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STRUCTURE OF CAYLEY GRAPH OVER GENERALIZED QUATERNION GROUP

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Abstract. We investigate the structure of undirected Cayley graphs on generalized quaternion groups constructed via inverse-closed connection sets excluding the identity element. Through an analysis of small valencies from one to four, we generalize the structural properties to arbitrary valency. We establish a complete classification theorem, proving that all resulting Cayley graphs are isomorphic to circulant graphs, regular bipartite graphs, or the edge-disjoint union of these two structures. These findings provide a definitive characterization of Cayley graph structures on generalized quaternion groups and establish their connectivity and algebraic properties.

Keywords. Cayley Graph, Generalized Quaternion Group, Valency
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1. Introduction and Preliminaries

1.1. **Introduction.** Graphs provide an effective framework for representing and studying algebraic structures, particularly those arising from group theory. The interplay between algebra and graph theory has led to the development of numerous graph constructions that encode algebraic properties through combinatorial structures, making algebraic graph theory an active area of research [14, 16]. Among the various graph constructions, the *Cayley graph* is one of the most fundamental, representing the elements of a group together with the adjacencies induced by a generating set. Introduced by Arthur Cayley in 1878 [4], Cayley graphs establish a natural connection between group theory and graph theory, allowing algebraic properties of groups to be investigated through combinatorial methods.

In this paper, all Cayley graphs are assumed to be undirected. This is ensured by requiring the connection set $S \subseteq G$ of a finite group G to satisfy $S = S^{-1}$. We also assume that $e \notin S$ so that the graph contains no loops. The Cayley graph of G with respect to S, denoted by $\mathrm{Cay}(G,S)$, is defined as the graph with vertex set G, where two vertices $x, y \in G$ are adjacent if and only if $xy^{-1} \in S$ [10]. Since every vertex in $\mathrm{Cay}(G,S)$ has degree $|S|$, the graph is $|S|$-regular [7].

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The number $|S|$ is called the *valency* of the graph. Because of their symmetry and regularity, Cayley graphs have been widely studied in algebraic graph theory and have applications in network design, coding theory, and computational group theory [3]. Therefore, studying the structural and spectral properties of Cayley graphs over different classes of groups remains an active area of research.

Cayley graphs over finite groups have been widely studied, particularly in terms of their spectral and structural properties. One important class is that of integral Cayley graphs, whose adjacency spectra consist entirely of integers. Integral Cayley graphs have been characterized for several classes of groups and generating sets [10]. For non-abelian groups, this topic has been investigated for dihedral groups [12], generalized quaternion groups Q_{4n} [5], and the group U_{6n} [17]. The spectrum of a Cayley graph can be computed from the complex character values of the underlying group, which also provides information about the number of paths between vertices [6]. For abelian groups, the integrality of Cayley graphs is closely related to generating sets belonging to the Boolean algebra generated by subgroups, leading to a complete characterization for cyclic groups [9]. These studies have contributed to a better understanding of the spectral and structural properties of Cayley graphs over finite groups.

The spectral properties of a Cayley graph are closely related to its structure, which is strongly influenced by the valency. Therefore, understanding the structure of Cayley graphs for different valencies is an important problem. Several studies have focused on Cayley graphs with fixed valency, since the valency affects important graph properties such as connectivity, symmetry, and the spectrum. In [1], the authors characterized Cayley graphs of dihedral groups with valencies 1, 2, and 3. This result was later extended to valency 4 in [15]. A similar study was carried out for generalized quaternion groups, where the structures of Cayley graphs with valencies 1, 2, and 3 were determined [18]. We note that the results in [18] are currently available as a *preprint* and were obtained independently and concurrently with the present work.

Despite these previous results, most existing studies focus only on Cayley graphs of small valencies and do not provide a general description of their structure. In particular, the results for valencies 2 and 3 are obtained using different approaches and cannot be expressed within a common framework. In this paper, we revisit the cases of valencies 2 and 3, extend the analysis to valency 4, and then develop a unified framework for describing Cayley graphs of arbitrary valency. This framework is built systematically from the cases of valencies 1–4, which reveal the structural patterns needed to describe the general case. As a result, the known constructions for small valencies are unified and naturally extended to arbitrary valencies. This approach yields a more comprehensive view of the structural development of Cayley graphs and contributes new insights to the existing literature.

1.2. **Preliminaries.** This section introduces the notation and basic concepts used throughout the paper. Only the definitions and results needed in the subsequent sections are presented, while further details can be found in the cited references.

A *circulant graph*, denoted by $C_n(S)$, is the graph with vertex set $\mathbb{Z}_n$, where two vertices are adjacent if and only if their difference modulo n belongs to the generating set S [11]. Throughout this paper, we also use the notation

$$C_n(s_1, s_2, \ldots, s_k),$$

where $S = \{s_1, s_2, \ldots, s_k\}$ and $1 \leq s_1 < s_2 < \cdots < s_k \leq \left\lfloor \frac{n}{2} \right\rfloor$.

The prism graph is the Cartesian product $C_n \Box K_2$ [2]. In this paper, we introduce a natural generalization by allowing multiple offset edges between the two cycles. We refer to this graph as the *generalized prism graph with multiple offsets*, which is defined as follows.

