^{\eta_{n+1}}$$
$$= \int_{\eta_n}^{\eta_{n+1}} \theta(\eta)v(\eta)d\eta, \;\; \forall v(\eta) \in \Pi_h, \;\; \rho_h(0) = \rho_{0,h}. \quad (4.1)$$

We seek an approximate solution to problem (4.1) in the form of a Hermitian spline of third degree

$$\rho_h(\eta) \approx \zeta(\eta) = \zeta^n\phi_{00}^n(\eta) + \dot{\zeta}^n\phi_{10}^n(\eta) + \zeta^{n+1}\phi_{01}^n(\eta) + \dot{\zeta}^{n+1}\phi_{11}^n(\eta), \quad (4.2)$$

where

$$\zeta^n = \zeta(\eta_n),\ \ \dot{\zeta}^n = \frac{d\zeta}{d\eta}(\eta_n),\ \ \phi^n_{00}(\eta) = 2y^3 - 3y^2 + 1,\ \ \phi^n_{01}(\eta) = 3y^2 + 2y^3,$$
$$\phi^n_{10}(\eta) = \tau(y^3 - 2y^2 + y),\ \phi^n_{11}(\eta) = \tau(y^3 - y^2),\ \ y = (\eta - \eta_n)/\tau.$$

Here, ζ approximates the semi-discrete function $\rho_h \in \Pi_h$.

With equation (4.2) in mind, and choosing different weight functions $v(\eta)$ from formula (4.1), similar to [27], we obtain the following three-parameter vector difference scheme for unknowns $\overset{\wedge}{\zeta}$, $\overset{\wedge}{\dot{\zeta}}$:

$$\begin{aligned}
&\mathrm{D}\frac{\overset{\wedge}{\zeta} - \zeta}{\tau} - \frac{\tau^2}{12}A\frac{\overset{\wedge}{\dot{\zeta}} - \dot{\zeta}}{\tau} + A\frac{\overset{\wedge}{\zeta} + \zeta}{2} = \phi_1,\\
&\gamma\mathrm{D}\frac{\overset{\wedge}{\dot{\zeta}} - \dot{\zeta}}{\tau} + \alpha A\frac{\overset{\wedge}{\zeta} - \zeta}{\tau} + \beta A\frac{\overset{\wedge}{\dot{\zeta}} + \dot{\zeta}}{2} = \phi_2,
\end{aligned} \tag{4.3}$$

$$\zeta^0 = \rho_0,\ \dot{\zeta}^0 = \rho_1, \tag{4.4}$$

where $\zeta = \zeta(\eta_n) \in \Pi_h$, $\overset{\wedge}{\zeta} = \zeta(\eta_{n+1}) \in \Pi_h$,

$$\phi_1 = \frac{1}{\tau}\int\limits_{\eta_n}^{\eta_{n+1}} \theta(\eta)d\eta, \quad \phi_2 = \frac{12}{\tau^3}\int\limits_{\eta_n}^{\eta_{n+1}} \theta(\eta)\left(c_1\nu_2^{(1)} + c_2\nu_2^{(3)}\right)d\eta,$$

$$c_1 = 15\gamma - 35\alpha/3,\ \ c_2 = 140\gamma - 350\alpha/3,$$

$$\nu_2^{(1)} = \tau(y + 1/2),\ \ \ \nu_2^{(3)} = \tau(y^3 - 3y^2/2 + y/2),\ \ \ y = (\eta - \eta_n)/\tau.$$

Scheme (4.3), (4.4) obeys the fourth-order approximation condition in time:

$$\alpha + \beta = \gamma,\ \ \alpha,\ \beta, \gamma\ =\ O(\tau^2). \tag{4.5}$$

5. Space discretization

We integrate identity (2.4) with respect to η from η_n to η_{n+1} and apply the integration by parts formula:

$$\int\limits_{\eta_n}^{\eta_{n+1}}\left[-(\rho(\eta), \dot{v}) + \sum_{\alpha=1}^{2} q_\alpha(\rho(\eta), v)\right]d\eta + \left.(\rho(\eta), v)\right|_{\eta_n}^{\eta_{n+1}}$$
$$= \int\limits_{\eta_n}^{\eta_{n+1}} (\theta(\eta), v)d\eta,\ \ \forall\, v \in \Pi. \tag{5.1}$$

