Gulf Journal of Mathematics ISSN: 2309-4966
Vol 23, Issue 2 (2026) 1-12 https://doi.org/10.56947/pqmghf22

KIRCHHOFF-TYPE PROBLEMS ON COMPACT MANIFOLDS

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ABSTRACT. In this paper, we consider p-biharmonic Kirchhoff-type equations on compact manifolds with Navier boundary conditions. Using the theory of Riemannian compact Sobolev spaces and the Browder-Minty theorem, we show the existence and uniqueness of a weak solution to this problem.

Keywords. p-biharmonic Kirchhoff-type equations,weak solution, compact manifolds.
2020 Mathematics Subject Classification. Primary 35D30 ; Secondary 35R01 ,58J05

1. Introduction and preliminaries

Let (M, g) be a compact smooth Riemannian manifold of dimension $N \geq 5$ with a smooth boundary ∂M. In this work, we consider a nonlocal fourth-order Kirchhoff-type problem subject to Navier boundary conditions.:

$$\begin{cases} \Delta_g\big(|\Delta_g u|^{p-2}\Delta_g u\big) - \mathcal{K}\Big(\displaystyle\int_M |\nabla_g u|^p\, dv_g\Big)\, \mathrm{div}_g\big(|\nabla_g u|^{p-2}\nabla_g u\big) + \lambda\, |u|^{p-2}u = \mu(x,u) & \text{in } M, \\ u = \Delta_g u = 0 & \text{on } \partial M. \end{cases} \tag{1.1}$$

Here, $-\mathrm{div}_g(|\nabla_g u|^{p-2}\nabla_g u)$ denotes the p-laplacian operator and $\Delta_g\big(|\Delta_g u|^{p-2}\Delta_g u\big)$ is a fourth-order operator called the p-biharmonic operator in (M, g), dv_g is the Riemannian volume element. More precisely, $dv_g = \sqrt{\det(g_{ij})}\, ds$, where g_{ij} are the components of the metric g in a local chart, and ds is the standard Lebesgue measure in $\mathbb{R}^N$.

Throughout this paper, we assume:

(H1) The functions $\mathcal{K} : [0, +\infty) \to [0, +\infty)$ are continuous and increasing, and satisfy
$$0 < \mathcal{K}_0 \leq \mathcal{K}(t) \leq \mathcal{K}_\infty, \quad \text{for all } t \in [0 + \infty),$$

(H2) The function $\mu : M \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that:
(i) μ is decreasing with respect to the second variable, that is,
$$\mu(x, \nu_2) \leq \mu(x, \nu_1), \quad \text{for a.e. } x \in M \text{ and } \nu_2 \geq \nu_1. \tag{1.2}$$

Date: Received: Jun 10, 2026; Revised: Jul 10, 2026; Accepted: Jul 17, 2026.
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(ii) For a.e. $x \in M$ and for all $\nu \in \mathbb{R}$, there exist a constant $\beta > 0$ and a function $\mu_0 \in L^{q'}(M)$ with $(\frac{1}{q'} + \frac{1}{q} = 1)$ such that

$$|\mu(x,\nu)| \leq \mu_0(x) + \beta|\nu|^{q-1}. \tag{1.3}$$

(H3) Let $\lambda > 0$ and $p < \frac{N}{2}$. Moreover, assume that

$$1 < q < p < p^* = \frac{Np}{N-2p}.$$

In [7] A. Bouali and R. Guefaifia studied the existence and uniqueness of a weak solution for the nonlocal problem involving the p-Laplacian,

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^p \, dx\right) \Delta_p u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

by using Brouwer's theorem. In (1.1), we investigate a problem of the same type in the biharmonic setting on a compact Riemannian manifold, and we use the same solution method.
In [17], the authors proved the existence of a positive weak solution for the nonlocal p-Kirchhoff

$$-\left[K\Big(\int_\Omega |\nabla u|^p \, dx\Big)\right]^{p-1} \Delta_p u = f(x,u), \quad x \in \Omega, \qquad u = 0, \quad x \in \partial\Omega, \tag{1.5}$$

using critical point theory. Here, $f \in CAR(\overline{\Omega} \times \mathbb{R}^+)$, and K is a continuous increasing function satisfying

$$K(t) \geq k_0 > 0, \quad \forall t \in \mathbb{R}^+.$$

Recently, in [12], Hebey studied the following stationary Kirchhoff equation on an N-dimensional closed Riemannian manifold (M,g):

