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MULTIVARIATE EXTREME CVAR BASED ON THE D-NORM

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Abstract. Measuring systemic risk is challenging in the presence of extreme events and tail dependence. This paper proposes an extension of Conditional Value-at-Risk based on Extreme Value Theory and D-norms. A multivariate excess aggregation variable is introduced through a D-norm, and its convergence to a Generalized Pareto Distribution is established. This framework preserves dependence structures and provides an analytical decomposition of systemic risk. Finally, a simulation study illustrates the theoretical results and the practical implementation of the proposed approach.

Keywords. Systemic risk, Conditional Value-at-Risk, D-norm, tail dependence, regularly varing.
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1. Introduction and preliminaries

1.1. **Introduction.** Over the past decades, and particularly since the 2008 global financial crisis, understanding systemic risk has become a central issue in quantitative finance and prudential regulation. Recent crises have demonstrated that extreme losses do not arise solely from the individual vulnerability of financial institutions, but rather from the dependence and contagion mechanisms linking the various components of the financial system.
Although Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR), introduced through optimization-based formulations in the early 2000 ([15]; [12]), have become standard tools in risk management, they remain inadequate for modeling multivariate extreme events. During periods of financial distress, linear correlations are no longer sufficient to describe the dependence structures observed in practice. Alternative approaches, such as CoVaR proposed by [3] and the Marginal Expected Shortfall (MES) introduced by [2], incorporate conditional dimensions of risk; however, their robustness tends to deteriorate in the presence of structural breaks characterizing extreme regimes.
In this context, Extreme Value Theory (EVT) provides a particularly suitable

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framework for the analysis of rare events. The work of [13] on multivariate regular variation and max-stable distributions offers powerful tools for modeling asymptotic joint tail behavior, while the D-norms developed by [8] provide a geometric characterization of tail dependence. More, recent contribution published in the Gulf Journal of Mathematics, including those of [6] and [9] , further illustrate the growing interest in Etreme Value Theory and its applications to the modeling of rare events.
Despite these advances, an important challenge remains: constructing a systemic risk measure that is simultaneously coherent, sensitive to tail dependence, and applicable in high-dimensional settings.
To address this challenge, we propose an extension of CVaR to the multivariate extreme-value framework through an aggregation variable based on threshold exceedances and constructed using a D-norm. We define a global systemic risk measure through Pflug variational formulation of CVaR [12] by introducing a novel exceedance aggregation variable whose convergence toward a Generalized Pareto Distribution (GPD) is established. This dimension-reduction mechanism makes it possible to extend the variational formulation of CVaR to a multivariate extreme setting.
Finally, we develop an exact decomposition methodology for this systemic risk measure, thereby providing a practical supervisory tool for regulators seeking to identify the marginal contributions to aggregate systemic risk. A simulation study is conducted to illustrate the theoretical results established in this paper.

1.2. **Preliminaries.** Before presenting our approach to extreme systemic risk, we briefly recall multivariate regular variation and the risk measures Value-at-Risk and Conditional Value-at-Risk, which form the basis of our framework.

Definition 1.1. Following [5], a sequence of positive Radon measures $\{\mu_n, n \in \mathbb{N}\}$ is said to converge vaguely to a Radon measure μ, denote by $\mu_n \xrightarrow{v} \mu$, if, for every continuous function $f : \mathbb{E} \longrightarrow \mathbb{R}_+$ with compact support,

$$\int_{\mathbb{E}} f d\mu_n \longrightarrow \int_{\mathbb{E}} f d\mu. \tag{1.1}$$

The following theorem adaptats Theorem 2.3 in [10] to the space $\mathbb{R}^d$.

