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A STUDY ON TOPOLOGICAL AND GEOMETRICAL CHARACTERISTICS OF NEW BANACH SEQUENCE SPACES

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ABSTRACT. A short extract of this study is to submit a new sequence space $\ell_p(\widehat{F}(r,s))$ under the domain of the matrix $\widehat{F}(r,s)$ constituted by using Fibonacci sequence and non-zero real number r and s, of ℓ_p previously defined. Additionally, we construct some inclusion relations concerning with this space and determine its the α-, β-, γ-duals. Moreover, we characterize some matrix classes on the space $\ell_p(\widehat{F}(r,s))$ and examine some geometric properties of this space.

1. Introduction and preliminaries

As usual, the symbol ω denotes the space of all real valued sequences. Also, we use the symbols ℓ_∞, c, c_0 and ℓ_p $(1 \leq p < \infty)$, to represent the sets of all bounded, convergent, null sequences and p-absolutely convergent series, respectively. Since any vector subspace of ω is called as a *sequence space*, each of these sets is a *sequence space*. Additionally, it will be used the conventions that $e = (1, 1, ...)$ and $e^{(n)}$ is the sequence whose only non-zero term is 1 in the n^{th} place for each $n \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, ...\}$. We also write $x = (x_k)$ and $A = (a_{nk})$ instead of $x = (x_k)_{k=0}^{\infty}$ and $A = (a_{nk})_{n,k=0}^{\infty}$, respectively.

Let η and ϑ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from η into ϑ and we denote it by writing $A : \eta \to \vartheta$, if for every sequence $x = (x_k) \in \eta$ the sequence $Ax = \{A_n(x)\}$, the A-transform of x, is in ϑ; where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k; \quad (n \in \mathbb{N}). \tag{1.1}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞.

Now, let us give the definition of matrix mapping which will be used especially in section 4. By (η, ϑ), we denote the class of all matrices A such that $A : \eta \to \vartheta$. Thus, $A \in (\eta, \vartheta)$ if and only if the series on the right side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in \eta$ and we have $Ax \in \vartheta$ for all $x \in \eta$.

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The matrix domain has fundamental importance for this study. So, the concept is stated in this paragraph. The matrix domain ψ_A of an infinite matrix A in a sequence space ψ is defined by

$$\psi_A = \{x = (x_k) \in \omega : Ax \in \psi\} \tag{1.2}$$

which is a sequence space.

In recent years, some researchers have addressed the approach to constructing a new sequence space by means of the matrix domain of a particular limitation method, see [2, 5, 6, 8, 10, 11, 12, 13, 14, 15, 16, 17, 42, 46, 49, 50, 51, 52, 53, 54], and the references therein.

The difference matrix $\Delta = (\delta_{nk})$ can be written in two different ways as follows

$$\delta_{nk} = \begin{cases} (-1)^{n-k} & (n-1 \le k \le n) \\ 0 & (0 \le k < n-1 \text{ or } k > n) \end{cases}$$

or

$$\delta_{nk} = \begin{cases} (-1)^{n-k} & (n \le k \le n+1) \\ 0 & (0 \le k < n \text{ or } k > n+1). \end{cases}$$

As already known, the matrix domain λ_Δ is said the *difference sequence space* whenever λ is a normed or paranormed sequence space. In 1981, Kızmaz [31] firstly defined difference operator $(\Delta x_k) = (x_k - x_{k+1})$ and examined difference sequence spaces established difference operator as follows:

$$\phi(\Delta) = \{x = (x_k) \in \omega : (x_k - x_{k+1}) \in \phi\}$$

for $\phi = \{\ell_\infty, c, c_0\}$. Now, let us give some fundamental study on this concept. The difference space bv_p as the set of all sequences whose $\Delta-$transforms is the space ℓ_p, i.e.,

$$bv_p = \{x = (x_k) \in \omega : (x_k - x_{k+1}) \in \ell_p\}.$$

It is obvious that $bv_p = (\ell_p)_\Delta$, see [3, 8, 20]. The paranormed difference sequence space

$$\Delta\lambda(p) = \{x = (x_k) \in \omega : (x_k - x_{k+1}) \in \lambda(p)\}$$

was examined by Ahmad and Mursaleen [1] and Malkowsky [37], where $\lambda(p)$ is any of the paranormed spaces $\ell_\infty(p)$, $c(p)$ and $c_0(p)$ defined by Simons [55] and Maddox [36].