$$\left(a + b\int_M |\nabla u|^2 \, dv_g\right) \Delta_g u + hu = |u|^{2^*-2}u. \tag{1.6}$$

where $2^* = \frac{2N}{N-2}, \quad N \geq 4$,with $a, b > 0$ and $h \in C^1(M)$.
In [16], Shapour Heidarkhani and Amjad Salari study a perturbed nonlocal fourth-order problem of Kirchhoff type under Navier boundary conditions. The problem is formulated as

$$\Delta\big(|\Delta u|^{p-2}\Delta u\big) - \left[M\Big(\int_\Omega |\nabla u|^p \, dx\Big)\right]^{p-1} \Delta_p u + \beta|u|^{p-2}u, = \lambda f(x,u) + \mu g(x,u) \quad \text{in } \Omega, \tag{1.7}$$

$\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded smooth domain, $p > \max\{1, \frac{N}{2}\}$, $\beta > 0$, $M : [0,+\infty) \to \mathbb{R}$ is continuous with constants $m_0, m_1 > 0$ such that $m_0 \leq M(t) \leq m_1$ for all $t \geq 0$, $\lambda > 0$, $\mu \geq 0$, and $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are L^1-Carathéodory functions. These classes have found numerous important applications in nonlinear analysis on compact Riemannian manifolds (see [4, 1, 5, 6]).

Problem (1.1) is related to the stationary version of a model known as the Kirchhoff equation (Kirchhoff, 1883;see[19]). More precisely, Kirchhoff established a model given by the following equation:

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2}=0,$$

Here, ρ denotes the mass density, P_0 is the initial tension, L is the length of the string, h is the area of the cross-section, and E is the Young modulus of the material . This equation extends the classical D'Alembert wave equation.
In recent years, differential equations and variational problems involving nonlocal operators have received increasing attention. This interest spans multiple fields including optimization, continuum mechanics,phase transition phenomena, finance, population (see [15, 2]).

This paper is structured as follows. Section 2 is devoted to recalling the necessary preliminaries concerning Sobolev spaces on compact Riemannian manifolds and present some useful auxiliary results.In Section 3,we establish several technical lemmas needed for the analysis.In Section 4,we apply Brouwer's theorem to establish the uniqueness of the weak solution.

2. Sobolev Spaces on Compact Riemannian Manifolds

In this section, we introduce the main properties and notations of Sobolev spaces on Riemannian manifolds that will be used in the analysis of problem (1.1). For more details, we refer to [13, 14, 3].

Definition 2.1. [3] Consider the Riemannian manifold (M,g), with dv_g the volume element associated with the metric g. Given $u:M\to\mathbb{R}$ a function of class $C^\infty(M)$, and k an integer, we denote by $\nabla^k u$ the k^{th} covariant derivative of u (with the convention $\nabla^0 w=w$) and $|\nabla^k u|$ the norm of $\nabla^k u$ defined by

$$\left|\nabla^k u\right|=g^{i_1j_1}\cdots g^{i_kj_k}\left(\nabla^k u\right)_{i_1\dots i_k}\left(\nabla^k u\right)_{j_1\dots j_k},$$

where the Einstein summation convention is adopted.

Let $p\geq 1$ and $k\in\mathbb{N}$. We introduce the following spaces:

$$L^p(M)=\left\{u:M\to\mathbb{R}\text{ measurable }:\int_M|u|^p\,dv_g<\infty\right\}$$

and

$$C_k^p(M)=\left\{u\in C^\infty:\forall j=0,\dots,k,\int_M\left|\nabla_g^j u\right|^p dv_g<\infty\right\}.$$

Proposition 2.2. *Let (M,g) be a Riemannian manifold, and let $1<q<\infty$ with*

$$\frac{1}{q}+\frac{1}{q'}=1.$$

Then for all $u\in L^q(M)$ and $v\in L^{q'}(M)$, we have

$$\int_M|u(x)v(x)|\,dv_g(x)\leq\|u\|_{L^q(M)}\,\|v\|_{L^{q'}(M)}.$$

Definition 2.3. Consider a smooth, compact Riemannian manifold (M, g) of dimension $N \geq 3$, with or without boundary. For $1 < p < \frac{N}{2}$, we denote by $W_0^{2,p}(M)$ and $W^{2,p}(M)$ the standard second-order Sobolev spaces. These are defined as the completions of $C_0^\infty(M)$ and $C^\infty(M)$, respectively, with respect to the norm

$$\|u\|_{W^{2,p}(M)} = \left(\int_M |\Delta_g u|^p \, dv_g + \int_M |\nabla_g u|^p \, dv_g + \int_M |u|^p \, dv_g \right)^{1/p}.$$

If we set $p^* = \frac{Np}{N-2p}$, then the Sobolev embedding theorem guarantees that $W_0^{2,p}(M) \hookrightarrow L^q(M)$ is compact for $1 < q < p^*$ and continuous for $q = p^*$ (see [6]).