Theorem 1.2. *Let $\mathbb{R}^d \setminus \{\mathbf{0}\}$ and $]0, \infty[$ be equipped with their respective Borel σ-fields $\mathcal{B}$ and $\mathcal{B}'$.*
Let $h : \mathbb{R}^d \setminus \{\mathbf{0}\} \longrightarrow]0, \infty[$ be a measurable mapping such that $h^{-1}(A')$ is bounded away from 0, for every set $A' \in \mathcal{B}' \cap h\big(\mathbb{R}^d \setminus \{\mathbf{0}\}\big)$ bounded away from 0.
Then, the mapping $\hat{h} : M_+\big(\mathbb{R}^d \setminus \{\mathbf{0}\}\big) \longrightarrow M_+(]0, \infty[)$ defined by $\hat{h}(\nu) = \nu \circ h^{-1}$ is continuous with respect to the topology of vague convergence at μ, provided that $\mu(D_h) = 0$, where D_h denotes the set of discontinuity points of h.
In particular, if $\{\mu_n, n \in \mathbb{N}\}$ is a sequence of measures in $M_+(\mathbb{R}^d \setminus \{\mathbf{0}\})$, then

$$\mu_n \xrightarrow{v} \mu \Longrightarrow \mu_n \circ h^{-1} \xrightarrow{v} \mu \circ h^{-1} \tag{1.2}$$

This result allows the transfer of multivariate convergence properties to a univariate framework through the D-norm.
The following theorem is taken from [11].

Theorem 1.3. *The distribution function F of a random variable X is regularly varying with index $\rho > 0$ ($F \in VR_\rho$) if and only if there exists a scale function $\theta(u)$ such that*

$$\lim_{u\to\infty} \sup_x \left|F_u(x) - G_{\rho,\theta(u)}(x)\right| = 0, \qquad \theta(u) = \rho u \quad \textit{(in the standard case)} \tag{1.3}$$

The following definition is taken from [13].

Definition 1.4. Let $\mathbf{X} = (X_1, \dots, X_d) \in \mathbb{R}^d \backslash \{\mathbf{0}\}$. The random vector $\mathbf{X}$ is said to be multivariate regularly varying with index $\gamma > 0$ if there exist a scaling function $b(t) \to \infty$ and a non-null Radon measure μ on $\mathbb{R}^d \backslash \{\mathbf{0}\}$ such that, for every Borel set $\mathbf{B} \subset \mathbb{R}^d \backslash \{\mathbf{0}\}$ bounded away from the origin,

$$t\mathbb{P}\left(\frac{\mathbf{X}}{b(t)} \in \mathbf{B}\right) \xrightarrow[t\to\infty]{v} \mu(\mathbf{B}), \tag{1.4}$$

where the limit measure μ satisfies the homogeneity property

$$\mu(t\mathbf{B}) = t^{-\gamma}\mu(\mathbf{B}), \qquad \forall t > 0 \tag{1.5}$$

In the univariate case, this reduces to the classical tail behavior

$$\lim_{t\longrightarrow\infty} t\mathbb{P}\left(\frac{X}{b(t)} > x\right) = \mu(x, \infty) = \lambda x^{-\gamma}, \ \ , \forall x > 0. \tag{1.6}$$

Definition 1.5. Following [4], a risk measure ρ is a mapping defined on the set of random variables and taking values in $\mathbb{R}$.

Following [4], ρ is said to be coherent if it satisfies the following properties: Translation invariance, positive homogeneity, monotonicity, and subadditivity. Moreover, ρ is convex if, for any random variables Y_1 and Y_2 and any $\lambda \in]0, 1[$,

$$\rho\big(\lambda Y_1 + (1 - \lambda)Y_2\big) \le \lambda\rho(Y_1) + (1 - \lambda)\rho(Y_2) \tag{1.7}$$

In this work, we focus on the two risk measures most relevant to our framework.

Definition 1.6. Following [12], the Value-at-Risk at level α is defined as the α-quantile of a continuous loss variable X:

$$VaR_\alpha(X) = inf\{x \in \mathbb{R} : \mathbb{P}\big(X \le x\big) \ge \alpha\} \tag{1.8}$$

The Conditional Value-at-Risk, a coherent risk measure, is defined by

$$CVaR_\alpha(X) = \mathbb{E}\big(X|X \ge VaR_\alpha(X)\big) \tag{1.9}$$

and admits the equivalent variational representation

$$CVaR_\alpha(X) = \inf_{a\in\mathbb{R}} \left\{a + \frac{1}{1-\alpha}\mathbb{E}\big[(X - a)_+\big]\right\}. \tag{1.10}$$

We also recall the following result, known as Euler's theorem.