Similar actions for identity (3.1) yield:

$$\int\limits_{\eta_n}^{\eta_{n+1}} \left[-(\rho_h, \dot{v}_h) + \sum_{\alpha=1}^{2} q_\alpha(\rho_h, v_h) \right] d\eta + \left.(\rho_h, v_h)\right|_{\eta_n}^{\eta_{n+1}}$$

$$= \int\limits_{\eta_n}^{\eta_{n+1}} (\theta, v_h) d\eta, \ \forall\, v_h \in \Pi_h. \tag{5.2}$$

Here and hereinafter, $\dot{\rho} = \partial\rho/\partial\eta$. Subtracting identity (5.2) from (5.1), we obtain

$$\int\limits_{\eta_n}^{\eta_{n+1}} \left[-(\rho - \rho_h, \dot{v}_h) + \sum_{\alpha=1}^{2} q_\alpha(\rho - \rho_h, v_h) \right] d\eta$$

$$+ \left.(\rho - \rho_h, v_h)\right|_{\eta_n}^{\eta_{n+1}} = 0, \ \ \forall\, v_h \in \Pi_h. \tag{5.3}$$

Let us set $z_h = \rho - \rho_h, \ \ e_h = \rho - \rho_I, \ \ y_h = \rho_I - \rho_h$, where $\rho_I = \rho_I(\xi, \eta)$ is the interpolant of solution $\rho(\xi, \eta)$ with respect to ξ (see [14]). We choose the trial function:

$$v_h(\eta) = -\int\limits_{\eta}^{c} y_h(\eta') d\eta' \in \Pi_h, \ \eta < c; \ \ \ v_h(\eta) = 0, \ \ \eta \geq c, \dot{v}_h(\eta) = \ y_h(\eta), \ \ v_h(c) = 0.$$

Then, considering $z_h = \rho - \rho_h = e_h + y_h$ from equality (5.3), we obtain the energy identity:

$$\int\limits_{\eta_n}^{\eta_{n+1}} \left[-(y_h, y_h) + \sum_{\alpha=1}^{2} q_\alpha(\dot{v}_h, v_h) \right] d\eta + \left.(y_h, v_h)\right|_{\eta_n}^{\eta_{n+1}}$$

$$= \int\limits_{\eta_n}^{\eta_{n+1}} \left[(e_h, y_h) - \sum_{\alpha=1}^{2} q_\alpha(e_h, v_h) \right] d\eta - \left.(e_h, v_h)\right|_{\eta_n}^{\eta_{n+1}}.$$

Since $q(\dot{v}_h, v_h) = \frac{1}{2}\frac{d}{d\eta} q(v_h, v_h)$, the last identity can be written as:

$$-\int\limits_{\eta_n}^{\eta_{n+1}} (y_h, y_h)\, d\eta + \frac{1}{2}\sum_{\alpha=1}^{2} q_\alpha(v_h, v_h)(\eta_{n+1}) + \left.(y_h, v_h)\right|_{\eta_n}^{\eta_{n+1}} = -\left.(e_h, v_h)\right|_{\eta_n}^{\eta_{n+1}}$$

$$+ \frac{1}{2}\sum_{\alpha=1}^{2} q_\alpha(v_h, v_h)(\eta_n) + \int\limits_{\eta_n}^{\eta_{n+1}} \left[(e_h, y_h) - \sum_{\alpha=1}^{2} q_\alpha(e_h, v_h) \right] d\eta. \tag{5.4}$$

We sum equation (5.4) with respect to $n = \overline{1, m-1}$, where m corresponds to the time point $c = m\tau$:

$$-\int\limits_{0}^{c} (y_h, y_h) d\eta + \frac{1}{2}\sum_{\alpha=1}^{2} q_\alpha(v_h, v_h)(c) + \left.(y_h, v_h)\right|_{0}^{c} = -\left.(e_h, v_h)\right|_{0}^{c}$$

$$+\frac{1}{2}\sum_{\alpha=1}^{2} q_\alpha(v_h,v_h)(0)+\int_0^c \left[(e_h,y_h)-\sum_{\alpha=1}^{2} q_\alpha(e_h,v_h)\right]d\eta.$$

