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RINGS IN WHICH EVERY PRINCIPAL IDEAL IS FINITELY PRESENTED

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Abstract. In this paper we introduce and investigate a class of those rings in which every principal ideal is finitely presented. We establish the transfer of this notion to the trivial ring extension, direct product and homomorphic image, and then generate new and original families of rings satisfying this property.

1. Introduction and preliminaries

Throughout this paper, all rings are commutative with identity element, and all modules are unitary.

For a nonnegative integer n, an R-module E is called n-presented if there is an exact sequence of R-modules:

$$F_n \to F_{n-1} \to \dots F_1 \to F_0 \to E \to 0$$

where each F_i is a finitely generated free R-module. In particular, 0-presented and 1-presented R-modules are, respectively, finitely generated and finitely presented R-modules.

A ring R is coherent if every finitely generated ideal of R is finitely presented; equivalently, if $(0 : a)$ and $I \cap J$ are finitely generated for every $a \in R$ and any two finitely generated ideals I and J of R. Examples of coherent rings are Noetherian rings, Boolean algebras, von Neumann regular rings, and Prüfer/semihereditary rings. For instance see [13].

Let A be a ring and E be an A-module. The trivial ring extension of A by E (also called the idealization of E over A) is the ring $R = A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + a'e)$. Recall that if I is an ideal of A and E' is a submodule of E such that $IE \subseteq E'$, then $J = I \propto E'$ is an ideal of R. However, prime (resp., maximal) ideals of R have the form $P \propto E$, where P is a prime (resp., maximal) ideal of A (see [1, Theorem 3.2]). Suitable background on commutative trivial ring extensions is [1, 3, 13, 16, 17].

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In this paper, we are interested in those rings in which every principal ideal is finitely presented and which we called a $p-$coherent rings. Domains and coherent rings are examples of $p-$coherent rings.

The purpose of this paper is to give some simple methods in order to construct $p-$coherent rings. For this, we investigate the transfer of $p-$coherent property to the trivial ring extension, direct product and homomorphic image.

2. Main results

A ring is called $p-$coherent if every principal ideal is finitely presented. We begin with the following remark:

Remark 2.1. *(1) A ring R is $p-$coherent if and only if the annihilator of every principal ideal of R is finitely generated by the following exact sequence:*

$$0 \longrightarrow Ann\,(a) \longrightarrow \quad R \longrightarrow \quad Ra \longrightarrow 0$$

where $u\,(x) = ax$.

(2) If R is coherent or an integral domain, then R is a $p-$coherent.

We start with the transfer of the p-coherent property in direct product of rings.

Theorem 2.2. *Let R be a finite direct product of some family of rings $(A_i)_{i=1,\dots,n}$. Then R is a $p-$coherent ring if and only if A_i is a $p-$coherent ring for each $i = 1, ..., n$.*

Proof. The proof is done by induction on n and it suffices to check it for $n = 2$. Assume that R is $p-$coherent and let $a \neq 0 \in A_1$, then $(a, 1) \in R$. We have $Ann\,((a, 1))$ is a finitely generated ideal of R such that:

$$\begin{aligned} Ann\,((a,1)) &= \{(x,y) \in R / \,(a,1)\,(x,y) = (0,0)\} \\ &= Ann\,(x) \times \{0\} \end{aligned}$$

Hence, $Ann\,(x)$ is a finitely generated ideal of A_1. Therefore, A_1 is $p-$coherent.

Likewise A_2 is an $p-$coherent.

Conversely, assume that A_1 and A_2 are $p-$coherents and let $(a, b) \in R$. We have:

$$\begin{aligned} Ann\,((a,b)) &= \{(x,y) \in A \times B / \,(a,b)\,(x,y) = (0,0)\} \\ &= Ann\,(a) \times Ann\,(b) \end{aligned}$$

Hence, $Ann\,((a, b))$ is finitely generated ideal of R since $Ann\,(a)$ and $Ann\,(b)$ are finitely generated, as desired. □

Now, we study the transfer of p-coherent property in localisation.

Proposition 2.3. *Let R be a $p-$coherent ring and S be a multiplicative subset of R. Then $S^{-1}R$ is $p-$coherent.*

Proof. Let $J = (S^{-1}R)\left(\frac{a}{s}\right)$ be a principal ideal of $S^{-1}R$ and set $I = Ra$. Then I is a finitely presented ideal of R since R is $p-$coherent and so $J = S^{-1}I$ is a finitely presented ideal of $S^{-1}R$, as desired. □

The homomorphic image of a $p-$coherent ring is not always a $p-$coherent ring as the following example shows:

Example 2.4. *Let $R = \mathbb{Z}+X\mathbb{Q}[X]$ and $I = X^2\mathbb{Q}[X]$ be an ideal of R. Then:*
(1) R is $p-$coherent.
(2) R/I is not $p-$coherent.