Theorem 2.4 ([14]). *Let (M, g) be a compact Riemannian manifold of dimension n. Given $1 \leq q < p$ two real numbers, and $0 \leq m < k$ two integers such that $\frac{1}{p} = \frac{1}{q} - \frac{k-m}{n}$, then the Sobolev space $W^{k,q}(M)$ is continuously embedded into $W^{m,p}(M)$, i.e.,*

$$W^{k,q}(M) \hookrightarrow W^{m,p}(M).$$

Definition 2.5. The Sobolev space $W_0^{k,p}(M)$ denotes the closure of $\mathcal{D}(M)$ in $W^{k,p}(M)$.

Proposition 2.6. *If $p > 1$ then $W^{k,p}(M)$ is a reflexive Banach space.*

3. Technical lemmas

In this section, we present the main definitions and results related to the operators used in this work.

Definition 3.1. Let X be a real Banach space and let $A : X \to X^*$ be an operator. For any $u, v, u_n \in X$, we define the following properties:

(i) **Boundedness:** An operator is said to be bounded if it maps every bounded subset of X into a bounded subset of X^*.; i.e.,
$$\forall R > 0, \ \exists M(R) > 0 : \|u\|_X \leq R \implies \|A(u)\|_{X^*} \leq M(R).$$

(ii) **Coercivity:** A is coercive if
$$\lim_{\|u\|_X \to \infty} \frac{\langle A(u), u \rangle_{X^* \times X}}{\|u\|_X} = +\infty.$$

(iii) **Monotonicity:** A is monotone if
$$\big\langle A(u) - A(v), \, u - v \big\rangle_{X^* \times X} \geq 0, \quad \forall u, v \in X.$$

(iv) **Strict monotonicity:** A is strictly monotone if
$$\big\langle A(u) - A(v), \, u - v \big\rangle_{X^* \times X} > 0, \quad \forall u \neq v.$$

(v) **Strong monotonicity:** A is strongly monotone if there exists a constant $\alpha > 0$ such that
$$\big\langle A(u) - A(v), \, u - v \big\rangle_{X^* \times X} \geq \alpha \, \|u - v\|_X^2, \quad \forall u, v \in X.$$

(vi) **Continuity:** A is continuous if
$$u_n \to u \text{ in } X \quad \Longrightarrow \quad A(u_n) \to A(u) \text{ in } X^*.$$

(vii) **Hemicontinuity:** A is hemicontinuous if for every $u, v \in X$, the real-valued function $t \mapsto \langle A(u+tv), v\rangle_{X^*\times X}$ is continuous on $[0,1]$.

Remark 3.2. The following relationships hold:

- Any continuous operator is hemicontinuous .
- Any strongly monotone operator is strictly monotone .

Theorem 3.3 ([20])**.** *Let $T : \mathcal{H} \to \mathcal{H}^*$ be an operator on a reflexive real Banach space $\mathcal{H}$. If T is bounded, hemicontinuous, monotone, and coercive, then for each $f \in \mathcal{H}^*$, the equation $T(u) = f$ admits at least one solution $u \in \mathcal{H}$. Moreover, if T is strictly monotone, then the solution is uniquely determined.*

Proposition 3.4. [21] *Let (x_n) be a sequence in a Banach space X. The following characterizations hold:*

(1) ***Strong convergence.*** *Let $x \in X$. If every subsequence of (x_n) contains a further subsequence that converges strongly to x, then the entire sequence (x_n) converges strongly to x.*
(2) ***Weak convergence.*** *Let $x \in X$. If every subsequence of (x_n) contains a further subsequence that converges weakly to x, then the entire sequence (x_n) converges weakly to x.*

Lemma 3.5 ([9])**.** *Let $1 < p < \infty$. Then there exist two positive constants $\beta_p, \tau_p > 0$ such that, for all $z, y \in \mathbb{R}^n$, one has*

$$\beta_p \left(|z| + |y|\right)^{p-2}|z-y|^2 \;\le\; \left\langle |z|^{p-2}z - |y|^{p-2}y,\, z-y\right\rangle \;\le\; \tau_p \left(|z| + |y|\right)^{p-2}|z-y|^2.$$

