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ON A (k,μ)-PARACONTACT MANIFOLD SATISFYING SOME CONDITIONS ON THE PROJECTIVE CURVATURE TENSOR

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ABSTRACT. In the present paper, we study the curvature tensors of (k,μ) - Paracontact manifold satisfying the conditions $P(\xi,Y)\widetilde{Z}=0$, $P(\xi,Y)P=0$, $P(\xi,Y)S=0$ and $P(\xi,Y)R=0$. According these cases, (k,μ)-Paracontact manifolds have been characterized. We obtain the necessary and sufficient conditions of a (k,μ)-Paracontact manifold to be (η-)Einsteins.

1. INTRODUCTION

The study of paracontact geometry was initiated by Kaneyuki and Williams [9] a systematic study of paracontact metric manifolds and their subclasses was carried out by Zamkovoy [23]. Since then several geometers studied paracontact metric manifolds and obtain various important properties of these manifolds [3, 4, 11, 18, 19]. [7] introduced a new type of paracontact geometry, so called paracontact metric $(k,\mu)-$spaces, where k and μ are real constants. Such manifolds are known as $(k,\mu)-$paracontact metric manifolds.

The class of $(k,\mu)-$paracontact metric manifolds contains para-Sasakian manifolds. As a generalization of locally symmetric spaces, many authors have studied semi-symmetric spaces. A semi-Riemannian manifold (M^{2n+1},g), $n\geq 1$, is said to be semi-symmetric if its curvature tensor R satisfies $R(X,Y).R=0$ for all vector fields X,Y on M^{2n+1}[10, 14]. The notion of locally $\phi-$symmetric was introduced by Takahashi [15] in Sasakian geometry as a weaker versionn of locally symmetric manifolds.

In [20], authors studied the Weyl projective curvature tensor in an $N(k)$-contact metric manifold and classified $N(k)$-contact metric manifolds. De. U. C. and Sarkar A. [8] studied properties of projective curvature tensor to generalized Sasakian space form Atçeken M. [2] studied generalized Sasakian space form satisfying certain conditions on the concircular curvature tensor. Özgür C. and De U. C. [13] researched some certain curvature conditions satisfied by quasi-conformal curvature tensor in Kenmotsu manifolds. Arslan K., Murathan C. and Özgür C. produced the works on contact manifold curvature tensor [1].

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Motivated by the studies of the above authors, in this paper we classify (k,μ)-paracontact manifolds, which satify the curvature conditions $P(\xi,Y)\widetilde{Z}=0$, $P(X,Y)P=0$, $P(\xi,Y)R=0$ and $P(\xi,Y)S=0$, where P is the projective curvature tensor, $\widetilde{Z}$ is the concircular curvature tensor, S is the Ricci tensor and R is the Riemannian curvature tensor.

2. PRELIMINARIES

A contact manifold is a $C^{\infty}-(2n+1)$ manifold M^{2n+1}equipped with a global 1-form η such that $\eta\wedge(d\eta)^{n}\neq 0$ everywhere on M^{2n+1}. Given such a form η, it is well known that there exists a unique vector field ξ, called the characteristic vector field, such that $\eta(\xi)=1$ and $d\eta(X,\xi)=0$ for every vector field X on M^{2n+1}. A Riemannian metric g is said to be associated metric if there exists a tensor field ϕ of type $(1,1)$ such that

$$\phi^{2}X=X-\eta(X)\xi,\qquad \eta(\xi)=1,\qquad \eta\circ\phi=0,\qquad \phi\xi=0, \tag{2.1}$$

$$g(\phi X,\phi Y)=-g(X,Y)+\eta(X)\eta(Y),\quad g(X,\xi)=\eta(X) \tag{2.2}$$

for all vector fields X,Y on M. Then the structure (ϕ,ξ,η,g) on M is called a paracontact metric structure and the manifold equipped with such a structure is called a paracontact metric manifold[23]. We now define a $(1,1)$ tensor field h by $h=\frac{1}{2}L_{\xi}\phi$, where L denotes the Lie derivative, then h is symmetric and satisfies[23].