Theorem 1.7. *Following [16], if $f : U \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ is continuously differentiable function of degree $k > 0$, then*

$$kf(x_1, ..., x_d) = \sum_{i=1}^{d} x_i \frac{\partial f}{\partial x_i}, \ \ \forall x_1, ..., x_d \in U \tag{1.11}$$

In this work, Euler's theorem is used to obtain an additive decomposition of the proposed risk measure into marginal contributions.
The preceding concepts provide the theoretical foundation for introducing our measure of extreme systemic risk.

2. Main results

We first define an aggregated multivariate exceedance variable based on a D-norm, which will serve as the foundation for the construction of our risk measure. Let $\mathbf{X} = (X_1, \dots, X_d)$ be a vector of losses associated with a financial system, and let $\mathbf{u} = (u_1, \dots, u_d)$ denote a vector of high thresholds. We define the aggregated exceedance variable as follows:

$$Y = \|(\mathbf{X} - \mathbf{u})_+\|_D \tag{2.1}$$

where $(\mathbf{X} - \mathbf{u}, \mathbf{0})_+ = \big(max(X_1 - u_1, 0), ..., max(X_d - u_d, 0)\big)$ denotes the vector of exceedances.
Throughout this work, $\|\cdot\|_D$ denotes a D-norm induced by a max-stable distribution and characterizing the extreme dependence structure among the components. Vectors will be denoted in boldface throughout the paper.
The aggregated variable Y provides a one-dimensional representation of multivariate exceedances. This aggregation enables the introduction of the risk measures associated with the proposed framework. We define Value-at-risk assciated with Y.

Definition 2.1. The Value-at-Risk of the aggregated exceedance variable Y at level $\alpha \in]0, 1[$ is given by

$$\mathrm{VaR}^D_\alpha(Y) = \inf\big\{y \in \mathbb{R} \mid \mathbb{P}(Y \leq y) \geq \alpha\big\} \tag{2.2}$$

Building upon this aggregated variable, we define our systemic risk measure as follows.

Theorem 2.2. *Let Y be an aggregated variable constructed through a D-norm induced by a max-stable distribution. Then*

$$\mathrm{CVaR}^D_\alpha(Y) = \inf_{t\in\mathbb{R}} \left\{ t + \frac{1}{1-\alpha}\, \mathbb{E}\big[(Y - t)_+\big] \right\} \tag{2.3}$$

defines a coherent and convex risk measure in the sense of Artzner and al.

Proof. The proof is obtained by adapting the standard CVaR arguments to the aggregated variable Y. It relies on the variational representation of CVaR given by [12] and the coherent risk measure framework of [4]. The details are omitted. □

The following conditional and integral representations are direct consequences of the previous formulation and correspond to the classical properties of CVaR applied to the aggregated variable Y.

Corollary 2.3. *Let Y be a continuous aggregated loss variable such that $\mathbb{E}(Y) < \infty$ and $\alpha \in]0, 1[$. Then:*

(1)

$$\mathrm{CVaR}^D_\alpha(Y) = \mathbb{E}\big[Y \,\big|\, Y > \mathrm{VaR}^D_\alpha(Y)\big] \tag{2.4}$$

(2)

$$\mathrm{CVaR}_{\alpha}^{D}(Y) = \frac{1}{1-\alpha}\int_{\alpha}^{1} \mathrm{VaR}_{\alpha}^{D}(Y)du \tag{2.5}$$

Proof. See [12] for the proof. □

The evaluation of $\mathrm{CVaR}_{\alpha}^{D}$ depends entirely on the conditional tail behavior of Y. Therefore, the investigation of $\mathrm{CVaR}_{\alpha}^{D}$ in the extreme-value framework amounts to characterizing the limiting distribution of the excesses of Y.