4. Existence Result

Definition 4.1. A function $u \in W_0^{2,p}(M)$ is said to be a weak solution of problem (1.1) if, for all $v \in W_0^{2,p}(M)$, the following identity holds:

$$\begin{aligned}\int_M |\Delta_g u|^{p-2}\Delta_g u\, \Delta_g v\, dv_g + \mathcal{K}\Big(\int_M |\nabla_g u|^p\, dv_g\Big) \int_M |\nabla_g u|^{p-2}\nabla_g u \nabla_g v\, dv_g \\ + \lambda \int_M |u|^{p-2} u\, v\, dv_g = \int_M \mu(x,u)\, v\, dv_g.\end{aligned} \tag{4.1}$$

Theorem 4.2. *Assume that hypotheses* (H1) - (H3) *hold. Then problem* (1.1) *admits a unique weak solution.*

proof Let

$$\mathcal{H} = W_0^{2,p}(M),$$

equipped with the norm

$$\|u\|_{\mathcal{H}} = \left(\int_M |\Delta_g u|^p\, dv_g + \int_M |\nabla_g u|^p\, dv_g + \int_M |u|^p\, dv_g\right)^{1/p}. \tag{4.2}$$

Then $\mathcal{H}$ is a reflexive Banach space.(where $\mathcal{H}^*$ denotes the dual space of $\mathcal{H}$.) We define the operator $\boldsymbol{\xi} : \mathcal{H} \to \mathcal{H}^*$ by

$$\boldsymbol{\xi} = \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 + \lambda \boldsymbol{\xi}_3 - \boldsymbol{\xi}_4,$$

where for all $u, v \in \mathcal{H}$,

$$\langle \boldsymbol{\xi}_1(u), v\rangle = \int_M |\Delta_g u|^{p-2}\Delta_g u\, \Delta_g v\, dv_g, \qquad \langle \boldsymbol{\xi}_2(u), v\rangle = \mathcal{K}\Big(\int_M |\nabla_g u|^p\, dv_g\Big)\int_M |\nabla_g u|^{p-2}\nabla_g u\cdot\nabla_g v\, dv_g.$$

$$\langle \boldsymbol{\xi}_3(u), v\rangle = \int_M |u|^{p-2}u\, v\, dv_g, \qquad \langle \boldsymbol{\xi}_4(u), v\rangle = \int_M \mu(x,u)\, v\, dv_g.$$

By Definition 4.1, to find weak solutions of (1.1), it suffices to determine $u \in \mathcal{H}$ satisfying the operator equation $\boldsymbol{\xi}(u) = 0$.
The proof proceeds in the following steps.
Step 1. The operator $\boldsymbol{\xi}$ is bounded.
Let $u \in \mathcal{H}$, then $\|u\|_{\mathcal{H}} \leq L$.
For each $v \in \mathcal{H}$, applying Hölder's inequality once more, we obtain

$$\begin{aligned}
\big|\langle \boldsymbol{\xi}_1(u), v\rangle\big| &= \left|\int_M |\Delta_g u|^{p-2}\Delta_g u)(\Delta_g v\, dv_g\right| \\
&\leq \int_M |\Delta_g u|^{p-1}|\Delta_g v|\, dv_g \\
&\leq \left(\int_M |\Delta_g u|^p\, dv_g\right)^{\frac{p-1}{p}}\left(\int_M |\Delta_g v|^p\, dv_g\right)^{\frac{1}{p}} \\
&= \|\Delta_g u\|^{p-1}_{L^p(M)}\,\|\Delta_g v\|_{L^p(M)} \\
&\leq \|u\|^{p-1}_{\mathcal{H}}\,\|v\|_{\mathcal{H}}.
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
\big|\langle \boldsymbol{\xi}_3(u), v\rangle\big| &= \left|\int_M |u|^{p-2}u\, v\, dv_g\right| \\
&\leq \int_M |u|^{p-1}|v|\, dv_g \\
&\leq \|u\|^{p-1}_{L^p(M)}\|v\|_{L^p(M)} \\
&\leq \|u\|^{p-1}_{\mathcal{H}}\|v\|_{\mathcal{H}}.
\end{aligned} \tag{4.4}$$