Given the properties of function $v_h(\eta)$ and initial condition $y_h(0)=0$, from the last identity we obtain:

$$\int_0^c (y_h,y_h)d\eta+\frac{1}{2}\sum_{\alpha=1}^{2} q_\alpha(v_h,v_h)(0)=-(e_h,v_h)(0)-\int_0^c \left[(e_h,y_h)-\sum_{\alpha=1}^{2} q_\alpha(e_h,v_h)\right]d\eta.$$

Let us introduce function

$$\omega_h(\eta)=\int_0^\eta y_h(\eta')d\eta'\in\Pi_h,\ \ \eta<c;\ \ \omega_h(\eta)=0,\ \ \eta\geq c.$$

Since, $v_h(\eta)=\omega_h(\eta)-\omega_h(c)$, from the last equality, we obtain:

$$\int_0^c (y_h,y_h)d\eta+\frac{1}{2}\sum_{\alpha=1}^{2} q_\alpha(\omega_h,\omega_h)(c)=(e_h(0),\omega_h(c))$$
$$-\int_0^c \left[(e_h,y_h)-\sum_{\alpha=1}^{2} q_\alpha(e_h,\omega_h(\eta)-\omega_h(c))\right]d\eta. \qquad (5.5)$$

The following estimates hold for the right-hand side of equation (5.5):

$$(e_h(0),\omega_h(c))\leq\varepsilon_1(\omega_h(c),\omega_h(c))+\frac{1}{4\varepsilon_1}(e_h(0),e_h(0)),$$

$$\left|\int_0^c (e_h,y_h)d\eta\right|\leq\varepsilon_2\int_0^c (y_h,y_h)d\eta+\frac{1}{4\varepsilon_2}\int_0^c (e_h,e_h)d\eta,$$

$$\left|\int_0^c q_\alpha(e_h,\omega_h(\eta)-\omega_h(c))d\eta\right|$$

$$\leq\varepsilon_3\int_0^c q_\alpha(\omega_h(\eta),\omega_h(\eta))d\eta+c\varepsilon_3 q_\alpha(\omega_h(c),\omega_h(c))+\frac{1}{2\varepsilon_3}\int_0^c q_\alpha(e_h,e_h)d\eta.$$

Let $\varepsilon_1=\ \varepsilon_2=1/2$, $0.5\varepsilon_1+\varepsilon_3 T\leq 0.25$. Then from equation (5.5), we obtain:

$$\int_0^c (y_h,y_h)d\eta+\sum_{\alpha=1}^{2} q_\alpha(\omega_h,\omega_h)(c)\leq M_1\left(\sum_{\alpha=1}^{2}\int_0^c q_\alpha(\omega_h,\ \omega_h)(\eta)d\eta+(e_h(0),e_h(0))\right.$$
$$\left.+(\omega_h(c),\omega_h(c))\ +\int_0^c (e_h,e_h)d\eta+\sum_{\alpha=1}^{2}\int_0^c q_\alpha(e_h,e_h)d\eta\right),$$

where $M_1 = \max(8,\ 1/T,\ 16T)$. Applying Grönwall's Lemma, we obtain:

$$\int_0^c (y_h, y_h)d\eta + \sum_{\alpha=1}^{2} q_\alpha(\omega_h, \omega_h)(c) \le M_1(e_h(0), e_h(0)) + (\omega_h(c), \omega_h(c)) + \int_0^c (e_h, e_h)d\eta + \sum_{\alpha=1}^{2} \int_0^c q_\alpha(e_h, e_h)d\eta.$$