Theorem 2.4. *Let $\mathbf{X} \in \mathbb{R}_{+}^{d}$ be a loss vector exhibiting multivariate regular variation with index $\gamma > 0$ and limit measure ν. Let $||\cdot||_D$ denote a D-norm associated with a spectral measure μ supported on the unit simplex $S_d = \{\boldsymbol{z} \in \mathbb{R}_{+}^{d} : ||\boldsymbol{z}||_D = 1\}$ and Y be the systemic aggregation variable .*
As $u \to \infty$, the conditional excess distribution of Y converges to a Generalized Pareto Distribution (GPD) with tail index $\dfrac{1}{\gamma}$.

Proof. By assumption, $\mathbf{X}$ exhibits multivariate regular variation. Consequently, there exist a scaling function $b(n)$ and a Radon measure ν such that the sequence of measures $\nu_n(\cdot)$ is defined by

$$\nu_n(.) = n\mathbb{P}\big(b(n)^{-1}\mathbf{X} \in \cdot\big) \xrightarrow[t\to\infty]{vague} \nu.$$

Define

$$\mathbf{Z} = (\mathbf{X}\text{-}\mathbf{u})_{+}, \tag{2.6}$$

$$r = ||\mathbf{Z}||_D \tag{2.7}$$

and

$$\theta = \frac{\mathbf{Z}}{||\mathbf{Z}||_D} \tag{2.8}$$

Let $T : \mathbf{z} \mapsto (r, \theta)$ be the pseudo-polar transformation. Under this representation, the intensity measure ν admits the unique decomposition following [13]

$$\nu\big(dr, d\theta\big) = \gamma r^{-\gamma-1} dr\, \mu(d\theta),$$

where μ denotes the spectral measure on S_d, which encodes the tail dependence structure.
Let $O = \mathbb{R}_{+}^{d} \setminus \{\mathbf{0}\}$ and $O' = \mathbb{R}_{+} \setminus \{\mathbf{0}\}$, equipped with their respective Borel σ-algebras $\mathcal{B}$ and $\mathcal{B}'$. Consider the mapping $h : (O, \mathcal{B}) \to (O', \mathcal{B}')$ given by $h(\cdot) = \|\cdot\|_D$. Since the D-norm is continuous on $\mathbb{R}^d$, its restriction to O is continuous, and thus h is measurable with empty discontinuity set $D_h = \emptyset$. Hence, for any limit measure $\nu \in M_{+}(O')$, it follows that $\nu(D_h) = 0$.
Hence, for any $A' \in \mathcal{B}'$ bounded away from 0, there exists $y > 0$ such that $A' \subset (y, +\infty)$. It follows that:

$$h^{-1}\big((y, +\infty)\big) = \{\mathbf{z} \in O : \|\mathbf{z}\|_D > y\}.$$

By norm equivalence, (i.e, $\|\mathbf{z}\|_D \leq k|\mathbf{z}|$), it follows that the following inclusion holds:

$$\{\mathbf{z} \in O : \|\mathbf{z}\|_D > y\} \subset \{\mathbf{z} \in O : \|\mathbf{z}\| > \frac{y}{k}\}.$$

The infimum of the Euclidean distance from this set to the origin $\mathbf{0}$ is bounded below by $\dfrac{y}{k} > 0$. Consequently, $h^{-1}(A')$ is bounded away from 0.

Hence, Theorem (1.2) holds and the induced mapping $\hat{h} : M_+(O) \to M_+(O')$, defined by $\hat{h}(\nu) = \nu \circ h^{-1}$, is continuous at ν. Consequently, since the sequence of loss measures converges in $M_+(O)$, the associated sequence of measures for Y also converges in $M_+(O')$:

$$\nu_{Y,n} = \nu_n o h^{-1} \xrightarrow{M_+(O')} \nu_Y = \nu o h^{-1}.$$

Using (2.6), (2.8), (2.7) and the homogeneity property of the D-norm, we obtain the following representation of Y:

$$Y = ||(\mathbf{X}\text{-}\mathbf{u})_+||_D = ||\mathbf{Z}||_D = ||r.\theta||_D = r.||\theta||_D.$$