by applying Hölder's inequality and (1.2) we have

$$\begin{aligned}
\big|\langle \boldsymbol{\xi}_2(u), v\rangle\big| &= \left|\mathcal{K}\Big(\int_M |\nabla_g u|^p\, dv_g\Big)\int_M |\nabla_g u|^{p-2}\nabla_g u\cdot\nabla_g v\, dv_g\right| \\
&\leq \mathcal{K}_\infty \int_M |\nabla_g u|^{p-1}|\nabla_g v|\, dv_g \\
&\leq \mathcal{K}_\infty\|\nabla_g u\|^{p-1}_{L^p(M)}\|\nabla_g v\|_{L^p(M)} \\
&\leq \mathcal{K}_\infty\|u\|^{p-1}_{\mathcal{H}}\|v\|_{\mathcal{H}}.
\end{aligned} \tag{4.5}$$

By condition (1.3) and Hölder's inequality, $\mathcal{H} \hookrightarrow L^q(M)$, we obtain

$$\begin{aligned} \big|\langle \boldsymbol{\xi}_4(u), v\rangle\big| &= \Big|\int_M \mu(x,u)\, v\, dv_g\Big| \\ &\le \int_M |\mu(x,u)|\,|v|\, dv_g \\ &\le \int_M \big(\mu_0(x) + \beta|u|^{q-1}\big)|v|\, dv_g \\ &= \int_M \mu_0(x)|v|\, dv_g + \beta\int_M |u|^{q-1}|v|\, dv_g \\ &\le \|\mu_0\|_{L^{q'}(M)}\,\|v\|_{L^q(M)} + \beta\,\|u\|^{q-1}_{L^q(M)}\,\|v\|_{L^q(M)} \\ &\le C_q\Big(\|\mu_0\|_{L^{q'}(M)} + \beta C_q^{q-1}\|u\|^{q-1}_{\mathcal{H}}\Big)\,\|v\|_{\mathcal{H}}, \end{aligned} \tag{4.6}$$

Combining of (4.3),(4.4), (4.5) and (4.6), we derive

$$\begin{aligned} \big|\langle \boldsymbol{\xi} u, v\rangle\big| &\le \big|\langle \boldsymbol{\xi}_1 u, v\rangle\big| + \big|\langle \boldsymbol{\xi}_2 u, v\rangle\big| + \lambda\big|\langle \boldsymbol{\xi}_3 u, v\rangle\big| + \big|\langle \boldsymbol{\xi}_4 u, v\rangle\big| \\ &\le (\lambda + 1 + \mathcal{K}_\infty)\|u\|^{p-1}_{\mathcal{H}}\|v\|_{\mathcal{H}} + C_q\Big(\|\mu_0\|_{L^{q'}(M)} \ + \beta C_q^{q-1}\|u\|^{q-1}_{\mathcal{H}}\Big)\,\|v\|_{\mathcal{H}}. \end{aligned}$$

Therefore $\boldsymbol{\xi}$ is bounded.

Step 2. The operator $\boldsymbol{\xi}$ is strictly monotone.

Let $u_1, u_2 \in \mathcal{H}$ with $u_1 \neq u_2$ Using the lemma 3.5, we obtain

$$\begin{aligned} \langle \boldsymbol{\xi}_1(u_1) - \boldsymbol{\xi}_1(u_2), u_1 - u_2\rangle &= \int_M \Big(|\Delta_g u_1|^{p-2}\Delta_g u_1 - |\Delta_g u_2|^{p-2}\Delta_g u_2\Big)(\Delta_g u_1 - \Delta_g u_2)\, dv_g \\ &\ge \beta_p \int_M \big(|\Delta_g u_1| + |\Delta_g u_2|\big)^{p-2}|\Delta_g u_1 - \Delta_g u_2|^2\, dv_g \\ &\ge \beta_p \int_M |\Delta_g u_1 - \Delta_g u_2|^{p-2}|\Delta_g u_1 - \Delta_g u_2|^2\, dv_g \\ &= \beta_p \int_M |\Delta_g u_1 - \Delta_g u_2|^p\, dv_g, \end{aligned}$$

and similarly,

$$\langle \boldsymbol{\xi}_3(u_2) - \boldsymbol{\xi}_3(u_2), u_1 - u_2\rangle \ge \beta_p \int_M |\ u_1 - u_2|^p\, dv_g.$$