Definition 1.1. The generalized prism graph with multiple offsets, denoted by $\mathrm{GPr}(n; k_1, k_2, \ldots, k_m)$, consists of two disjoint n-cycles with vertex sets

$$U = \{u_0, u_1, \ldots, u_{n-1}\} \text{ and } V = \{v_0, v_1, \ldots, v_{n-1}\},$$

together with the following edges:

(1) For each $0 \leq i \leq n-1$, the intra-cycle edges are

$$(u_i, u_{(i+1) \bmod n}) \quad \text{and} \quad (v_i, v_{(i+1) \bmod n}).$$

(2) For each $0 \leq i \leq n-1$ and each offset k_j, where $1 \leq j \leq m$, the inter-cycle edges are

$$(u_i, v_{(i+k_j) \bmod n}).$$

The parameters $k_1, k_2, \ldots, k_m$ are distinct integers satisfying $0 \leq k_j < n$. When $m = 1$ and $k_1 = 0$, the graph is isomorphic to the prism graph $C_n \Box K_2$.

We next recall several properties of the generalized quaternion group that will be used throughout the paper. The generalized quaternion group, denoted by Q_{4n}, is defined by

$$Q_{4n} = \langle a, b \mid a^{2n} = 1,\ a^n = b^2,\ b^{-1}ab = a^{-1} \rangle.$$

As shown in [13], Q_{4n} has order $4n$ and is abelian if and only if $n = 1$. Every element of Q_{4n} can be written uniquely in the form $a^i b^j$, where $0 \leq i < 2n$ and $j \in \{0, 1\}$.

The order of an element in Q_{4n} depends on its canonical form. In particular, the order of $a^i b^j$ is given by [13]

$$o(a^i b^j) = \begin{cases} \dfrac{2n}{\gcd(i, 2n)}, & \text{if } j = 0, \\ 4, & \text{if } j = 1. \end{cases}$$

These properties are used throughout the paper to determine the connection sets and to analyze the corresponding Cayley graphs.

2. Structure of Cayley Graph for Valency 1 and 2

This section examines the graph structures obtained from Cayley graphs over the generalized quaternion group for $|S| = 1$ and $|S| = 2$. Based on the fundamental properties regarding the order of each element in the generalized quaternion group, it is clear that the only self-inverse element in Q_{4n} is a^n. Consequently, there exists only one type of graph that can be formed with valency 1, namely when $S = \{a^n\}$.

Theorem 2.1. *Let Q_{4n} be the generalized quaternion group of order $4n$. If $|S| = 1$, then the corresponding Cayley graph is a disjoint union of $2n$ copies of the complete graph on two vertices, that is* $\mathrm{Cay}(Q_{4n}, S) = 2nK_2$.

Proof. Since the only element of order 2 in Q_{4n} is a^n, it follows that $S = \{a^n\}$. Let $H = \{a^n, e\}$, which is a cyclic subgroup of order 2 in Q_{4n}. Because $[Q_{4n} : H] = 2n$, there exist exactly $2n$ distinct right cosets, namely $He, Ha, \ldots, Ha^{n-1}, Hb, Hab, \ldots, Ha^{n-1}b$. Consequently,

$$Q_{4n} = He \cup Ha \cup \ldots \cup Ha^{n-1}b = \{e, a^n\} \cup \{a, a^{n+1}\} \ldots \cup \{a^{n-1}b, a^{2n-1}b\},$$

and thus there are exactly $2n$ edges, namely $e - a^n,\ a - a^{n+1},\ \ldots,\ a^{n-1}b - a^{2n-1}b$, as illustrated in Figure 1. $\square$

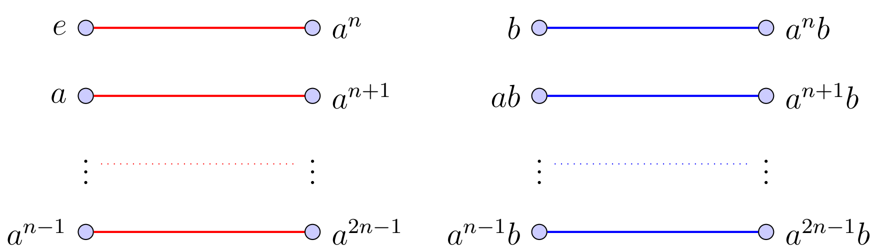

FIGURE 1. Structure of $\mathrm{Cay}(Q_{4n}, \{a^n\})$

Since $S^{-1} = S$ and $e \notin S$, there are two possible choices for the connection set S when the valency is 2. The first is $S = \{a^i, a^{2n-i}\}$, where $i \neq n$, while the second is $S = \{a^ib, (a^ib)^{-1}\} = \{a^ib, a^{n+i}b\}$. We first consider the former case. The following theorem shows that the corresponding Cayley graph is the disjoint union of cycles, where both the number and the length of the cycles are determined by the subgroup generated by a^i.

Theorem 2.2. *Let Q_{4n} be the generalized quaternion group of order $4n$. If $S = \{a^i, a^{2n-i}\}$, where $i \neq n$, then*

$$\mathrm{Cay}(Q_{4n}, S) \cong 2\gcd(i, 2n)\, C_{\frac{2n}{\gcd(i,2n)}}.$$

Proof. Consider the cyclic subgroup of Q_{4n} generated by a^i, where $i \neq n$, denoted by H, namely

$$H = \langle a^i \rangle = \{a^i, a^{2i \pmod{2n}}, a^{3i \pmod{2n}}, \ldots, e\} = \langle a^{\gcd(i,2n)} \rangle.$$