With $k_0 \|\omega_h(c)\|_1^2 \le q_\alpha(\omega_h, \omega_h)(c) \le k_1 \|\omega_h(c)\|_1^2$, $(y_h, y_h)(c) = \|y_h(c)\|_0^2$, the following estimate holds:

$$\int_0^c \|y_h(\eta)\|_0^2 d\eta + \sum_{\alpha=1}^{2} \left\| \int_0^c y_h(\eta)d\eta \right\|_\alpha^2 \le M_1 \left(\|e_h(0)\|_0^2 + \int_0^c \|e_h(\eta)\|_0^2 \, d\eta + \sum_{\alpha=1}^{2} \int_0^c \|e_h(\eta)\|_\alpha^2 \, d\eta \right). \tag{5.6}$$

To solve $\rho(\xi, \eta) \in H_2^{k+1}(G) \quad \forall \eta \in [0, T]$, the following estimates hold [14]: $\|e_h(0)\|_0 \le M_2 h^{k+1} \|\rho_0\|_k$, $\|e_h(\eta)\|_0 \le M_2 h^{k+1} \|\rho(\eta)\|_k$, $\|e_h(\eta)\|_\alpha \le M_2 h^k \|\rho(\eta)\|_{k+1}$. Therefore, based on equation (5.6) and the triangle inequality $\|z_h\| \le \|e_h\| + \|y_h\|$, the following assertion holds.

Theorem 5.1. *Let $\rho(\xi, \eta) \in L_2 \left\{ [0, T]; \quad H_2^{k+1}(G) \cap \overset{\circ}{H}{}_2^1(G) \right\}$. If the reduction of space Π_h to a single finite element is a polynomial of degree k, then the following accuracy estimate holds for the solution to problem (3.3):*

$$\sqrt{\int_0^\eta \|\rho(\xi, \eta') - \rho_h(\xi, \eta')\|_0^2 \, d\eta'} + \sum_{\alpha=1}^{2} \left\| \int_0^\eta [\rho(\xi, \eta') - \rho_h(\xi, \eta')] \, d\eta' \right\|_\alpha \le M_1 h^k \left\{ \|\rho(\xi, 0)\|_k + 2\sqrt{\int_0^\eta \|\rho(\xi, \eta')\|_{k+1}^2 \, d\eta'} \right\}, \tag{5.7}$$
$$\forall \eta \in [0, T], \ M_1 = M_1(k_0, k_1, T).$$

Let us proceed to estimating the discretization error of problem (3.3) in time.

We approximate problem (3.3) with the difference scheme (4.3), (4.4). It corresponds to the following weak formulation:

$$\int_{\eta_n}^{\eta_{n+1}} \left[-(\zeta(\eta), \dot{v}_\tau) + \sum_{\alpha=1}^{2} q_\alpha(\zeta(\eta), v_\tau) \right] d\eta + (\zeta(\eta), v_\tau)\Big|_{\eta_n}^{\eta_{n+1}} = \int_{\eta_n}^{\eta_{n+1}} (\theta, v_\tau)d\eta, \quad \forall v_\tau(\xi) \in \Pi_h^\tau. \tag{5.8}$$

We choose in equation (5.8)

$$v_\tau(\eta) = -\int\limits_{\eta}^{c} y_\tau(\eta) d\eta', \;\; \eta < c; \;\; v_\tau(\eta) = 0, \;\; \eta \geq c.$$

Clearly, $\dot{v}_\tau(\eta) = y_\tau(\eta), \;\; \eta < c$ and $v_\tau(c) = 0$. Substituting function $v_\tau(\eta)$ into formula (5.8) and applying the same transformations as in estimating $z_h = \rho - \rho_h = e_h + y_h$, we obtain:

$$\int\limits_0^c (y_\tau, y_\tau) d\eta + \frac{1}{2}\sum_{\alpha=1}^{2} q_\alpha(v_\tau, v_\tau)(0)$$

$$= (e_\tau, v_\tau)(0) - \int\limits_0^c \left[(e_\tau, y_\tau) - \sum_{\alpha=1}^{2} q_\alpha(e_\tau, v_\tau)\right] d\eta. \tag{5.9}$$

We introduce

$$\omega_\tau(\eta) = \int\limits_0^{\eta} y_\tau(\eta') d\eta' \in \Pi_h, \;\; \eta < c, \;\; \omega_\tau(\eta) = 0\,, \;\; \eta \geq c.$$