By condition (1.2) and the lemma 3.5, we get

$$\begin{aligned} \langle \boldsymbol{\xi}_2(u_1) - \boldsymbol{\xi}_2(u_2), u_1 - u_2\rangle &\ge \mathcal{K}_0 \int_M \big(|\nabla_g u_1|^{p-2}\nabla_g u_1 - |\nabla_g u_2|^{p-2}\nabla_g u_2\big)\cdot(\nabla_g u_1 - \nabla_g u_2)\, dv_g \\ &\ge \mathcal{K}_0\, \beta_p \int_M |\nabla_g u_1 - \nabla_g u_2|^p\, dv_g \end{aligned}$$

Further, since μ iis non-increasing with respect to the second variable,we have

$$\langle \boldsymbol{\xi}_4(u_1) - \boldsymbol{\xi}_4(u_2), u_1 - u_2\rangle = \int_M \big(\mu(x,u_1) - \mu(x,u_2)\big)(u_1 - u_2)\, dv_g \le 0.$$

This has the consequence that

$$\langle \boldsymbol{\xi}(u_1) - \boldsymbol{\xi}(u_2), u_1 - u_2 \rangle \geq \beta_{min} \|u_1 - u_2\|_{\mathcal{H}}$$

where $\beta_{\min} = \min\{\beta_p, \beta_p \mathcal{K}_0, \lambda\beta_p\}$
Therefore, the operator $\boldsymbol{\xi}$ is strictly monotone,
Step 3. The operator $\boldsymbol{\xi}$ is coercive.
By condition (1.3) and Hölder's inequality, $\mathcal{H} \hookrightarrow L^q(M)$, we obtain

$$\begin{aligned}
|\langle \mu(u), u\rangle| &\leq \int_M \left(|\mu_0| + \beta |u|^{q-1}\right) |u| \, dv_g \\
&\leq \left(\int_M |\mu_0|^{q'} \, dv_g\right)^{\frac{1}{q'}} \left(\int_M |u|^q \, dv_g\right)^{\frac{1}{q}} + \beta \int_M |u|^q \, dv_g \\
&= \|\mu_0\|_{q'} \|u\|_q + \beta \|u\|_q^q \\
&\leq C_q \|\mu_0\|_{q'} \|u\|_{\mathcal{H}} + \beta C_q^q \|u\|_{\mathcal{H}}^q \\
&= C_1 \|u\|_{\mathcal{H}} + C_2 \|u\|_{\mathcal{H}}^q
\end{aligned} \tag{4.7}$$

where $C_1 = C_q \|\mu_0\|_{q'}$ and $C_2 = \beta C_q^q$
According to (4.7) and by condition (1.2), we obtain

$$\begin{aligned}
\langle \boldsymbol{\xi}(u), u\rangle &= \int_M |\Delta_g u|^{p-2} \Delta_g u \Delta_g u \, dv_g + \mathcal{K}\left(\int_M |\nabla_g u|^p \, dv_g\right) \int_M |\nabla_g u|^{p-2} \nabla_g u \nabla_g u \, dv_g \\
&\quad + \lambda \int_M |u|^{p-2} u \, u \, dv_g - \int_M \mu(x, u) \, u \, dv_g \\
&\geq \int_M |\Delta_g u|^p \, dv_g + \lambda \int_M |u|^p \, dv_g + \mathcal{K}_0 \int_M |\nabla u|^{p-2} \nabla u \, \nabla u \, dv_g - C_1 \|u\|_{\mathcal{H}} - C_2 \|u\|_{\mathcal{H}}^q \\
&\geq \min(1, \lambda, \mathcal{K}_0) (\int_M |\Delta_g u|^p \, dv_g + \int_M |\nabla_g u|^p \, dv_g + \int_M |u|^p \, dv_g) - C_1 \|u\|_{\mathcal{H}} - C_2 \|u\|_{\mathcal{H}}^q \\
&= \mathcal{K}_{min} \|u\|_{\mathcal{H}}^p - C_1 \|u\|_{\mathcal{H}} - C_2 \|u\|_{\mathcal{H}}^q.
\end{aligned}$$

where $\mathcal{K}_{min} = \min(1, \lambda, \mathcal{K}_0)$
then

$$\frac{\langle \boldsymbol{\xi}(u), u\rangle}{\|u\|_{\mathcal{H}}} \geq \mathcal{K}_{\min} \|u\|_{\mathcal{H}}^{p-1} - C_1 - C_2 \|u\|_{\mathcal{H}}^{q-1}. \tag{4.8}$$

Therefore, since $p > q > 1$, we have

$$\frac{\langle \boldsymbol{\xi}(u), u\rangle}{\|u\|_{\mathcal{H}}} \longrightarrow +\infty \quad \text{as } \|u\|_{\mathcal{H}} \longrightarrow +\infty.$$