Proof. 1) R is $p-$coherent since R is an integral domain.
2) We have:

$$\begin{aligned} R/I &= \left\{\left(a + X\left(\sum_{i=0}^{n} a_i X^i\right)\right) + X^2\mathbb{Q}[X] / a \in \mathbb{Z} \text{ and } a_i \in \mathbb{Q}\right\} \\ &= \left\{a + a_0X + X^2\mathbb{Q}[X] / a \in \mathbb{Z} \text{ and } a_0 \in \mathbb{Q}\right\} \\ &= \left\{a + a_0\overline{X} / a \in \mathbb{Z} \text{ and } a_0 \in \mathbb{Q}\right\} \\ &\cong \mathbb{Z} \propto \mathbb{Q} \end{aligned}$$

which is not $p-$coherent by Corollary 2.7 below, as desired. □

Also if I is an ideal of a ring R, we don't have R/I is $p-$coherent implies that R is $p-$coherent in general as the following example shows:

Example 2.5. *Let $R = \mathbb{Z} \propto \mathbb{R}$ and $I = 0 \propto \mathbb{R}$. Then:*
(1) R/I is a $p-$coherent ring since $R/I \cong \mathbb{Z}$ is an integral domain.
(2) R is not $p-$coherent by Corollary 2.7 below.

Now, to the main result which study the possible transfer of the p-coherent ring property between a ring A and a trivial ring extension $A \ltimes E$.

Theorem 2.6. *Let A be a ring and E be an $A-$module. Let $R = A \propto E$ be the trivial ring extension of A by E. Then:*

(1) *Assume that R is $p-$coherent. Then:*
   (a) A is $p-$coherent.
   (b) $Ann_A(f) = \{a \in A / af = 0\}$ is finitely generated ideal of A for each $f \in E$.
   (c) Assume that A is an integral domain and $\exists e \in E$ such that $\forall a \in A$, $ae = 0 \Longrightarrow a = 0$. Then $E' = \{f \in E / af = 0\}$ is a finitely generated $A-$submodule of E for every $a \in A$.
   (d) Assume that $\exists e \in E$ such that $\forall a \in A$, $ae = 0 \Longrightarrow a = 0$. Then E is a finitely generated $A-$module.
(2) *Assume that A is an integral domain and $\exists e \in E$ such that $\forall a \in A$, $ae = 0 \Longrightarrow a = 0$. Then:*
   R is $p-$coherent if and only if $\forall f \in E$ $Ann_A(f)$ is a finitely generated ideal of A and $\forall a \in A$, $E' = \{f \in E / af = 0\}$ is a finitely generated $A-$module.

(3) *Assume that* (A, M) *is a local ring such that* $ME = 0$. *Then:*
R is p−coherent if and only if A is p−coherent, M is finitely generated and E is a finitely generated A−module.

Proof. 1) Assume that R is a $p-$coherent ring.

a) Let $a \in A$. Then:

$$\begin{aligned} Ann\left((a,0)\right) &= \{(b,f) \in R/\left(b,f\right)(a,0) = (0,0)\} \\ &= \{(b,f) \in R/ba = 0 \text{ and } af = 0\} \\ &= Ann\left(a\right) \propto E' \end{aligned}$$

where $E' = \{f \in E/af = 0\}$. Then $Ann\left(a\right) \propto E'$ is a finitely generated ideal of R since R is $p-$coherent. Hence, $Ann\left(a\right)$ is a finitely generated ideal of A, as desired.

b) Let $f \in E$. Then:

$$\begin{aligned} Ann\left((0,f)\right) &= \{(b,e) \in R/\left(b,e\right)(0,f) = (0,0)\} \\ &= \{(b,e) \in R/bf = 0\} \\ &= Ann_A\left(f\right) \propto E \end{aligned}$$

is a finitely generated ideal of R. Then $Ann_A\left(f\right)$ is a finitely generated ideal of A, as desired.

c) Assume that A is an integral domain and there exist $e \in E$ such that $\forall a \in A$, $ae = 0 \Longrightarrow a = 0$.

Let $a \in A$. Then:

$$\begin{aligned} Ann\left((a,0)\right) &= \{(b,f) \in R/\left(b,f\right)(a,0) = (0,0)\} \\ &= \{(b,f) \in R/ab = 0 \text{ and } af = 0\} \\ &= 0 \propto E' \end{aligned}$$

where $E' = \{f \in E/af = 0\}$. Then $0 \propto E'$ is a finitely generated ideal of R. Hence, E' is a finitely generated $A-$submodule.

d) Assume that $\exists e \in E$ such that $\forall a \in A$, $ae = 0 \Longrightarrow a = 0$. Then:

$$\begin{aligned} Ann\left((0,e)\right) &= \{(a,f) \in R/\left(a,f\right)(0,e) = (0,0)\} \\ &= \{(a,f) \in R/ae = 0\} \\ &= 0 \propto E \end{aligned}$$

is a finitely generated ideal of R since R is $p-$coherent. Hence, E is a finitely generated $A-$module.

2) Assume that A is an integral domain and $\exists e \in E$ such that $Ann_A\left(e\right) = \{0\}$.

Assume that R is $p-$coherent. $\forall f \in E$,we have $Ann_A\left(f\right)$ is a finitely generated ideal of A by (1)(b). $\forall a \in A$, we have $E' = \{f \in E/af = 0\}$ is a finitely generated $A-$module by (1)(c).