Since, $e_\tau(0) = \rho_h(0) - \rho_I^\tau(0) = \rho_{0,h} - \rho_{0,h} = 0$, equation (5.9) has the following form:

$$\int\limits_0^{\eta} (y_\tau, y_\tau) d\eta + \frac{1}{2}\sum_{\alpha=1}^{2} q_\alpha(\omega_\tau, \omega_\tau)(c)$$

$$= -\int\limits_0^c \left[(e_\tau, y_\tau) - \sum_{\alpha=1}^{2} q_\alpha(e_\tau, \omega_\tau(\eta) - \omega_\tau(c))\right] d\eta.$$

Hence, applying the Cauchy-Bunyakovsky inequality, the ε-inequality, and Grönwall's Lemma, we obtain:

$$\int\limits_0^c \|y_\tau(\eta)\|_0^2 \, d\eta + \sum_{\alpha=1}^{2} \left\| \int\limits_0^c y_\tau(\eta) d\eta \right\|_\alpha^2$$

$$\leq M_1 \left(\int\limits_0^c \|e_\tau(\eta)\|_0^2 \, d\eta + \sum_{\alpha=1}^{2} \int\limits_0^c \|e_\tau(\eta)\|_\alpha^2 \, d\eta \right). \tag{5.10}$$

Let us now estimate the error of scheme (4.3), (4.4) $x_\tau(\eta) = y_\tau(\eta) + e_\tau(\eta)$. By virtue of the triangle inequality and $(a+b)^2 \leq 2(a^2+b^2)$, we obtain:

$$\int\limits_0^c \|x_\tau(\eta)\|_0^2 \, d\eta + \sum_{\alpha=1}^{2} \left\| \int\limits_0^s x_\tau(\eta) d\eta \right\|_\alpha^2$$

$$\leq 2\left(\int_0^c \|y_\tau(\eta)\|_0^2 d\eta + \sum_{\alpha=1}^{2}\left\|\int_0^c y_\tau(\eta)d\eta\right\|_\alpha^2\right)$$

$$+2\left(\int_0^c \|e_\tau(\eta)\|_0^2\, d\eta + \sum_{\alpha=1}^{2}\left\|\int_0^c e_\tau(\eta)d\eta\right\|_\alpha^2\right).$$

Based on the Cauchy-Bunyakovsky inequality, we have:

$$\left\|\int_0^c e_\tau(\eta')d\eta'\right\|_1^2 \leq \left\|\sqrt{\int_0^c 1d\eta'}\ \sqrt{\int_0^s e_\tau^2(\eta')d\eta'}\right\|_1^2 \leq c\int_0^c \|e_\tau(\eta')\|_1^2\, d\eta'.$$

With this inequality, from formula (5.10), we obtain:

$$\int_0^c \|x_\tau(\eta)\|_0^2\, d\eta + \sum_{\alpha=1}^{2}\left\|\int_0^c x_\tau(\eta)d\eta\right\|_\alpha^2$$

$$\leq M_1\left(\|e_\tau(c)\|_0^2 + \int_0^c \|e_\tau(\eta)\|_0^2\, d\eta + \sum_{\alpha=1}^{2}\int_0^c \|e_\tau(\eta)\|_\alpha^2\, d\eta\right).$$

Now we consider $e_\tau(\rho_h) = \rho_h - \rho_I^\tau$. The change of variable $\eta = \eta_n + \sigma\tau,\ 0 < \sigma < 1$ gives:

$$\tilde{e}_\tau(\tilde{\rho}_h(\sigma)) = e_\tau(\rho_h) = \rho_h(\eta_n + \sigma\tau) - \rho_I^\tau(\eta_n + \sigma\tau) = \tilde{\rho}_h(\sigma) - \tilde{\rho}_I^\tau(\sigma),$$

bounded for function $\tilde{\rho}_h(\sigma) \in C[0,1]$. Therefore, it is bounded for $\tilde{\rho}_h(\sigma) \in H_2^4[0,1]$. Then functional