Recently, Başar et al. [4] have defined the sequence spaces $bv(u,p)$ and $bv_\infty(u,p)$ by

$$bv(u,p) = \{x = (x_k) \in \omega : \sum_k |u_k(x_k - x_{k-1})|^{p_k} < \infty\}$$

and

$$bv_\infty(u,p) = \{x = (x_k) \in \omega : \sup_{k\in\mathbb{N}} |u_k(x_k - x_{k-1})|^{p_k} < \infty\},$$

where $u = (u_k)$ is an arbitrary fixed sequence and $0 < p_k \le H < \infty$ for all $k \in \mathbb{N}$. These spaces are generalization of the space bv_p for $1 \le p \le \infty$. Quite recently, Kirişçi and Başar [32] have introduced and studied the generalized difference sequence spaces

$$\widehat{X} = \{x = (x_k) \in \omega : B(r,s)x \in X\}$$

for $X = \ell_\infty$, ℓ_p, c and c_0, where $1 \le p < \infty$ and $B(r,s)x = (sx_{k-1} + rx_k)$ $(r, s \neq 0)$. Following Kirişçi and Başar [32], Candan [10] has examined the sequence space $X(\widetilde{B})$ as the set of all sequences whose $\widetilde{B}(\tilde{r}, \tilde{s})$-transforms are in the space $X \in \{\ell_\infty, \ell_p, c, c_0\}$, where $\widetilde{B}(\tilde{r}, \tilde{s})$ denotes the double sequential band matrix $\widetilde{B}(\tilde{r}, \tilde{s}) = \{b_{nk}(r_n, s_n)\}$ defined by

$$b_{nk}(\widetilde{r}, \widetilde{s}) = \begin{cases} r_n & , \quad (k = n), \\ s_n & , \quad (k = n-1), \\ 0 & , \quad (0 \le k < n-1 \quad \text{or} \quad k > n), \end{cases}$$

for all $k, n \in \mathbb{N}$, where $\tilde{r} = (r_n)_{n=0}^\infty$ and $\tilde{s} = (s_n)_{n=0}^\infty$ be given convergent sequences of positive real numbers and $1 \le p < \infty$.

Also in [9, 19, 23, 24, 25, 28, 38, 39, 40, 45, 48], the authors studied some difference sequence spaces.

In this study, we have introduce the generalized Fibonacci difference matrix $\widehat{F}(r,s)$ constituted by using Fibonacci sequence $\{f_n\}$ and non-zero real numbers r and s, earlier defined by Candan [15]. Also, we have defined new sequence spaces $\ell_p(\widehat{F}(r,s))$ and $\ell_\infty(\widehat{F}(r,s))$ related to matrix domain of $\widehat{F}(r,s)$ in the sequence spaces ℓ_p and ℓ_∞, respectively; where $1 \le p < \infty$.

The rest of this study is organized as follows:

At the beginning of the next section, it is given some historical developments about Fibonacci numbers and served some notations and basic concepts including BK-space which are going to be needed in later sections. After then, we consider a band matrix, constituted by using Fibonacci sequences and non-zero real number r and s. Furthermore, we introduce the new brand sequence spaces $\ell_p(\widehat{F}(r,s))$ and $\ell_\infty(\widehat{F}(r,s))$. Additionally, we find out some inclusion relations concerning with these spaces and establish the basis of the space $\ell_p(\widehat{F}(r,s))$ for $1 \le p < \infty$. In Section 3, we compute the α-, β-, γ-duals of the spaces $\ell_p(\widehat{F}(r,s))$ and $\ell_\infty(\widehat{F}(r,s))$. In Section 4, we characterize the classes $(\ell_p(\widehat{F}(r,s)), X)$ and $(\ell_\infty(\widehat{F}(r,s)), X)$, where $1 \le p < \infty$ and X is any of the spaces ℓ_∞, ℓ_1, c and c_0. In the final section of the study, we devote to investigate some geometric properties of the space $\ell_p(\widehat{F}(r,s))$ for $1 < p < \infty$.