Therefore, the order of H is $\frac{2n}{\gcd(i,2n)}$. From the choice of S, it follows that adjacent vertices form edges of the form $e - a^i$, $a^i - a^{2i \pmod{2n}}$, $a^{2i \pmod{2n}} - a^{3i \pmod{2n}}$, and so on. These edges form a cycle of length $\frac{2n}{\gcd(i,2n)}$, namely

$$e - a^i - a^{2i \pmod{2n}} - a^{3i \pmod{2n}} - \cdots - e.$$

Since H is a cyclic subgroup of order $\frac{2n}{\gcd(i,2n)}$, its index in Q_{4n} is

$$[Q_{4n} : H] = \frac{4n}{\frac{2n}{\gcd(i,2n)}} = 2\gcd(i, 2n).$$

Hence, there are exactly $2\gcd(i, 2n)$ distinct right cosets of H, namely

$$He,\ Hg_1,\ Hg_2,\ \ldots,\ Hg_{2\gcd(i,2n)-1}.$$

Each right coset induces a disjoint cycle of length $\frac{2n}{\gcd(i,2n)}$. Therefore,

$$\mathrm{Cay}(Q_{4n}, S) \cong 2\gcd(i, 2n)\, C_{\frac{2n}{\gcd(i,2n)}}.$$

Figure 2 illustrates this graph. $\square$

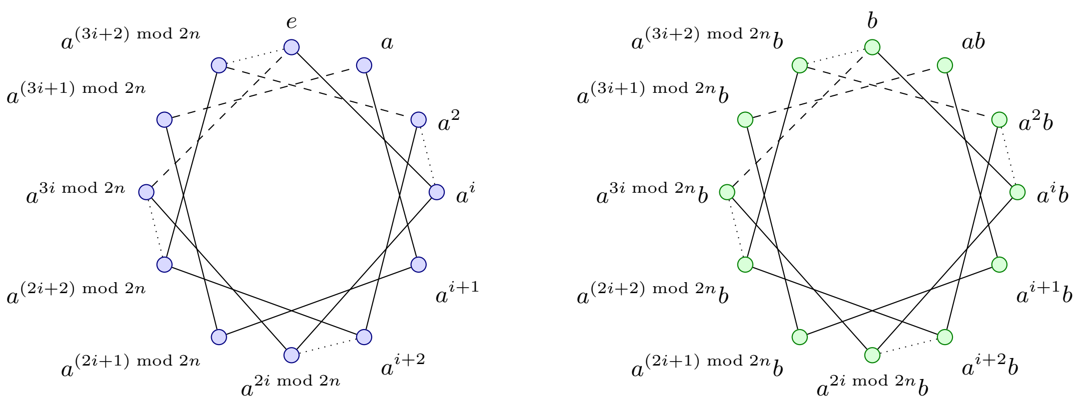

FIGURE 2. Structure of $Cay(Q_{4n}, \{a^i, a^{2n-i}\})$

Next, we consider the second case for valency 2, where the connection set is $S = \{a^ib, a^{n+i}b\}$. Unlike the previous case, where the graph decomposes into disjoint cycles within a subgroup, choosing elements from the coset $\langle a\rangle b$ changes the adjacency relations. Since the product of two elements involving b belongs to the cyclic subgroup $\langle a\rangle$, the vertex set naturally splits into two distinct subsets. The following theorem shows that this property gives rise to a balanced 2-regular bipartite graph.

Theorem 2.3. *Let Q_{4n} be the generalized quaternion group of order $4n$. If $S = \{a^ib, a^{n+i}b\}$, then $\mathrm{Cay}(Q_{4n}, S)$ is a 2-regular bipartite graph.*

Proof. Consider the connection set $S = \{a^ib, a^{n+i}b\}$, where $i \neq n$. Based on this choice of S, we analyze the adjacency of the vertices. For $k \leq i$, each vertex of the form a^k is adjacent to exactly two vertices of the form $a^{i-k}b$ and $a^{n+i-k}b$ because

$$a^k(a^{i-k}b)^{-1} = a^k(a^{n+i-k}b) = a^{n+i}b \in S,$$
$$a^k(a^{n+i-k}b)^{-1} = a^k(a^{i-k}b) = a^ib \in S.$$

Next, for each vertex of the form a^k with $i < k < 2n$, it is adjacent to exactly two vertices of the form $a^{2n-k+i}b$ and $a^{n+i-k}b$ because

$$a^k(a^{2n-k+i}b)^{-1} = a^k(a^{n+i-k}b) = a^{n+i}b \in S,$$
$$a^k(a^{n+i-k}b)^{-1} = a^k(a^{2n+i-k}b) = a^ib \in S.$$

Furthermore, for any $i, j = 0, 1, \ldots, 2n-1$, the group operations satisfy $(a^i)(a^j) = a^{i+j}$ and $(a^ib)(a^jb) = a^{n+i-j}$. This implies that no edge connects two vertices of the form a^i and a^j. Similarly, no edge connects two vertices of the form a^ib and a^jb.

Based on these properties, partitioning the vertex set into

$$E_1 = \{e, a, a^2, \ldots, a^{2n-1}\} \text{ and } E_2 = \{b, ab, a^2b, \ldots, a^{2n-1}b\}$$

yields a bipartite graph. Since every vertex has degree 2, the graph is 2-regular, completing the proof. □

3. Structure of Cayley Graph for Valency 3 and 4

In this section, we analyze the structure of Cayley graphs over the generalized quaternion group for $|S| = 3$ and $|S| = 4$. For the case $|S| = 3$, based on the membership conditions of S, there are two possible structures for the generating set: $S = \{a^n, a^i, a^{2n-i}\}$ with $i \neq n$, and $S = \{a^n, a^ib, a^{n+i}b\}$.