That is, $\boldsymbol{\xi}$ is coercive.
Step 4. The operator $\boldsymbol{\xi}$ is continuous .
We need to show that if $u_k \to u$ in $\mathcal{H}$, then $\boldsymbol{\xi} u_k \to \boldsymbol{\xi} u$ in $\mathcal{H}^*$.
we recall that if $u_k \to u$ in $\mathcal{H}$ as $k \to \infty$, then

$$u_k \to u \quad \text{in } L^p(M), \qquad \nabla_g u_k \to \nabla_g u \quad \text{in } L^p(M), \quad \Delta_g u_k \to \Delta_g u \quad \text{in } L^p(M).$$

Hence, there exists a subsequence (u_{k_i}) of (u_k) and three measurable functions $\psi_1, \psi_2, \psi_3 \in L^p(M)$ such that

$$\begin{aligned} u_{k_i}(x) &\to u(x), && \text{for a.e. } x \in M, \\ |u_{k_i}(x)| &\leq \psi_1(x), && \text{for a.e. } x \in M, \\ \nabla_g u_{k_i}(x) &\to \nabla_g u(x), && \text{for a.e. } x \in M, \\ |\nabla_g u_{k_i}(x)| &\leq \psi_2(x), && \text{for a.e. } x \in M, \\ \Delta_g u_{k_i}(x) &\to \Delta_g u(x), && \text{for a.e. } x \in M, \\ |\Delta_g u_{k_i}(x)| &\leq \psi_3(x), && \text{for a.e. } x \in M. \end{aligned} \tag{4.9}$$

Let $u_k \to u$ in $\mathcal{H}$ as $k \to \infty$.
Clearly, as $k \to \infty$, we have

$$|u_{k_i}|^{p-2}u_{k_i} \to |u|^{p-2}u \quad \text{a.e. in } M.$$

Moreover,

$$\big||u_{k_i}|^{p-2}u_{k_i}\big| \leq |\psi_1|^{p-1},$$

and

$$\int_M \left(|\psi_1|^{p-1}\right)^{p'}\, dv_g = \int_M |\psi_1|^p \, dv_g = \|\psi_1\|^p_{L^p(M)}.$$

Hence,

$$|\psi_1|^{p-1} \in L^{p'}(M).$$

Applying the Lebesgue dominated convergence theorem, we obtain

$$|u_{k_i}|^{p-2}u_{k_i} \to |u|^{p-2}u \quad \text{in } L^{p'}(M).$$

By the convergence principle in Banach spaces (see Proposition 3.4), we obtain

$$|u_k|^{p-2}u_k \to |u|^{p-2}u \quad \text{in } L^{p'}(M).$$

Therefore, by Hölder's inequality, we have

$$\begin{aligned} \|\boldsymbol{\xi}_3(u_n) - \boldsymbol{\xi}_3(u)\|_{\mathcal{H}^*} &= \sup_{\|v\|_{\mathcal{H}} \leq 1} \left| \int_M \left(|u_n|^{p-2}u_n - |u|^{p-2}u\right) v \, dv_g \right| \\ &\leq \sup_{\|v\|_{\mathcal{H}} \leq 1} \left(\int_M \big||u_n|^{p-2}u_n - |u|^{p-2}u\big|^{p'} d\sigma_g \right)^{\frac{1}{p'}} \left(\int_M |v|^p \, dv_g \right)^{\frac{1}{p}} \\ &\leq \left(\int_M \big||u_n|^{p-2}u_n - |u|^{p-2}u\big|^{p'} dv_g \right)^{\frac{1}{p'}} \longrightarrow 0. \end{aligned}$$

Using (4.9) and the same technique as above, we conclude that

$$\begin{aligned}\|\boldsymbol{\xi}_2(u_k)-\boldsymbol{\xi}_2(u)\|_{\mathcal{H}^*} &= \sup_{\|v\|_{\mathcal{H}}\le 1}\left|\mathcal{K}\Big(\int_M |\nabla_g u|^p\,dv_g\Big)\int_M \left(|\nabla_g u_k|^{p-2}\nabla_g u_k-|\nabla_g u|^{p-2}\nabla_g u\right)\nabla_g v\,dv_g\right|\\ &\le \sup_{\|v\|_{\mathcal{H}}\le 1}\mathcal{K}_\infty\left(\int_M \left||\nabla_g u_k|^{p-2}\nabla_g u_k-|\nabla_g u|^{p-2}\nabla_g u\right|^{p'}dv_g\right)^{\frac{1}{p'}}\left(\int_M |\nabla_g v|^p dv_g\right)^{\frac{1}{p}}\\ &\le \mathcal{K}_\infty\left(\int_M \left||\nabla_g u_k|^{p-2}\nabla_g u_k-|\nabla_g u|^{p-2}\nabla_g u\right|^{p'}dv_g\right)^{\frac{1}{p'}}\longrightarrow 0.\end{aligned}$$