For $y > 0$, the tail measure of Y on the Borel set $A' = (y, +\infty)$ is obtained by integrating the joint measure ν over the corresponding region:

$$A_y = \{(r,\theta) : r||\theta||_D > y\} :$$

$$\begin{aligned}
\nu_Y\big((y,\infty)\big) &= \int_{S_d} \int_{y/||\theta||_D}^{\infty} \gamma r^{-(\gamma+1)} dr\, \mu(d\theta) \\
&= \int_{S_d} \lim_{x \longrightarrow \infty} \big[- r^{-\gamma} \big]_{y/||\theta||_D}^{x} \mu(d\theta) \\
&= y^{-\gamma} \int_{S_d} ||\theta||_D^{\gamma}\, \mu(d\theta) \\
\nu_Y\big((y,\infty)\big) &= C.y^{-\gamma} \ \ with \ \ C = \int_{S_d} ||\theta||_D^{\gamma}\, \mu(d\theta).
\end{aligned}$$

The scale parameter C explicitly involves both the D-norm and the spectral measure. Since the limit measure ν_Y on O' is a pure power measure with exponent γ, we deduce that Y is univariate regularly varying. Hence, by the Pickands-Balkema and Haan theorem [11], the excesses of Y are asymptotically governed by a Generalized Pareto Distribution with shape parameter $\dfrac{1}{\gamma}$ □

Once the limiting law of the excesses of Y is characterized, the CVaR_α^D in the extreme regime can be approximated by substituting the true tail of Y with its asymptotic Generalized Pareto representation, leading to a closed-form expression in terms of the parameters of the limiting distribution.

Corollary 2.5. *Let Y be a continuous aggregated loss variable. If the exceedances of Y above a sufficiently high threshold follow GPD(ξ,σ) with $\xi < 1$ then,*

$$\mathrm{CVaR}_\alpha^D(Y) = u_Y + \frac{\sigma}{1-\xi}\left[1 + \frac{1}{\xi}\left[\left(\frac{1-\alpha}{\bar{F}_Y(u_Y)}\right)^{-\xi} - 1\right]\right], \forall \alpha > F_Y(u_Y) \qquad (2.9)$$

Proof. The analytical expression follows from the CVaR representation for a Generalized Pareto Distribution established by [17]. Applying this result to the limiting distribution of the aggregated variable Y yields the stated expression. □

This approach provides a coherent tail risk measure based on Extreme Value Theory, combining regular variation and Generalized Pareto Distribution convergence to quantify multivariate risks.

However, the usefulness of a coherent risk measure also relies on its ability to disentangle individual sources of vulnerability. By exploiting the positive 1-homogeneity of the D-norm, Euler's theorem allows for a decomposition of CVaR_α^D into marginal risk contributions, thereby projecting the complexity of the spectral measure onto each component of the system.

Theorem 2.6. *Let $\mathbf{X} \in \mathbb{R}_+^d \setminus \{\mathbf{0}\}$ be a loss vector with max-stable distribution admitting a density with respect to the Lebesgue measure and Y be the aggregation variable. Then CVaR_α^D can be decomposed into individual contributions as follows:*

$$CVaR_\alpha^D(Y) = \sum_{i=1}^{d} \mathbb{E}\left[(X_i - u_i)_+ \cdot \frac{\partial \|(\boldsymbol{X} - \boldsymbol{u})_+\|_D}{\partial X_i} \;\middle|\; Y > VAR_\alpha^D(Y)\right] \tag{2.10}$$

where each institution i contributes to the aggregate risk through the conditional expectation of its weighted loss, conditional on the system being in a distress state, given by:

$$C_i = \mathbb{E}\left[(X_i - u_i)_+ \cdot \frac{\partial \|(\boldsymbol{X} - \boldsymbol{u})_+\|_D}{\partial X_i} \;\middle|\; Y > VAR_\alpha(Y)\right]. \tag{2.11}$$