We first examine the structures where $S = \{a^n, a^i, a^{2n-i}\}$ with $i \neq n$. The combination of the cycle structures generated by the inverse pair $\{a^i, a^{2n-i}\}$ and the perfect matching induced by the involution $\{a^n\}$ yields a family of periodic graphs. The interaction between the components generated by these subsets establishes a circulant structure within each cycle. This leads to the formulation of the following theorem.

Theorem 3.1. *Let Q_{4n} be the generalized quaternion group of order $4n$. If $S = \{a^n, a^i, a^{2n-i}\}$ with $i \neq n$, then the Cayley graph $\mathrm{Cay}(Q_{4n}, S)$ is a circulant graph isomorphic to*

$$2\gcd(i, 2n)\, C_{\frac{2n}{\gcd(i,2n)}}\left(1, \frac{n}{\gcd(i, 2n)}\right).$$

Proof. Let $S = \{a^n, a^i, a^{2n-i}\}$. Suppose that $S_1, S_2 \subset S$ with $S_1 = \{a^n\}$ and $S_2 = \{a^i, a^{2n-i}\}$. If the generating set S_1 is chosen, then based on Theorem 2.1, the resulting Cayley graph is $2nK_2$, where each vertex of the form a^k is connected to the vertex a^{n+k}, and each vertex of the form a^kb is connected to the vertex $a^{n+k}b$, for $k = 0, 1, 2, \ldots, n-1$.

Next, for the generating set S_2, by Theorem 2.2, the resulting graph consists of disjoint cycle graphs that depend on the value of i, specifically

$$2\gcd(i, 2n)\, C_{\frac{2n}{\gcd(i,2n)}}.$$

These cycle graphs are formed through the cyclic subgroup generated by the element $a^{\gcd(i,2n)}$. Thus, adding the generator $S_1 = \{a^n\}$ to this structure gives additional edges that connect vertices of the form

$$a^{\frac{k}{\gcd(i,2n)}} \quad \text{to} \quad a^{\frac{n+k}{\gcd(i,2n)}},$$

where $k = l \cdot \gcd(i, 2n)$. A similar connection pattern also holds for the other cycles. Consequently, within the cycle graphs $C_{\frac{2n}{\gcd(i,2n)}}$, the resulting graph forms a circulant graph with jump sizes of 1 and $\frac{n}{\gcd(i,2n)}$, which completes the proof. $\square$

The remaining structural possibility for connection sets of valency 3 arises when $S = \{a^n, a^i b, a^{n+i} b\}$. In this case, the structure of the resulting Cayley graph can be easily understood by observing its subsets.

Theorem 3.2. *Let Q_{4n} be the generalized quaternion group of order $4n$. If the connection set is $S = \{a^n, a^i b, a^{n+i} b\}$, then the Cayley graph $\mathrm{Cay}(Q_{4n}, S)$ is the edge-disjoint union of a 2-regular bipartite graph and $2n$ copies of K_2.*

Proof. Let $S = \{a^n, a^i b, a^{n+i} b\}$. We partition the connection set S into two disjoint subsets: $S_1 = \{a^n\}$ and $S_2 = \{a^i b, a^{n+i} b\}$. By Theorem 2.3, the subset S_2 generates a 2-regular bipartite graph in which each vertex a^k is adjacent to the vertices of the form $a^{i-k} b$ and $a^{n+i-k} b$ for $k = 1, 2, \ldots, 2n$.

On the other hand, by Theorem 2.1, the subset S_1 generates a perfect matching isomorphic to $2nK_2$, wherein each vertex a^k is adjacent to a^{n+k}, and each vertex $a^k b$ is adjacent to $a^{n+k} b$ for $k = 1, 2, \ldots, n$.

Since these two subgraphs are defined over the same vertex set but are formed by strictly disjoint edge sets, the resulting Cayley graph $\mathrm{Cay}(Q_{4n}, S)$ is precisely the edge-disjoint union of a 2-regular bipartite graph and $2n$ copies of the complete graph K_2. This completes the proof. $\square$

Next, we discuss the case of valency 4. For this valency, there are three possible structural configurations for the connection set S:

(1) All elements of S belong to the cyclic subgroup $\langle a \rangle$, namely
$$S = \{a^i, a^{2n-i}, a^j, a^{2n-j}\},$$
under the condition that $i \neq j$ and $i, j \neq n$.

(2) All elements of S are contained within the coset $\langle a \rangle b$, namely
$$S = \{a^i b, a^{n+i} b, a^j b, a^{n+j} b\},$$
where $i \neq j$.

(3) The connection set S consists of a combination of elements from both $\langle a \rangle$ and $\langle a \rangle b$, namely
$$S = \{a^i, a^{2n-i}, a^j b, a^{n+j} b\},$$
with the condition that $i \neq n$.

First, consider the configuration where $S = \{a^i, a^{2n-i}, a^j, a^{2n-j}\}$ with $i, j \neq n$. By Theorem 2.2, the subsets $\{a^i, a^{2n-i}\}$ and $\{a^j, a^{2n-j}\}$ generate disjoint cycle graphs of lengths $\frac{2n}{\gcd(i,2n)}$ and $\frac{2n}{\gcd(j,2n)}$, respectively. If $\gcd(i, 2n) \neq \gcd(j, 2n)$, these 2-regular subgraphs span the same vertex set, and their edge-disjoint union forms a circulant graph, as given in the following theorem.