$$|\tilde{e}_\tau(\tilde{\rho}_h)| = |\tilde{\rho}_h(\sigma) - \tilde{\rho}_I^\tau(\sigma)| \leq M_3\sum_{m=0}^{4}\left(\int_0^1\left(\frac{d^m\tilde{\rho}_h}{d\sigma^m}\right)^2 d\sigma\right)^{1/2}$$

vanishes on polynomials up to and including the third degree in variable σ, since on the interval $[0,1]$, $\tilde{\rho}_I^\tau$ is the third-degree polynomial that interpolates $\tilde{\rho}_h$. Hence, based on the Bramble-Hilbert Lemma [20], we obtain:

$$|\tilde{e}_\tau(\tilde{\rho}_h)| = |\tilde{\rho}_h(\sigma) - \tilde{\rho}_I^\tau(\sigma)| \leq \bar{M}_3\left(\int_0^1\left(\frac{d^4\tilde{\rho}_h}{d\sigma^4}\right)^2 d\sigma\right)^{1/2}.$$

Returning to the former variables, we have the following estimate

$$|e_\tau(\rho_h(\eta))| = |\rho_h(\eta) - \rho_I^\tau(\eta)| \leq \bar{M}_3\tau^{7/2}\left(\int_{\eta_n}^{\eta_{n+1}}\left(\frac{d^4\rho_h}{d\eta^4}\right)^2 d\eta\right)^{1/2},\ \forall\eta\in[\eta_n,\eta_{n+1}].$$

Therefore,

$$\int\limits_0^c \|e_\tau(\eta')\|_1^2\, d\eta' = \sum_{n=0}^{m-1} \int\limits_{\eta_n}^{\eta_{n+1}} \|e_\tau(\eta')\|_1^2\, d\eta' \leq \sum_{n=0}^{m-1} \int\limits_{\eta_n}^{\eta_{n+1}} \bar{M}_3^2 \tau^7 \int\limits_{\eta_n}^{\eta_{n+1}} \left\| \frac{d^4\rho_h}{d\eta^4}(\eta) \right\|_1^2 d\eta d\eta'$$

$$= \sum_{n=0}^{m-1} \bar{M}^2 \tau^7 \int\limits_{\eta_n}^{\eta_{n+1}} \left\| \frac{d^4\rho_h}{d\eta^4}(\eta) \right\|_1^2 d\eta \times \tau = \bar{M}_3^2 \tau^8 \int\limits_0^c \left\| \frac{d^4\rho_h}{d\eta^4}(\eta) \right\|_1^2 d\eta, \tag{5.11}$$

$$\int\limits_0^s \|e_\tau(\eta')\|_2^2\, d\eta' = \bar{M}_3^2 \tau^8 \int\limits_0^c \left\| \frac{d^4\rho_h}{d\eta^4}(\eta) \right\|_2^2 d\eta.$$

Similarly

$$\int\limits_0^c \|e_\tau(\eta)\|_0^2\, d\eta \leq \bar{M}_3^2 \tau^8 \int\limits_0^c \left\| \frac{d^4\rho_h}{d\eta^4}(\eta) \right\|_0^2 d\eta. \tag{5.12}$$

If boundedness $\left\| \frac{d^4\rho_h}{d\eta^4}(\eta) \right\|_0$ is required for any η, then

$$\|e_\tau(c)\|_0^2 \leq \bar{M}_3^2 \tau^7 \left\| \int\limits_{\eta_{m-1}}^{\eta_m} \left(\frac{d^4\rho_h}{d\eta^4} \right)^2 d\eta \right\|_0 \leq \bar{M}_3^2 \tau^8 \max_\eta \left\| \frac{d^4\rho_h}{d\eta^4} \right\|_0^2. \tag{5.13}$$

Based on estimates (5.11)–(5.13), the following result is obtained.