2. The Fibonacci Difference Sequence Spaces $\ell_p(\widehat{F}(r,s))$ and $\ell_\infty(\widehat{F}(r,s))$

Before we go to the main topic of our study, let us give a history about the Fibonacci numbers.

In 1202, Fibonacci numbers were first noted by the Italian mathematician Leonardo of Pisa whose nickname was *Fibonacci* in his *"Liber Abaci"* which famous in the world. For information on the important properties and uses of the Fibonacci numbers see [35] . Define the sequence $\{f_n\}_{n=0}^\infty$ of Fibonacci numbers given by the linear recurrence relations

$$f_0 = f_1 = 1 \text{ and } f_n = f_{n-1} + f_{n-2}; \;\; n \ge 2.$$

Fibonacci numbers have many interesting properties and applications in arts, sciences and architecture. For instance, the ratio sequences of Fibonacci numbers converges to the golden ratio which is important in sciences and arts. Additionally, some well-known basic properties of Fibonacci numbers are given as follows:

$$\lim_{n\to\infty}\frac{f_{n+1}}{f_n}=\frac{1+\sqrt{5}}{2}=\alpha \quad \text{(Golden Ratio)},$$

$$\sum_{k=0}^{n} f_k = f_{n+2}-1;\ (n\in\mathbb{N}),$$

$$\sum_k \frac{1}{f_k} \quad \text{converges},$$

$$f_{n-1}f_{n+1}-f_n^2=(-1)^{n+1};\ \ (n\geq 1) \quad \text{(Cassini Formula)}.$$

Substituting for f_{n+1} in Cassini's formula yields $f_{n-1}^2+f_nf_{n-1}-f_n^2=(-1)^{n+1}$.

Now, we are going to define the generalized Fibonacci band matrix $\widehat{F}(r,s)=(\widehat{f}_{nk}(r,s))$ constituted by using Fibonacci sequence and non-zero real numbers r and s. Additionally, we firstly introduce the sequence spaces $\ell_p(\widehat{F}(r,s))$ and $\ell_\infty(\widehat{F}(r,s))$, where $1\leq p<\infty$. Also, we present some inclusion theorems and construct Schauder basis of the space $\ell_p(\widehat{F}(r,s))$ for $1\leq p<\infty$.

Let f_n be the nth Fibonacci number for every $n\in\mathbb{N}$. Then, we define the infinite matrix $\widehat{F}(r,s)=(\widehat{f}_{nk}(r,s))$ by

$$\widehat{f}_{nk}(r,s)=\begin{cases} s\frac{f_{n+1}}{f_n} & , \quad (k=n-1), \\ r\frac{f_n}{f_{n+1}} & , \quad (k=n), \\ 0 & , \quad (0\leq k<n-1 \ \text{ or } \ k>n), \end{cases} \qquad (n,k\in\mathbb{N}).$$

Now, we introduce the Fibonacci difference sequence spaces $\ell_p(\widehat{F}(r,s))$ and $\ell_\infty(\widehat{F}(r,s))$ as the set of all sequences such that their $\widehat{F}(r,s)$-transforms are in the space ℓ_p and ℓ_∞, respectively, i.e.,

$$\ell_p(\widehat{F}(r,s))=\left\{x=(x_n)\in\omega:\ \sum_n\left|r\frac{f_n}{f_{n+1}}x_n+s\frac{f_{n+1}}{f_n}x_{n-1}\right|^p<\infty\right\};\ 1\leq p<\infty$$

and

$$\ell_\infty(\widehat{F}(r,s))=\left\{x=(x_n)\in\omega:\ \sup_{n\in\mathbb{N}}\left|r\frac{f_n}{f_{n+1}}x_n+s\frac{f_{n+1}}{f_n}x_{n-1}\right|<\infty\right\}.$$