Theorem 3.3. *Let Q_{4n} be the generalized quaternion group of order $4n$. If*
$$S = \{a^i, a^{2n-i}, a^j, a^{2n-j}\},$$

with $i, j \neq n$ *and* $\gcd(i, 2n) < \gcd(j, 2n)$*, then the Cayley graph generated by* S *is a circulant graph of the form*

$$2 \gcd(i, 2n)\ C_{\frac{2n}{\gcd(i,2n)}}\big(1, \gcd(j, 2n)\big).$$

Proof. Let $S = \{a^i, a^{2n-i}, a^j, a^{2n-j}\}$ with $i, j \neq n$ and $\gcd(i, 2n) < \gcd(j, 2n)$. Suppose S is partitioned into two disjoint subsets, $S_1, S_2 \subset S$, such that

$$S_1 = \{a^i, a^{2n-i}\} \quad \text{and} \quad S_2 = \{a^j, a^{2n-j}\}.$$

By Theorem 2.2, the Cayley subgraphs generated by S_1 and S_2 are given respectively by

$$\mathrm{Cay}(Q_{4n}, S_1) \cong 2 \gcd(i, 2n)\ C_{\frac{2n}{\gcd(i,2n)}} \quad \text{and} \quad \mathrm{Cay}(Q_{4n}, S_2) \cong 2 \gcd(j, 2n)\ C_{\frac{2n}{\gcd(j,2n)}}.$$

Given that $\gcd(i, 2n) < \gcd(j, 2n)$ and both subgraphs are defined over the identical vertex set, the overall Cayley graph is obtained via the edge-disjoint union of these two cycle structures. Specifically, on each cycle $C_{\frac{2n}{\gcd(i,2n)}}$ generated by S_1, the additional edges contributed by S_2 correspond to jump sizes of $\gcd(j, 2n)$. Hence, the resulting graph forms a circulant graph of the form

$$2 \gcd(i, 2n)\ C_{\frac{2n}{\gcd(i,2n)}}\big(1, \gcd(j, 2n)\big),$$

which completes the proof. □

Remark 3.4. In Theorem 3.3, if $\gcd(i, 2n) = \gcd(j, 2n)$, the resulting graph reduces to a union of cycle graphs, as described in Theorem 2.2.

The next case involves the connection set

$$S = \{a^i b, a^{n+i} b, a^j b, a^{n+j} b\},$$

with $i \neq j$. According to Theorem 2.3, a generating set of this form yields a 2-regular bipartite graph, since each vertex a^k is adjacent to exactly two vertices of the form $a^l b$. This leads to the following theorem.

Theorem 3.5. *Let* Q_{4n} *be the generalized quaternion group of order* $4n$*. If*

$$S = \{a^i b, a^{n+i} b, a^j b, a^{n+j} b\}$$

with $i \neq j$*, then the Cayley graph* $\mathrm{Cay}(Q_{4n}, S)$ *is a* 4*-regular bipartite graph.*

Proof. Let $S = \{a^i b, a^{n+i} b, a^j b, a^{n+j} b\}$ with $i \neq j$. Suppose $S_1, S_2 \subset S$ where

$$S_1 = \{a^i b, a^{n+i} b\} \quad \text{and} \quad S_2 = \{a^j b, a^{n+j} b\}.$$

According to Theorem 2.2, the Cayley subgraphs generated by S_1 and S_2 are both 2-regular bipartite graphs.

Since $i \neq j$, each vertex a^k in the graph is adjacent to exactly two vertices of the form $a^\ell b$ from S_1 and two vertices of the form $a^m b$ from S_2, along with their respective inverse connections. Clearly, the vertex set can be partitioned into two disjoint subsets:

$$E_1 = \{a^k \mid k = 0, 1, 2, \ldots, 2n-1\} \quad \text{and} \quad E_2 = \{a^k b \mid k = 0, 1, 2, \ldots, 2n-1\},$$

where no edges exist within E_1 or within E_2. Since every vertex in E_1 is adjacent to exactly four distinct vertices in E_2, and vice versa, the resulting Cayley graph $\mathrm{Cay}(Q_{4n}, S)$ is a 4-regular bipartite graph. □

Remark 3.6. In Theorem 3.5, if $i = j$, the resulting graph reduces to a 2-regular bipartite graph, as stated in Theorem 2.3.

The final configuration for valency 4 occurs when the connection set contains elements from both the cyclic subgroup $\langle a\rangle$ and the coset $\langle a\rangle b$, given by

$$S = \{a^i, a^{2n-i}, a^j b, a^{n+j} b\},$$

with $i \neq n$. By Theorems 2.2 and 2.3, combining these subsets yields a generalized prism graph, as formalized in the following theorem.

Theorem 3.7. *Let Q_{4n} be a generalized quaternion group of order $4n$. If $S = \{a^i, a^{2n-i}, a^j b, a^{n+j} b\}$ with $i \neq n$, then*

$$\mathrm{Cay}(Q_{4n}, S) = \gcd(i, 2n) \cdot \mathrm{GPr}\left(\frac{2n}{\gcd(i, 2n)}; 0, \frac{n}{\gcd(i, 2n)}\right).$$

Proof. Let $S = \{a^i, a^{2n-i}, a^j b, a^{n+j} b\}$ with $i \neq n$, and partition S into two disjoint subsets: $S_1 = \{a^i, a^{2n-i}\}$ and $S_2 = \{a^j b, a^{n+j} b\}$. By setting $\gcd(i, 2n) = k$, Theorem 2.2 tell us that the subset S_1 generates a cycle of length $\frac{2n}{k}$ within the subgroup $\langle a\rangle$, given by

$$e - a^k - a^{2k} - \cdots - a^{2n-k} - e.$$

Simultaneously, a corresponding cycle structure of identical length is formed within the coset $\langle a\rangle b$, which can be expressed as

$$a^j b - a^{j+k} b - a^{j+2k} b - \cdots - a^{j+2n-k} b - a^j b.$$

Furthermore, the subset S_2 defines the edges between these two cycles. Specifically, each element a^{lk} is connected to an element of the form $a^{j+2n-lk} b$ for $l = 1, 2, \ldots, \frac{2n}{k}$. This adjacency relation is verified by the group operation:

$$a^{lk} \left(a^{j+2n-lk} b\right)^{-1} = a^{lk} a^{j+n-lk} b = a^{n+j} b \in S.$$