$$h\phi=-\phi h,\qquad h\xi=0,\qquad Tr.h=Tr.\phi h=0. \tag{2.3}$$

If ∇ denotes the Levi-Civita connection of g, then we have the following relation

$$\nabla_{X}\xi=-\phi X+\phi hX \tag{2.4}$$

for all $X\in\chi(M)$[23]. For a para-contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$, if ξ is a killing vector field or equivalently, $h=0$ it is called a K-paracontact manifold.

A para-contact metric structure (ϕ,ξ,η,g) is normal, that is, satisfies $[\phi,\phi]+2d\eta\otimes\xi=0$, which is equivalent to

$$(\nabla_{X}\phi)Y=-g(X,Y)\xi+\eta(Y)X$$

for any $X,Y\in\chi(M)$[23]. Any para-Sasakian manifold is K-paracontact, and the converse holds when $n=1$, that is, for 3-dimensional spaces. Any para-Sasakian manifold satisfies

$$R(X,Y)\xi=-(\eta(Y)X-\eta(X)Y) \tag{2.5}$$

for any $X,Y\in\chi(M)$, but this is not a sufficient condition for a para-contact manifold to be para-Sasakian. It is clear that every para-Sasakian manifold is K-paracontact, but the converse is not always true[7].

Definition 2.1. A paracontact manifold M is said to be η-Einstein if its Ricci tensor S of type $(0,2)$ is of the from $S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y)$,where a,b are smooth functions on M. If $b=0$, then the manifold is called Einstein[5].

Definition 2.2. A paracontact metric manifold is said to be a $(k,\mu)-$paracontact manifold if the curvature tensor R satisfies

$$\widetilde{R}(X,Y)\xi = k\,[\eta(Y)X - \eta(X)Y] + \mu\,[\eta(Y)hX - \eta(X)hY] \tag{2.6}$$

for all $X, Y \in \chi(M)$ and k, μ are real constants. This class is very wide containing the para-Sasakian manifolds as well as the paracontact metric manifolds satisfying $R(X,Y)\xi = 0$ [24]. In particular, if $\mu = 0$, then the paracontact metric $(k,\mu)-$manifold is called paracontact metric $N(k)$-manifold . Thus for a paracontact metric $N(k)$-manifold the curvature tensor satisfies the following relation

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) \tag{2.7}$$

for all $X, Y \in \chi(M)$. Though the geometric behavior of paracontact metric $(k,\mu)-$spaces is different according as $k < -1$, or $k > -1$, but there are also some common results for $k < -1$ and $k > -1$.

Lemma 2.3. *There does not exist any paracontact $(k,\mu)-$manifold of dimension greater than 3 with $k > -1$ which is Einstein whereas there exits such manifolds for $k < -1$.*

In a paracontact metric $(k,\mu)-$manifold $(M^{2n+1}\phi, \xi, \eta, g)$, $n > 1$, the following relation hold :

$$h^2 = (k+1)\phi^2 \tag{2.8}$$

$$(\widetilde{\nabla}_X\phi)Y = -g(X - hX, Y)\xi + \eta(Y)(X - hX), \text{ for } k \neq -1, \tag{2.9}$$

$$\begin{aligned} S(X,Y) &= [2(1-n) + n\mu]g(X,Y) + [2(n-1) + \mu]g(hX,Y) \\ &\quad +[2(n-1) + n(2k-\mu)]\eta(X)\eta(Y), \end{aligned} \tag{2.10}$$

$$S(X,\xi) = 2nk\eta(X), \tag{2.11}$$

$$\begin{aligned} QY &= [2(1-n) + n\mu]Y + [2(n-1) + \mu]hY \\ &\quad +[2(n-1) + n(2k+\mu)]\eta(Y), \end{aligned} \tag{2.12}$$

$$Q\xi = 2nk\xi, \tag{2.13}$$

$$Q\phi - \phi Q = 2[2(n-1) + \mu]h\phi \tag{2.14}$$

for any vector fields X, Y on M^{2n+1} , where Q and S denotes the Ricci operator and Ricci tensor of (M^{2n+1}, g), respectively[7].