It is natural that the spaces $\ell_p(\widehat{F}(r,s))$ and $\ell_\infty(\widehat{F}(r,s))$ may be rewritten by with the notation of (1.2) that

$$\ell_p(\widehat{F}(r,s))=(\ell_p)_{\widehat{F}(r,s)}\ (1\leq p<\infty) \ \text{ and } \ \ell_\infty(\widehat{F}(r,s))=(\ell_\infty)_{\widehat{F}(r,s)}. \tag{2.1}$$

It is remarkable that the sequence $y=(y_n)$ which will be frequently used is the $\widehat{F}$-transform of a sequence $x=(x_n)$ everywhere in study, in other words,

$$y_n=\widehat{F}_n(r,s)(x)=\begin{cases} r\frac{f_0}{f_1}x_0=rx_0 & (n=0) \\ r\frac{f_n}{f_{n+1}}x_n+s\frac{f_{n+1}}{f_n}x_{n-1} & (n\geq 1) \end{cases}\ ;\ (n\in\mathbb{N}). \tag{2.2}$$

Now, we should state that the matrix $\widehat{F}(r,s)$ can be reduced to the matrix $\widehat{F}$ in case $r=1$ and $s=-1$. Therefore, the results related to the spaces $\ell_p(\widehat{F}(r,s))$ and $\ell_\infty(\widehat{F}(r,s))$ are more general and more comprehensive than the corresponding consequences of the spaces $\ell_p(\widehat{F})$ and $\ell_\infty(\widehat{F})$ more recently defined by Kara in [30]. For additional details may be refer to the following references [15, 17, 18].

Before presenting the next theorem, let us consider the following concepts.

A sequence space X is called *FK space* if it is a complete linear metric space with continuous coordinates $p_n : X \to \mathbb{R}$ $(n \in \mathbb{N})$, where $\mathbb{R}$ denotes the real field and $p_n(x) = x_n$ for all $x = (x_k) \in X$ and every $n \in \mathbb{N}$. A *BK space* is a normed *FK space*, that is, a *BK space* is a Banach space with continuous coordinates. The space ℓ_p $(1 \leq p < \infty)$ is BK space with $\|x\|_p = \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{1/p}$ and c_0, c and ℓ_∞ are BK spaces with $\|x\|_\infty = \sup_k |x_k|$.

Now, we may begin with the following theorem which is essential in the text.

Theorem 2.1. *When p satisfied the condition $1 \leq p \leq \infty$. The newly defined sequence space $\ell_p(\widehat{F}(r,s))$ is a BK-space with the norm $\|x\|_{\ell_p(\widehat{F}(r,s))} = \|\widehat{F}(r,s)x\|_p$, in other words*

$$\|x\|_{\ell_p(\widehat{F}(r,s))} = \left(\sum_n \left|\widehat{F}_n(r,s)(x)\right|^p\right)^{1/p} \quad ; \quad (1 \leq p < \infty)$$

and

$$\|x\|_{\ell_\infty(\widehat{F}(r,s))} = \sup_{n \in \mathbb{N}} \left|\widehat{F}_n(r,s)(x)\right|.$$

Proof. The proof can be directly reached when we assess as right both the hypothesis and well-know Theorem 4.3.12 of Wilansky [57]. Indeed, we know that (2.1) is valid and the spaces ℓ_p and ℓ_∞ recalled earlier section are BK-spaces with respect to their natural norms and the matrix $\widehat{F}(r,s)$ is a triangle; Theorem 4.3.12 of Wilansky results in the fact that the spaces $\ell_p(\widehat{F}(r,s))$ and $\ell_\infty(\widehat{F}(r,s))$ are BK-space with the given norms, where $1 \leq p < \infty$. This marks the end of the proof. $\square$