Conversely, assume that $\forall f \in E$ $Ann\,(f)$ is a finitely generated ideal of A and $\forall a \in A$, $E' = \{f \in E/af = 0\}$ is a finitely generated $A-$module and we must show that R is $p-$coherent.

Let $(a, e) \in R$. Two cases are possible:

Case 1: $a \neq 0$.

In this case, we have:

$$\begin{aligned} Ann\,((a,e)) =& \{(b,f) \in R/\,(b,f)\,(a,e) = (0,0)\} \\ =& \{(b,f) \in R/ab = 0 \text{ } and \text{ } af + be = 0\} \\ =& 0 \propto E' \end{aligned}$$

where $E' = \{f \in E/af = 0\}$. Hence, $Ann\,((a,e)) = 0 \propto E'$ is finitely generated by (2)(b), as desired.

Case 2: $a = 0$.

In this case, we have:

$$\begin{aligned} Ann\,((0,e)) =& \{(b,f) \in R/\,(b,f)\,(0,e) = (0,0)\} \\ =& Ann_A\,(e) \propto E \end{aligned}$$

which is a finitely generated ideal of R since $Ann_A\,(e)$ is a finitely generated ideal of A by (2)(a) and E is a finitely generated $A-$module by (2)(b). Therefore, R is a $p-$coherent ring.

3) Assume that (A, M) is local and $ME = 0$.

Assume that R is $p-$coherent.

Then A is $p-$coherent by (1)(a).

Let $e \in E \setminus \{0\}$. Then:

$$\begin{aligned} Ann\,((0,e)) =& \{(a,f) \in R/\,(a,f)\,(0,e) = (0,0)\} \\ =& M \propto E \end{aligned}$$

is a finitely generated ideal of R since R is$p-$coherent. Hence, M is finitely generated.

Further, assume that $M \propto E = \sum_{i=1}^{n} R\,(a_i, e_i)$, where $a_i \in M$ and $e_i \in E$ for each $i \in \{1, ..., n\}$. Then $E \subseteq \sum_{i=1}^{n} (A/M)\,e_i$, and hence E is an $A/M-$vector space of finite rank. Therefore, E is a finitely generated $A-$module, as desired.

Conversely, assume that A is $p-$coherent, M is finitely generated and E is a finitely generated $A-$module. Let $(a, e) \in R \setminus \{0\}$. If a is invertible in A, then (a, e) is invertible in R. Then, without loss of generality, we may assume that $a \in M$. Hence:

$$\begin{aligned} Ann\,((a,e)) =& \{(b,f) \in R/\,(a,e)\,(b,f) = (ab, be) = (0,0)\} \\ =& \{(b,f) \in M \propto E/ab = 0\} \end{aligned}$$

since if b is invertible in A, then (b, f) is invertible in R, and so $(a, e) = 0$, a contradiction. It is easy to see that if $a = 0$ then $Ann\,((a,e)) = M \propto E$, and if $a \neq 0$ then $Ann\,((a,e)) = Ann_A\,(a) \propto E$. In both cases, $Ann\,((a,e))$ is a finitely generated ideal of R since M and $Ann_A(a)$ are finitely generated ideals of A and E is a finitely generated $A-$module, completing the proof of Theorem 2.6. □

The next two corollaries are applications of Theorem 2.6.

Corollary 2.7. *Let A be a domain, $K = qf(A)$ and E be a $K-$vector space. Let $R = A \propto E$ be the trivial ring extension of A by E. Then:*
R is $p-$coherent if and only if A is a field and $dim_A E < \infty$.

Proof. Assume that R is $p-$coherent and let $e \in E \setminus \{0\}$. We have:

$$\begin{aligned} Ann((0,e)) &= \{(a,f) \in R / (a,f)(0,e) = (0,0)\} \\ &= \{(a,f) \in R / ae = 0\} \\ &= 0 \propto E \end{aligned}$$

which is finitely generated ideal of R. Then E is finitely generated $A-$module. Hence, K is finitely generated $A-$module since $E \cong K^{(I)}$. Therefore, $K = A$ is a field and $dim_K E < \infty$.

Conversely, assume that $R = K \propto E$ and $dim_K E < \infty$ and let $(a,e) \in R$. Two cases are possible:

Case 1: $a \neq 0$.

In this case, (a,e) is invertible. Then $Ann((a,e)) = \{0\}$ is finitely generated.

Case 2: $a = 0$.

In this case, we have:

$$\begin{aligned} Ann((0,e)) &= \{(x,f) \in R / (x,f)(0,e) = (0,0)\} \\ &= \{(x,f) \in R / xe = 0\} \\ &= \begin{cases} 0 \propto E & if\ e \neq 0 \\ R & if\ e = 0 \end{cases} \end{aligned}$$

which is finitely generated. □

Corollary 2.8. *Let $A \subseteq B$ be two integral domains and set $R = A \propto B$. Then:*
R is $p-$coherent if and only if B is a finitely generated $A-$module.

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