The concept of quasi-conformal curvature tensor was defined by K. Yano and S. Sawaki[17]. Quasi-conformal curvature tensor of a $(2n+1)$-dimensional Riemannian manifold is defined as

$$\begin{aligned} \widetilde{C}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y \\ &\quad +g(Y,Z)QX - g(X,Z)QY] \\ &- \frac{\tau}{2n+1}[\frac{a}{2n} + 2b][g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{2.15}$$

where a and b are two scalars, and r is the scalar curvature of the manifold. If $a = 1$ and $b = \frac{-1}{2n-1}$, then quasi conformal curvature tensor reduces to conformal curvature tensor is defined as

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &- g(X,Z)QY] + \frac{\tau}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{2.16}$$

Let (M, g) be an $(2n+1)$-dimensional Riemannian manifold. Then the concircular curvature tensor $\widetilde{Z}$ is defined by

$$\widetilde{Z}(X,Y)Z = R(X,Y)Z - \frac{\tau}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y], \tag{2.17}$$

for all $X, Y, Z \in \chi(M)$, where r is the scalar curvature of M [16].

Let (M, g) be an $(2n+1)$-dimensional Riemannian manifold. The projective curvature tensor field is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y] \tag{2.18}$$

for all $X, Y, Z \in \chi(M)$, where R denotes the Riemannian curvature tensor of M and Q is the Ricci operator given by $g(QX, Y) = S(X, Y)$[16].

3. A $(k, \mu)-$ Paracontact Manifold Satisfying Certain Conditions On The Projective Curvature Tensor

In this section, we will give the main results for this paper.

Let M be $(2n+1)-$dimensional $(k, \mu)-$paracontact metric manifold and we denote the Riemannian curvature tensor of R, then we have from (2.6) for $X = \xi$,

$$R(\xi, Y)Z = k(g(Y,Z)\xi - \eta(Z)Y) + \mu(g(hY,Z)\xi - \eta(Z)hY). \tag{3.1}$$

In (3.1), choosing $Z = \xi$, we obtain

$$R(\xi, Y)\xi = k(\eta(Y)\xi - Y) - \mu hY. \tag{3.2}$$

Also from (2.6), we obtain

$$\eta(R(\xi, Y)Z) = k(g(Y,Z) - \eta(Y)\eta(Z)) + \mu g(hY, Z). \tag{3.3}$$

In the same way, taking $X = \xi$ in (2.15), we have

$$\begin{aligned} \widetilde{C}(\xi, Y)Z &= [ak + 2nkb - \frac{r}{2n(2n+1)}(\frac{a}{2n} + 2b)][g(Y,Z)\xi - \eta(Z)Y] \\ &+a\mu[g(hY,Z)\xi - \eta(Z)hY] + b[S(Y,Z)\xi - \eta(Z)QY]. \end{aligned} \tag{3.4}$$

In (3.4), choosing $Z = \xi$, we obtain

$$\begin{aligned} \widetilde{C}(\xi, Y)\xi &= [ak + 2nkb - \frac{r}{2n(2n+1)}(\frac{a}{2n} + 2b)][\eta(Y)\xi - Y] \\ &-a\mu hY + b[2nk\eta(Y)\xi - QY]. \end{aligned} \tag{3.5}$$