Proof. Let $h : \mathbb{R}_+^d \to \mathbb{R}_+$ be given by $h(\mathbf{Z}) = \|\mathbf{Z}\|_D$. Since h is positively 1-homogeneous, Euler's theorem implies that, for any excess vector $\mathbf{Z} = (\mathbf{X} - \mathbf{u})_+$ at points of differentiability of h, we have:

$$h(\mathbf{Z}) = \sum_{i=1}^{d} Z_i \cdot \frac{\partial h}{\partial Z_i}(\mathbf{Z}). \tag{2.12}$$

Substituting the model components into (2.12), the aggregation variable Y admits the following almost sure representation:

$$Y = \sum_{i=1}^{d} (X_i - u_i)_+ \cdot \frac{\partial \|(\mathbf{X} - \mathbf{u})_+\|_D}{\partial X_i}. \tag{2.13}$$

Since $\mathbf{X}$ admits a density, the vector $(\mathbf{X} - \mathbf{u})_+$ intersects the non-differentiability set of the D-norm with probability zero, so that (2.13) holds almost surely.
The quantity $\dfrac{\partial \|(\mathbf{X} - \mathbf{u})_+\|_D}{\partial X_i}$ can be interpreted as a random weighting factor capturing the sensitivity of the system to institution i.
Taking conditional expectations in (2.13) yields:

$$\mathbb{E}\left[Y \mid Y > \mathrm{VAR}_\alpha(Y)\right] = \mathbb{E}\left[\sum_{i=1}^{d} (X_i - u_i)_+ \cdot \frac{\partial \|(\mathbf{X} - \mathbf{u})_+\|_D}{\partial X_i} \;\middle|\; Y > \mathrm{VAR}_\alpha(Y)\right].$$

Using the linearity of conditional expectation, we obtain that

$$CVaR^D = \sum_{i=1}^{d} \mathbb{E}\left[(X_i - u_i)_+ \cdot \frac{\partial \|(\mathbf{X}-\mathbf{u})_+\|_D}{\partial X_i} \,\middle|\, Y > VaR_\alpha^D(Y) \right].$$

□

This theorem shows that the risk of an institution is given by the product of its own severity term $(X_i - u_i)_+$ and its weight within the dependence structure, $\frac{\partial \|\cdot\|_D}{\partial X_i}$.

It thus transforms a complex risk measure into a decision-support tool. It provides a precise decomposition of systemic fragility into dependence-driven contributions and quantifies the exact cost attributable to each institution in stabilizing the system.

3. Simulation Study

To illustrate the theoretical results established in this paper, we conduct a simulation study based on a sample of 5000 observations of a trivariate random vector generated from a multivariate Student-t distribution with four degrees of freedom ($\nu = 4$) and a specified correlation matrix.

For each marginal component, threshold levels are set at the 95th empirical quantile, and exceedances above these thresholds are computed. The resulting exceedance vectors are then aggregated using a logistic D-norm with parameter $\theta = 0.5$ in order to construct the aggregated risk variable Y.

The exceedances of Y are subsequently fitted using a Generalized Pareto Distribution (GPD) within the Peaks-Over-Threshold (POT) framework, and the associated parameters are estimated. Finally, empirical and asymtotic (MPOT) estimates of $CVaR^D_{99\%}$, together with the marginal contributions to systemic risk, are computed.

The main results are reported in Tables 1 to 4, while Figure (1) illustrates the goodness-of-fit of the GPD approximation for the exceedances of the aggregated variable Y.

Table (1) reports the marginal threshold levels corresponding to the 95th empirical quantiles for the three simulated variables, as well as the threshold associated with the aggregated variable Y.

Table 1. Table of thresholds at the $95th$ quantile.

Thresholds	u_1	u_2	u_3	u_Y
Value	2.14	2	2.11	0.75

The threshold set for the aggregate variable Y results in 250 outliers, or 5% of the simulated observations.

Table (2) presents the parameter estimates of the GPD for the exceedances of Y.

TABLE 2. Estimation of the GPD.