Moreover, each element a^{lk} is also adjacent to the element $a^{j+n-lk} b$, since

$$a^{lk} \left(a^{j+n-lk} b\right)^{-1} = a^{lk} a^{j-lk} b = a^j b \in S.$$

The structure of the resulting Cayley graph is illustrated in Figure 3. Consequently, the graph forms a generalized prism graph with no step size and a jump size of $\frac{n}{k}$. Since the graph consists of exactly k isomorphic components, the complete Cayley graph generated by S is a collection of k copies of the generalized prism graph, namely

$$\gcd(i, 2n) \cdot \mathrm{GPr}\left(\frac{2n}{\gcd(i, 2n)}; 0, \frac{n}{\gcd(i, 2n)}\right).$$

This completes the proof. □

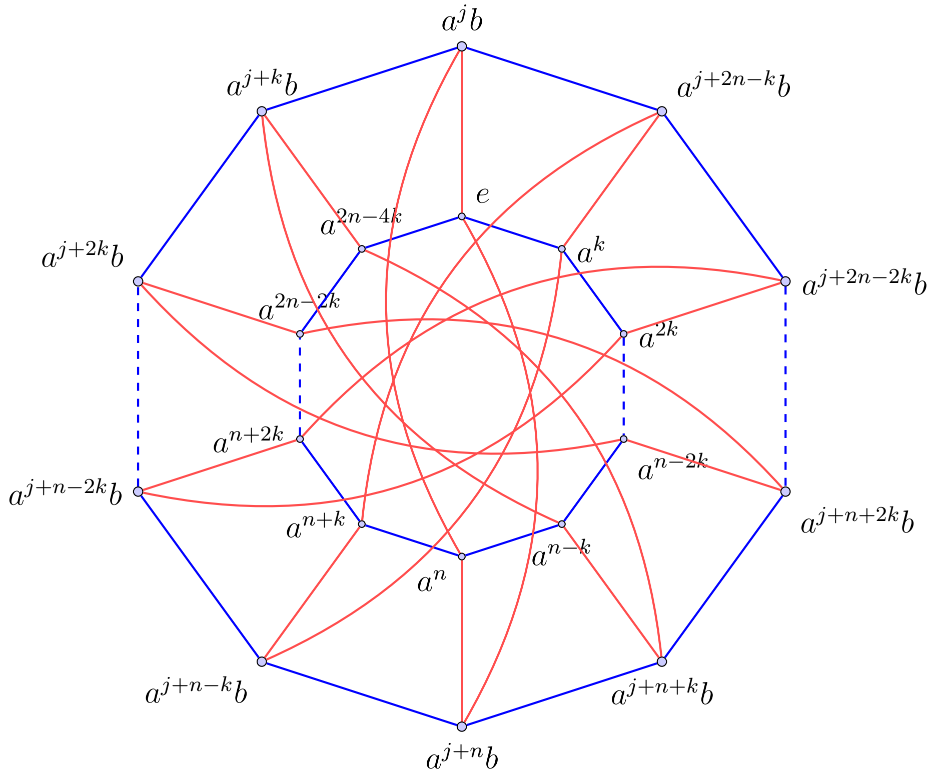

FIGURE 3. The structure of $\mathrm{Cay}(Q_{4n}, S)$ for $S = \{a^i, a^{2n-i}, a^j b, a^{n+j}b\}$.

4. STRUCTURE OF CAYLEY GRAPH FOR ANY VALENCY

In this section, we investigate the Cayley graphs constructed for an arbitrary valency m. We begin by analyzing the case where m is an even integer. Under this condition, there are three possible configurations for the connection set S: (1) S consists entirely of elements of the form a^i, (2) S consists entirely of elements of the form $a^k b$, and (3) S contains a combination of elements of both forms.

The first configuration occurs when the connection set S consists exclusively of elements of the form a^i, namely

$$S = \{a^{i_1}, (a^{i_1})^{-1}, a^{i_2}, (a^{i_2})^{-1}, \dots, a^{i_k}, (a^{i_k})^{-1}\}.$$

In this case, it is clear that $e, a^n \notin S$. Furthermore, as established in the valency-4 case, to ensure that the generated cycles have distinct lengths, we require that

$$\gcd(i_a, 2n) \neq \gcd(i_b, 2n)$$

for all distinct $a, b \in \{1, 2, \dots, k\}$. Under this requirement, the maximum number of pairs k is bounded by the number of positive divisors of $2n$, excluding the divisors corresponding to the identity element e and the central involution a^n. Consequently, the maximum value of k is given by

$$k = \tau(2n) - 2,$$

where $\tau(2n)$ denotes the total number of positive divisors of $2n$.

By Theorem 2.2, each pair $\{a^{i_l}, (a^{i_l})^{-1}\}$ generates a cycle of length
$$\frac{2n}{\gcd(i_l, 2n)}.$$
Assuming without loss of generality that
$$\gcd(i_1, 2n) < \gcd(i_2, 2n) < \cdots < \gcd(i_k, 2n),$$
we obtain the following theorem.