Also from (2.17), we have

$$\widetilde{Z}(\xi,Y)Z=(k-\frac{r}{2n(2n+1)})[g(Y,Z)\xi-\eta(Z)Y]+\mu(g(hY,Z)\xi-\eta(Z)hY) \quad (3.6)$$

and

$$\widetilde{Z}(\xi,Y)\xi=(k-\frac{r}{2n(2n+1)})[\eta(Y)\xi-Y]-\mu hY. \quad (3.7)$$

From (2.18), we get

$$P(\xi,Y)Z=kg(Y,Z)\xi+\mu g(hY,Z)\xi-\mu\eta(Z)hY-\frac{1}{2n}S(Y,Z)\xi. \quad (3.8)$$

Theorem 3.1. *Let M be $(2n+1)-$dimensional a (k,μ)-paracontact manifold. Then, $P(\xi,Y)R=0$ if and only if M is an Einstein manifold.*

Proof. Suppose that $P(\xi,Y)R=0$. This implies that

$$\begin{aligned}(P(\xi,Y)R)(U,W)Z &= P(\xi,Y)R(U,W)Z-R(P(\xi,Y)U,W)Z\\ &\quad -R(U,P(\xi,Y)W)Z-R(U,W)P(\xi,Y)Z=0, \quad (3.9)\end{aligned}$$

for all $Y,U,W,Z\in\chi(M)$. Using (3.8) in (3.9), we obtain

$$\begin{aligned}(P(\xi,Y)R)(U,W)Z &= kg(Y,R(U,W)Z)\xi+\mu g(hY,R(U,W)Z)\xi\\ &\quad -\mu\eta(R(U,W)Z)hY-\frac{1}{2n}S(Y,R(U,W)Z)\xi\\ &\quad -R(kg(Y,U)\xi+\mu g(hY,U)\xi-\mu\eta(U)hY\\ &\quad -\frac{1}{2n}S(Y,U)\xi,W)Z-R(U,kg(Y,W)\xi\\ &\quad +\mu g(hY,W)\xi-\mu\eta(W)hY-\frac{1}{2n}S(Y,W)\xi)Z\\ &\quad -R(U,W)(kg(Y,Z)\xi+\mu g(hY,Z)\xi\\ &\quad -\mu\eta(Z)hY-\frac{1}{2n}S(Y,Z)\xi)=0. \quad (3.10)\end{aligned}$$

Here setting $U=Z=\xi$, by using (2.6), (3.1) and inner product both sides of (3.10) by $\xi\in\chi(M)$, we can easily to see that

$$\frac{\mu}{2n}S(Y,hW)=k^2g(Y,W)+k\mu g(Y,hW)-\frac{k}{2n}S(Y,W). \quad (3.11)$$

Using the equations (2.1),(2.8) and replacing hW instead of W in (3.11), we have

$$\frac{k}{2n}S(Y,hW)=-\frac{\mu}{2n}(k+1)S(Y,W)+k^2g(Y,hW)+k\mu(1+k)g(Y,W) \quad (3.12)$$

Also from (3.11) and (3.12), we conclude that

$$S(Y,W)=2nkg(Y,W).$$

So, M is an Einstein manifold. The converse is obvious. □

Theorem 3.2. *Let M be $(2n+1)-$dimensional a (k,μ)-paracontact manifold. Then, $P(\xi,Y)\widetilde{Z}=0$ if and only if M is an Einstein manifold.*

Proof. Suppose that $P(\xi,Y)\widetilde{Z}=0$. This means that

$$\begin{aligned}(P(\xi,Y)\widetilde{Z})(U,W)Z &= P(\xi,Y)\widetilde{Z}(U,W)Z-\widetilde{Z}(P(\xi,Y)U,W)Z\\ &\quad -\widetilde{Z}(U,P(\xi,Y)W)Z-\widetilde{Z}(U,W)P(\xi,Y)Z=0\end{aligned} \tag{3.13}$$