Remark 2.2. One can easily check that the absolute property does not hold on the spaces $\ell_p(\widehat{F}(r,s))$ and $\ell_\infty(\widehat{F}(r,s))$, that is $\|x\|_{\ell_p(\widehat{F}(r,s))} \neq \||x|\|_{\ell_p(\widehat{F}(r,s))}$ and $\|x\|_{\ell_\infty(\widehat{F}(r,s))} \neq \||x|\|_{\ell_\infty(\widehat{F}(r,s))}$ for at least one sequence in the spaces $\ell_p(\widehat{F}(r,s))$ and $\ell_\infty(\widehat{F}(r,s))$, and this shows that $\ell_p(\widehat{F}(r,s))$ and $\ell_\infty(\widehat{F}(r,s))$ are the sequence spaces of non-absolute type, where $|x| = (|x_k|)$ and $1 \leq p < \infty$.

Theorem 2.3. *The generalized Fibonacci difference sequence space $\ell_p(\widehat{F}(r,s))$ of non-absolute type are linearly isomorphic to the space ℓ_p, that is $\ell_p(\widehat{F}(r,s)) \cong \ell_p$ for $1 \leq p \leq \infty$.*

Proof. To verify the fact that $\ell_p(\widehat{F}(r,s)) \cong \ell_p$, we need to show the existence of a linear bijection between the spaces $\ell_p(\widehat{F}(r,s))$ and ℓ_p when p satisfied the condition $1 \leq p \leq \infty$, from the definition of linear isomorphism. To do this, we consider the transformation T defined above, with the aid of notation of (2.2), from $\ell_p(\widehat{F}(r,s))$ to ℓ_p by $x \to y = Tx$. In that case $Tx = y = \widehat{F}(r,s)x \in \ell_p$ for every $x \in \ell_p(\widehat{F}(r,s))$. Since the linearity of the map T is not difficult to prove, we omit the detail. Further, it is trivial that $x = 0$ whenever $Tx = 0$ and hence T is injective.

Furthermore, let $y = (y_k) \in \ell_p$ for $1 \le p \le \infty$ and define the sequence $x = (x_k)$ by

$$x_k = \frac{1}{r}\sum_{j=0}^{k}\left(\frac{-s}{r}\right)^{k-j}\frac{f_{k+1}^2}{f_j f_{j+1}}y_j \ ; \ (k \in \mathbb{N}).$$

Thus, in the cases $1 \le p < \infty$ and $p = \infty$, we have

$$\begin{aligned}
\|x\|_{\ell_p(\widehat{F}(r,s))} &= \left(\sum_k \left| r\frac{f_k}{f_{k+1}}x_k + s\frac{f_{k+1}}{f_k}x_{k-1}\right|^p\right)^{1/p} \\
&= \left(\sum_k \left| \frac{f_k}{f_{k+1}}\sum_{j=0}^{k}\left(\frac{-s}{r}\right)^{k-j}\frac{f_{k+1}^2}{f_j f_{j+1}}y_j + \frac{s f_{k+1}}{r f_k}\sum_{j=0}^{k-1}\left(\frac{-s}{r}\right)^{k-j-1}\frac{f_k^2}{f_j f_{j+1}}y_j\right|^p\right)^{1/p} \\
&= \left(\sum_k |y_k|^p\right)^{1/p} = \|y\|_p < \infty
\end{aligned}$$

and

$$\|x\|_{\ell_\infty(\widehat{F}(r,s))} = \sup_{k\in\mathbb{N}}\left|\widehat{F}_k(r,s)(x)\right| = \|y\|_\infty < \infty,$$

respectively. Then, we get $x \in \ell_p(\widehat{F}(r,s))$ $(1 \le p \le \infty)$. Therefore, T is surjective and norm preserving. Consequently, T is a linear bijection which tells us the result that the spaces $\ell_p(\widehat{F}(r,s))$ and ℓ_p are linearly isomorphic for $1 \le p \le \infty$. This ends the proof. $\square$

Now, we give some inclusion relations concerning with the space $\ell_p(\widehat{F}(r,s))$.