Theorem 4.1. *Let Q_{4n} be a generalized quaternion group of order $4n$. If*
$$S = \{a^{i_1}, (a^{i_1})^{-1}, a^{i_2}, (a^{i_2})^{-1}, \dots, a^{i_k}, (a^{i_k})^{-1}\},$$
with $i_l \neq n$ for every l, and
$$\gcd(i_1, 2n) < \gcd(i_2, 2n) < \cdots < \gcd(i_k, 2n),$$
then the resulting Cayley graph is a circulant graph of the form
$$\mathrm{Cay}(Q_{4n}, S) = 2\gcd(i_1, 2n)\, C_{\frac{2n}{\gcd(i_1,2n)}} \left(1, \gcd(i_2, 2n), \gcd(i_3, 2n), \dots, \gcd(i_k, 2n)\right).$$

Proof. The proof follows by extending the arguments of Theorem 3.3 to k pairs of generators of the form a^i. □

Remark 4.2. Since the maximum length of the cycles generated in Theorem 4.1 is $\frac{2n}{\gcd(i_1,2n)}$, the maximum allowable jump size on these cycles is $\frac{n}{\gcd(i_1,2n)}$. It then follows that
$$\gcd(i_k, 2n) < \frac{n}{\gcd(i_1, 2n)}.$$

The second configuration covers the case where the connection set S consists elements of the form $a^i b$. Specifically, we consider a connection set containing $k+1$ pairs of such elements, expressed as
$$S = \{b, b^{-1}, ab, (ab)^{-1}, \dots, a^k b, (a^k b)^{-1}\},$$
where $0 \leq k \leq n-1$. By Theorem 2.3, each pair $\{a^i b, (a^i b)^{-1}\}$ in S generates a 2-regular bipartite subgraph. This structural property leads to the following generalization.

Theorem 4.3. *Let Q_{4n} be the generalized quaternion group of order $4n$. If*
$$S = \{b, b^{-1}, ab, (ab)^{-1}, \dots, a^k b, (a^k b)^{-1}\},$$
with $0 \leq k \leq n-1$, then the resulting Cayley graph $\mathrm{Cay}(Q_{4n}, S)$ is a $2(k+1)$-regular bipartite graph. Moreover, if $k = n-1$, then the graph is the complete bipartite graph $K_{2n,2n}$.

Proof. The proof proceeds by extending the arguments applied in Theorem 3.5. Each distinct pair $\{a^i b, (a^i b)^{-1}\} \subset S$ ensures that every vertex of the form a^m is adjacent to exactly two vertices within the coset $\langle a \rangle b$. The vertex set of $\mathrm{Cay}(Q_{4n}, S)$ can be partitioned into two disjoint subsets:
$$E_1 = \{a^m \mid m = 0, 1, 2, \dots, 2n-1\} \quad \text{and} \quad E_2 = \{a^m b \mid m = 0, 1, 2, \dots, 2n-1\}.$$
Since all elements in S belong strictly to the coset $\langle a \rangle b$, no edges exist within E_1 or within E_2. Given that S contains exactly $k+1$ pairs of generators, every

vertex in E_1 is adjacent to exactly $2(k+1)$ distinct vertices in E_2, and vice versa. Thus, the resulting graph is a $2(k+1)$-regular bipartite graph.

Furthermore, when $k = n-1$, the connection set S contains all $2n$ elements of the coset $\langle a \rangle b$. Under this condition, every vertex in E_1 is adjacent to every vertex in E_2, which implies that the graph is the complete bipartite graph $K_{2n,2n}$. □

The final configuration for the connection set S with an even number of elements consists of a combination elements from the cyclic subgroup $\langle a \rangle$ and the coset $\langle a \rangle b$. To ensure distinct cycle lengths, the elements selected from $\langle a \rangle$ must satisfy the conditions defined in Theorem 4.1. Under these constraints, the connection set S is expressed as

$$S = \{a^{i_1}, (a^{i_1})^{-1}, a^{i_2}, (a^{i_2})^{-1}, \ldots, a^{i_k}, (a^{i_k})^{-1}, b, b^{-1}, ab, (ab)^{-1}, \ldots, a^{n-1}b, (a^{n-1}b)^{-1}\}.$$

By Theorem 4.1 and Theorem 4.3, the generators from $\langle a \rangle$ determine a circulant subgraph, while the generators from $\langle a \rangle b$ determine a bipartite subgraph. An exact closed-form description for this combined structure, analogous to the one established in Theorem 3.7, remains an open problem. However, the resulting Cayley graph can be characterized as the edge-disjoint union of the two spanning subgraphs obtained in Theorem 4.1 and Theorem 4.3.

Corollary 4.4. *Let Q_{4n} be the generalized quaternion group of order $4n$. If*

$$S = \{a^{i_1}, (a^{i_1})^{-1}, a^{i_2}, (a^{i_2})^{-1}, \ldots, a^{i_k}, (a^{i_k})^{-1}, b, b^{-1}, ab, (ab)^{-1}, \ldots, a^{n-1}b, (a^{n-1}b)^{-1}\},$$

with $i_l \neq n$ for every l, and

$$\gcd(i_1, 2n) < \gcd(i_2, 2n) < \cdots < \gcd(i_k, 2n),$$

then the resulting Cayley graph $\mathrm{Cay}(Q_{4n}, S)$ is the edge-disjoint union of the circulant graph

$$2\gcd(i_1, 2n)\, C_{\frac{2n}{\gcd(i_1,2n)}}\, (1, \gcd(i_2, 2n), \gcd(i_3, 2n), \ldots, \gcd(i_k, 2n))$$

and the complete bipartite graph $K_{2n,2n}$.