for all $Y,U,W,Z\in\chi(M)$. Using (3.8) in (3.13), we reach at

$$\begin{aligned}(P(\xi,Y)\widetilde{Z})(U,W)Z &= kg(Y,\widetilde{Z}(U,W)Z)\xi+\mu g(hY,\widetilde{Z}(U,W)Z)\xi\\ &\quad -\mu\eta(\widetilde{Z}(U,W)Z)hY-\frac{1}{2n}S(Y,\widetilde{Z}(U,W)Z)\xi\\ &\quad -\widetilde{Z}(kg(Y,U)\xi+\mu g(hY,U)\xi-\mu\eta(U)hY\\ &\quad -\frac{1}{2n}S(Y,U)\xi,W)Z-\widetilde{Z}(U,kg(Y,W)\xi\\ &\quad +\mu g(hY,W)\xi-\mu\eta(W)hY-\frac{1}{2n}S(Y,W)\xi)Z\\ &\quad -\widetilde{Z}(U,W)(kg(Y,Z)\xi+\mu g(hY,Z)\xi\\ &\quad -\mu\eta(Z)hY-\frac{1}{2n}S(Y,Z)\xi)=0.\end{aligned} \tag{3.14}$$

Putting $U=Z=\xi$ in (3.14), by using (3.6), (3.7) and inner product both sides of (3.14) by $\xi\in\chi(M)$, we observe

$$\frac{\mu}{2n}S(Y,hW)=\alpha kg(Y,W)+\mu kg(Y,hW)-\frac{a}{2n}S(Y,W). \tag{3.15}$$

where $(k-\frac{r}{2n(2n+1)})=\alpha$. Using (2.1),(2.8) and substituting hW into W in (3.15), we arrive

$$\frac{a}{2n}S(Y,hW)=-\frac{\mu}{2n}(k+1)S(Y,W)+\alpha kg(Y,hW)+\mu k(k+1)g(Y,W). \tag{3.16}$$

Also from (3.15) and (3.16), we reach at

$$S(Y,W)=2nkg(Y,W).$$

Thus, M is an Einstein manifold. The converse is obvious. □

Theorem 3.3. *Let M be $(2n+1)-$dimensional a (k,μ)-paracontact manifold. Then, $P(\xi,Y)S=0$ if and only if M is an Einstein manifold.*

Proof. Suppose that $P(\xi,Y)S=0$. Then we have,

$$S(P(\xi,Y)U,W)+S(U,P(\xi,Y)W)=0, \tag{3.17}$$

for all $Y,U,W\in\chi(M)$. By making use of (3.8) and (3.17), we obtain

$$\begin{aligned}&S(kg(Y,U)\xi+\mu g(hY,U)\xi-\mu\eta(U)hY-\frac{1}{2n}S(Y,U)\xi,W)\\ &+S(U,kg(Y,W)\xi+\mu g(hY,W)\xi-\mu h\eta(W)Y-\frac{1}{2n}S(Y,W)\xi)=0.\end{aligned} \tag{3.18}$$

Choosing $U=\xi$, using (2.11) and inner product both sides of (3.18) by $\xi\in\chi(M)$, we have

$$\mu S(hY,W) = 2nk^2 g(Y,W) - 2nk\mu g(hY,W) - kS(Y,W). \tag{3.19}$$

Substituting hY into Y in (3.19) and taking into account (2.1) and (2.8), we get

$$kS(hY,W) = -\mu(1+k)S(Y,W) + 2nk^2 g(hY,W) + 2nk\mu(1+k)g(Y,W). \tag{3.20}$$

Also from (3.19) and (3.20), we conclude

$$S(Y,W) = 2nkg(Y,W).$$

Thus, M is an Einstein manifold. The converse is obvious. $\square$

Theorem 3.4. *Let M be $(2n+1)-$dimensional a (k,μ)-paracontact manifold. Then, $P(\xi,Y)P = 0$ if and only if M is an $\eta-$Einstein manifold.*