Indicator	Value
Shape parameter ξ	0.18
Scale Parameter β	1.05

The positive estimate of the shape paramerter ($\xi = 0.18$) indicates that the exceedances of Y exhibit heavy-tailed behavior, which is consistent with the Student-t distribution used in the simulation. The scale parameter ($\beta = 1.05$) reflects the intensity of the exceedances.
Figure (1) displays the QQ-plot of exceedances of Y against the fitted GPD.

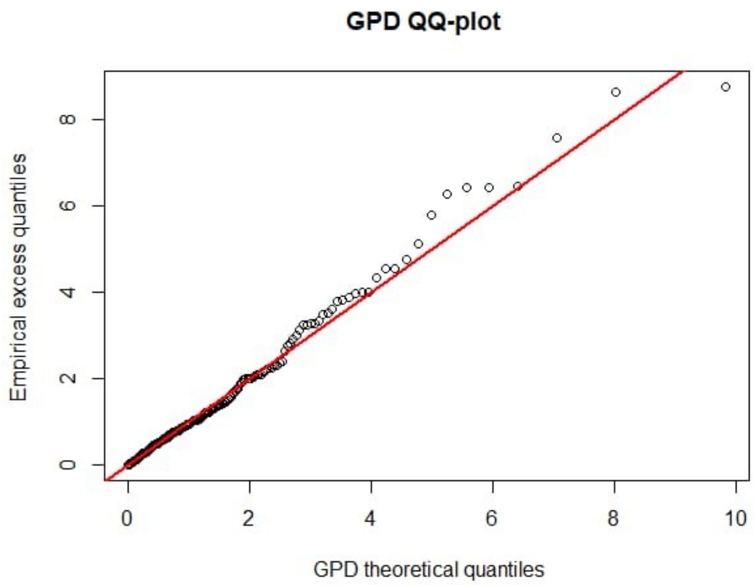

FIGURE 1. QQ-plot of the exceedances of the aggregated variable Y against the GPD.

The overall alignment of the empirical and theoretical quantiles indicates a satisfactory fit of the GPD model for the considered exceedances.
Table (3) compares the empirical risk measures with the asymptotic estimates obtained from the MPOT model.

TABLE 3. Comparison of risk measures.

Measure	Empirical	Asymptotic MPOT
$CVaR^D_{99\%}$	4.43	4.40

The MPOT estimation provides a value very close to the baseline empirical estimate, indicating that the POT approach effectively estimates the aggregated $CVaR^D_{99\%}$ within the simulated framework.

Table (4) presents the marginal contributions of the three variables to the overall systemic risk.

Table 4. Marginal risk contributions.

Variables	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	Sum of contributions
Contributions	1.74	1.61	1.08	4.43

The sum of marginal contributions exactly reproduces the overall $CVaR^D_{99\%}$, which confirms the adopted decomposition property. The results show that all three components contribute positively to systemic risk. However, $\mathbf{X}_1$ exhibits the highest contribution, followed by $\mathbf{X}_2$, while $\mathbf{X}_3$ contributes to a lesser extent. This difference reflects the relative influence of individual components in shaping the aggregated extreme risk.

The simulation results are consistent with the theoretical framework developed in this paper. They illustrate the relevance of the proposed approach for modeling extreme events and assessing systemic risk.

4. Conclusion

This work establishes a unified analytical framework for the study of extreme systemic risk by combining D-norm theory with tail risk measures. We have shown that the aggregation of excesses using a D-norm asymptotically converges to a univariate Generalized Pareto Distribution, which simplifies the analysis of multivariate risks while preserving their dependency structure. This approach thus provides a theoretical tool for better assessing extreme financial risks and identifying individual contributions to systemic risk. The simulation study illustrates the consistency of the proposed approach and highlights its ability to characterize extreme losses and individual contributions to systemic risk. This framework provides a theoretical tool for better assessing extreme financial risks. Future research will focus on the empirical validation of the model using financial data of featuring extreme events, its comparison with other measures of systemic risk, and the development of methods for estimating the D-norm in real-world applications.

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