We next consider the Cayley graphs obtained when the connection set S has an odd cardinality. These configurations are constructed by adding the element a^n to each of the three even-cardinality cases discussed previously. Consequently, the first configuration arises when S consists of elements of the form a^i (where $i \neq n$) with the addition of a^n. The second configuration occurs when S consists of elements of the form $a^i b$ along with the additional element a^n. Finally, the third configuration occurs when S contains a combination of elements of both forms, with the additional element a^n included.

We begin by analyzing the case where S consists of elements of the form a^i together with a^n, where $i \neq n$, namely

$$S = \{a^{i_1}, (a^{i_1})^{-1}, a^{i_2}, (a^{i_2})^{-1}, \ldots, a^{i_k}, (a^{i_k})^{-1}, a^n\}.$$

By Theorem 4.1, the elements of S of the form a^i with $i \neq n$ generate a circulant subgraph. Meanwhile, by Theorem 2.1, the additional element a^n induces a 1-regular spanning subgraph isomorphic to $2nK_2$, where each vertex a^m is adjacent to a^{n+m} and each vertex $a^m b$ is adjacent to $a^{n+m}b$ for all $0 \leq m \leq 2n-1$. This yields the following result.

Corollary 4.5. *Let Q_{4n} be the generalized quaternion group of order $4n$. If*
$$S = \{a^{i_1}, (a^{i_1})^{-1}, a^{i_2}, (a^{i_2})^{-1}, \dots, a^{i_k}, (a^{i_k})^{-1}, a^n\},$$
where $i_l \neq n$ for all l, and
$$\gcd(i_1, 2n) < \gcd(i_2, 2n) < \cdots < \gcd(i_k, 2n),$$
then the resulting Cayley graph is a circulant graph of the form
$$2\gcd(i_1, 2n)\, C_{\frac{2n}{\gcd(i_1,2n)}} \left(1, \gcd(i_2, 2n), \gcd(i_3, 2n), \dots, \gcd(i_k, 2n), \frac{n}{\gcd(i_1, 2n)}\right).$$

Proof. The inclusion of a^n in the set S introduces additional edges that connect a^i to a^{n+i} and $a^i b$ to $a^{n+i}b$. Within the cycle of length $\frac{2n}{\gcd(i_1,2n)}$ described in Theorem 4.1, this implies that any elements a^k is connected to the elements $a^{k+\frac{2n}{\gcd(i_1,2n)}}$, and similarly, any elements $a^k b$ is connected to the elements $a^{k+\frac{2n}{\gcd(i_1,2n)}} b$. □

The second configuration combines elements of the form $a^i b$ with the element a^n. According to Theorem 4.3, selecting a connection set S consisting exclusively of elements of the form $a^i b$ yields a $2(k+1)$-regular bipartite spanning subgraph. Since the inclusion of the element a^n introduces an additional set of edges connecting a^m to a^{n+m} and $a^m b$ to $a^{n+m} b$ for all $0 \leq m \leq 2n-1$, this selection of S leads to the following result.

Corollary 4.6. *Let Q_{4n} be the generalized quaternion group of order $4n$. If*
$$S = \{b, b^{-1}, ab, (ab)^{-1}, \dots, a^k b, (a^k b)^{-1}, a^n\},$$
with $0 \leq k \leq n-1$, then the resulting Cayley graph $\mathrm{Cay}(Q_{4n}, S)$ is the edge-disjoint union of a $2(k+1)$-regular bipartite graph and the 1-regular graph $2nK_2$. Furthermore, if $k = n-1$, the resulting graph is the edge-disjoint union of the complete bipartite graph $K_{2n,2n}$ and $2nK_2$.

The final configuration occurs when the connection set S contains a combination of elements of the forms a^i, $a^j b$, and a^n. The resulting structure is characterized as the edge-disjoint union of the subgraphs described in Corollary 4.5 and Theorem 4.3.

Corollary 4.7. *Let Q_{4n} be the generalized quaternion group of order $4n$. Define*
$$S_1 = \{a^{i_1}, (a^{i_1})^{-1}, a^{i_2}, (a^{i_2})^{-1}, \dots, a^{i_k}, (a^{i_k})^{-1}, a^n\},$$
$$S_2 = \{b, b^{-1}, ab, (ab)^{-1}, \dots, a^{n-1}b, (a^{n-1}b)^{-1}\},$$
where $i_l \neq n$ for all l, and
$$\gcd(i_1, 2n) < \gcd(i_2, 2n) < \cdots < \gcd(i_k, 2n).$$
If $S = S_1 \cup S_2$, then the resulting Cayley graph $\mathrm{Cay}(Q_{4n}, S)$ is the edge-disjoint union of the circulant graph
$$2\gcd(i_1, 2n)\, C_{\frac{2n}{\gcd(i_1,2n)}} \left(1, \gcd(i_2, 2n), \gcd(i_3, 2n), \dots, \gcd(i_k, 2n), \frac{n}{\gcd(i_1, 2n)}\right)$$
and the complete bipartite graph $K_{2n,2n}$.

5. Conclusion

The structural properties of the Cayley graph constructed on the generalized quaternion group Q_{4n} are determined by the configuration of the connection set S. Specifically, generators of the form a^i, where $i \neq n$, induce a circulant spanning subgraph, while the element a^n induces a subgraph isomorphic to $2nK_2$. In contrast, elements of the coset $\langle a \rangle b$ generate a regular bipartite spanning subgraph. Furthermore, when the connection set S consists of a combination of these different types of generators, the resulting Cayley graph is precisely characterized as the edge-disjoint union of the spanning subgraphs corresponding to each individual type.

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