Proof. Suppose that $P(\xi,Y)P = 0$. Then we have,

$$\begin{aligned}(P(\xi,Y)P)(U,V)Z &= P(\xi,Y)P(U,V)Z - P(P(\xi,Y)U,V)Z \\ &\quad -P(U,P(\xi,Y)V)Z - P(U,V)P(\xi,Y)Z = 0.\end{aligned} \tag{3.21}$$

Choosing $Z = \xi$ and using (2.18) in (3.21), we get

$$\begin{aligned}&P(\xi,Y)\mu(\eta(V)hU - \eta(U)hV) - P(kg(Y,U)\xi + \mu g(hY,U)\xi \\ &-\mu\eta(U)hY - \frac{1}{2n}S(Y,U)\xi, V)\xi - P(U, kg(Y,V)\xi + \mu g(hY,V)\xi \\ &-\mu\eta(V)hY - \frac{1}{2n}S(Y,V)\xi)\xi + \mu P(U,V)hY = 0.\end{aligned} \tag{3.22}$$

Using (2.6), (3.8) and inner product both sides of (3.22) by $Z \in \chi(M)$, we arrive

$$\begin{aligned}&2ng(P(U,V)hY,Z) + 2nk(\eta(V)\eta(Z)g(Y,hU) - \eta(U)\eta(Z)g(Y,hV)) \\ &+2n\mu(1+k)(\eta(V)\eta(Z)g(Y,U) - \eta(U)\eta(Z)g(Y,V)) + (\eta(U)\eta(Z)S(Y,hV) \\ &-\eta(V)\eta(Z)S(Y,hU)) + 2nk(g(Y,U)g(hV,Z) - g(Y,V)g(hU,Z)) \\ &+2n\mu(g(hY,U)g(hV,Z) - g(hY,V)g(hU,Z)) + (S(Y,V)g(hU,Z) \\ &-S(Y,U)g(hV,Z)) = 0.\end{aligned} \tag{3.23}$$

Making use of (2.18) and choosing $V = Y = e_i$, ξ $1 \le i \le n$, for orthonormal basis of $\chi(M)$ in (3.23), we observe

$$\begin{aligned}&2nS(U,hZ) + (2n(1+k)[2(n-1)+\mu] + 2n\mu(1+k))g(U,Z) \\ &-(4n^2k + r)g(U,hZ) + (2n(1+k)[2(n-1)+\mu] \\ &-2n\mu(1+k)(2n+1))\eta(U)\eta(Z) = 0.\end{aligned} \tag{3.24}$$

Replacing hZ of Z in (3.24) and taking into account (2.8), we get

$$\begin{aligned}
&2n(1+k)S(U,Z)-4n^2k(1+k)\eta(U)\eta(Z)+(2n(1+k)[2(n-1)+\mu]\\
&+2n\mu(1+k))g(U,hZ)-(4n^2k+r)(1+k)g(U,Z)\\
&+(4n^2k+r)(1+k)\eta(U)\eta(Z)=0. \qquad (3.25)
\end{aligned}$$

From (3.25) and by using (2.10), for the sake brevity, we set

$$\begin{aligned}
A &= (2n(1+k)[2(n-1)+\mu]+2n\mu(1+k)])[2(1-n)+n\mu]\\
&\quad +(4n^2k+r)(1+k)[2(n-1)+\mu],\\
B &= 2n(1+k)[2(n-1)+\mu]+(2n(1+k)[2(n-1)+\mu]+2n\mu(1+k)]),\\
C &= (2n(1+k)[2(n-1)+\mu]+2n\mu(1+k)])[2(n-1)\\
&\quad +n(2k-\mu)]-r(1+k)[2(n-1)+\mu]
\end{aligned}$$

and we conclude

$$BS(U,Z)=Ag(U,Z)+C\eta(U)\eta(Z).$$

So, M is an $\eta-$Einstein manifold. The converse is obvious. □

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