Theorem 5.2. *If $A^* = A > 0$, $D^* = D > 0$ and conditions (4.5) and $\alpha = \tau^2/12$, $\beta > 0$ are satisfied, then, for scheme (4.3), (4.4), approximating the solution to problem (3.3) for $\frac{d^4\rho_h}{d\eta^4}(\eta) \in L_2[0,T]$, the following accuracy estimate holds*

$$\sqrt{\int\limits_0^\eta \|\rho_h(\eta') - \zeta(\eta')\|_0^2\, d\eta'} + \sum_{\alpha=1}^{2} \left\| \int\limits_0^\eta [\rho_h(\eta') - \zeta(\eta')]\, d\eta' \right\|_\alpha^2$$

$$\leq M_1 \tau^4 \left\{ \|\rho_h(0)\|_0 + \sum_{\alpha=1}^{2} \sqrt{\int\limits_0^\eta \left\| \frac{d^4\rho_h}{d\eta^4}(\eta') \right\|_\alpha^2 d\eta'} \right\}.$$

For $k = 0{,}1$ the following estimates hold (see [20]):

$$\|\rho_h\|_k = \|\rho - \rho + \rho_h\|_k \leq \|\rho\|_k + \|\rho - \rho_h\|_k$$
$$\leq \|\rho\|_k + M_2 h \|\rho\|_{k+1} \leq \bar{M}_2 \|\rho\|_{k+1}. \tag{5.14}$$

Based on inequality (5.14), the last estimate has the following form:

$$\sqrt{\int\limits_0^\eta \|\rho_h(\eta') - \zeta(\eta')\|_0^2\, d\eta'} + \sum_{\alpha=1}^{2} \left\| \int\limits_0^\eta [\rho_h(\eta') - \zeta(\eta')]\, d\eta' \right\|_\alpha^2$$

$$\leq M_1\tau^4\left\{\|\rho(0)\|_1+\sum_{\alpha=1}^{2}\sqrt{\int_0^\eta\left\|\frac{d^4\rho}{d\eta^4}(\eta')\right\|_\alpha^2 d\eta'}\right\}.$$

Based on Theorems 5.1 and 5.2, we obtain the following result.

Theorem 5.3. *If* $A^*=A>0$, $D^*=D>0$, *and conditions* (4.5) *and* $\alpha=\tau^2/12$, $\beta>0$ *are met, then for scheme* (4.3), (4.4), *approximating the solution to problem* (2.1)–(2.3), *for* $\rho(\xi,\eta)\in L_2\left\{[0,T];\ H_2^{k+1}(G)\cap \overset{\circ}{H}{}_2^1(G)\right\}$, $\frac{\partial^4\rho}{\partial t^4}(\xi,\eta)\in L_2\left\{[0,T];\ H_2^2(G)\right\}$, *the following accuracy estimate holds:*

$$\sqrt{\int_0^\eta\|\rho(\xi,\eta')-\zeta(\xi,\eta')\|_0^2\,d\eta'+\sum_{\alpha=1}^{2}\left\|\int_0^\eta[\rho(\xi,\eta')-\zeta(\xi,\eta')]\,d\eta'\right\|_\alpha^2}$$

$$\leq M_1\left\{\tau^4\left(\|\rho(\xi,0)\|_1+2\sqrt{\int_0^\eta\left\|\frac{\partial^4\rho}{\partial\eta^4}(\xi,\eta')\right\|_2^2 d\eta'}\right)\right.$$

$$\left.+\,h^k\left(\|\rho(\xi,0)\|_k+2\sqrt{\int_0^\eta\|\rho(\xi,\eta')\|_{k+1}^2\,d\eta'}\right)\right\}.$$

If we choose a polynomial of degree $k=3$ at each finite element, we obtain third-order accuracy in space variables and fourth-order accuracy in time variable, i.e., $O(h^3+\tau^4)$.

6. Conclusions

High accuracy difference schemes are constructed for the multidimensional convection-diffusion equation in classes of generalized solutions. Piecewise polynomial approximation schemes of the finite element method are used to approximate the spatial variables and the time variable. The constructed difference schemes have third-order accuracy in spatial variables and fourth-order accuracy in the time variable. Stability conditions and a priori estimates for the solution of the scheme in classes of generalized solutions are obtained. Using a special technique, theorems are obtained in a norm weaker than L_2. Theorems on convergence and accuracy are derived. The obtained results can be applied to the numerical solution of various initial-boundary value problems of convection-diffusion and convection-diffusion-reaction with constant and variable coefficients.

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