and

$$\begin{aligned}\|\boldsymbol{\xi}_1(u_k)-\boldsymbol{\xi}_1(u)\|_{\mathcal{H}^*} &= \sup_{\|v\|_{\mathcal{H}}\le 1}\left|\int_M \left(|\Delta_g u_k|^{p-2}\Delta_g u_k-|\Delta_g u|^{p-2}\Delta_g u\right)\Delta_g v\,dv_g\right|\\ &\le \sup_{\|v\|_{\mathcal{H}}\le 1}\left(\int_M \left||\Delta_g u_k|^{p-2}\Delta_g u_k-|\Delta_g u|^{p-2}\Delta_g u\right|^{p'}dv_g\right)^{\frac{1}{p'}}\left(\int_M |\Delta_g v|^p dv_g\right)^{\frac{1}{p}}\\ &\le \left(\int_M \left||\Delta_g u_k|^{p-2}\Delta_g u_k-|\Delta_g u|^{p-2}\Delta_g u\right|^{p'}dv_g\right)^{\frac{1}{p'}}\longrightarrow 0.\end{aligned}$$

Hence, $\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3$ are continuous.
Let $u_k\to u$ in $\mathcal{H}$ as $k\to\infty$.
Using (4.9) and the assumptions (1.3), we obtain the following:

$$|\mu(y,u_{k_i}(y))|\le \mu_0+\beta|\psi_1(y)|^{q-1},\quad \text{for a.e. } x\in M.$$

On the other hand, by (H_2), we deduce that, as $k\to\infty$,

$$\mu(x,u_{k_i}(x))\to\mu(x,u(x))\quad \text{a.e. in } M.$$

Moreover,

$$\begin{aligned}\int_M |\mu_0+\beta|\psi_1|^{q-1}|^{q'}\,dv_g &\le \int_M \left(|\mu_0|+\beta|\psi_1|^{q-1}\right)^{q'}dv_g\\ &\le C_{q'}\int_M \left(|\mu_0|^{q'}+\beta^{q'}|\psi_1|^q\right)dv_g\\ &= C_{q'}\left(\|\mu_0\|^{q'}_{L^{q'}(M)}+\beta^{q'}\|\psi_1\|^q_{L^q(M)}\right),\end{aligned}$$

Hence,

$$\mu_0+\beta|\psi_1|^{q-1}\in L^{q'}(M).$$

Applying the Lebesgue dominated convergence theorem, one obtains

$$\mu(x,u_{k_i}(x))\to\mu(x,u(x))\quad \text{in } L^{q'}(M).$$

By the convergence principle in Banach spaces (see Proposition 3.4), we obtain

$$\mu(x,u_k)\to\mu(x,u)\quad \text{in } L^{q'}(M).$$

By Hölder's inequality and $\mathcal{H} \hookrightarrow L^q(M)$, we obtain

$$\begin{aligned}\|\boldsymbol{\xi}_4(u_k) - \boldsymbol{\xi}_4(u)\|_{\mathcal{H}'} &= \sup_{\|v\|_{\mathcal{H}} \le 1} \left| \int_M \big(\mu(x,u_k) - \mu(x,u)\big) v \, dv_g \right| \\ &\le \sup_{\|v\|_{\mathcal{H}} \le 1} \left(\int_M \left|\mu(x,u_k) - \mu(x,u)\right|^{q'} dv_g \right)^{\frac{1}{q'}} \left(\int_M |v|^q \, dv_g \right)^{\frac{1}{q}} \\ &\le C_q \left(\int_M \left|\mu(x,u_k) - \mu(x,u)\right|^{q'} dv_g \right)^{\frac{1}{q'}} \longrightarrow 0.\end{aligned}$$

Hence, $\boldsymbol{\xi}_4$ is continuous.
Therefore $\boldsymbol{\xi}$ is continuous. Moreover, $\boldsymbol{\xi}$ is hemicontinuous.
By Theorem 3.3, Hence, the problem (1.1) has a unique solution $u \in \mathcal{H}$.

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