Theorem 2.4. *The inclusion $\ell_p \subset \ell_p(\widehat{F}(r,s))$ strictly holds for $1 \le p \le \infty$.*

Proof. The proof will be done in two steps. In the first step, we will show that the inclusion $\ell_p \subset \ell_p(\widehat{F}(r,s))$ is valid for $1 \le p \le \infty$. If every x lies in ℓ_p, then it is sufficient to prove the existence of a number $M > 0$ such that $\|x\|_{\ell_p(\widehat{F}(r,s))} \le M\|x\|_p$. To do this, let us assume that $x \in \ell_p$ and $1 < p \le \infty$. Since the inequalities $\frac{f_k}{f_{k+1}} \le 1$ and $\frac{f_{k+1}}{f_k} \le 2$ always hold for every $k \in \mathbb{N}$, we can deduce with the notation of (2.2),

$$\begin{aligned}
\sum_k \left|\widehat{F}_k(r,s)(x)\right|^p &\le \sum_k 2^{p-1}\left(|rx_k|^p + |2sx_{k-1}|^p\right) \\
&\le 2^{2p-1}max\{|r|,|s|\}\left(\sum_k |x_k|^p + \sum_k |x_{k-1}|^p\right)
\end{aligned}$$

and

$$\sup_{k\in\mathbb{N}}\left|\widehat{F}_k(r,s)(x)\right| \le 3max\{|r|,|s|\}\sup_{k\in\mathbb{N}}|x_k|,$$

which together result in, as expected,

$$\|x\|_{\ell_p(\widehat{F}(r,s))} \leq 4max\{|r|,|s|\}\,\|x\|_p \tag{2.3}$$

for $1 < p \leq \infty$.

In the last step, we must find an element which belong to $\ell_p(\widehat{F}(r,s))$ but which does not belong to ℓ_p. Since the sequence $x = (x_k) = (1/r(-s/r)^k f_{k+1}^2)$ is in $\ell_p(\widehat{F}(r,s)) - \ell_p$, the inclusion $\ell_p \subset \ell_p(\widehat{F}(r,s))$ is strictly valid for $1 < p \leq \infty$. Similarly, one can easily prove that inequality (2.3) also holds in the case $p = 1$ and so we omit the details. This step finishes the proof. □

Theorem 2.5. *If $1 \leq p < s$, then $\ell_p(\widehat{F}(r,s)) \subset \ell_s(\widehat{F}(r,s))$.*

Proof. The proof of the theorem is quite standard and easy. If two real numbers p and s satisfy the condition $1 \leq p < s$ and $x \in \ell_p(\widehat{F}(r,s))$. Then we get from Theorem 2.1 that $y \in \ell_p$, where y is the sequence given by (2.2). Therefore, the well-known inclusion $\ell_p \subset \ell_s$ results in $y \in \ell_s$. This shows that $x \in \ell_s(\widehat{F}(r,s))$ and then, the inclusion $\ell_p(\widehat{F}(r,s)) \subset \ell_s(\widehat{F}(r,s))$ is valid. In fact, this is exactly what we want to prove. □

After, we remember the concept of Schauder basis, we are going to give a sequence of the points of the space $\ell_p(\widehat{F}(r,s))$ which forms a basis for the space $\ell_p(\widehat{F}(r,s))$ $(1 \leq p < \infty)$.

A sequence (b_n) in a normed space X is called a *Schauder basis* for X if for every $x \in X$ there is a unique sequence (α_n) of scalars such that $x = \sum_n \alpha_n b_n$, i.e., $\lim_{m\to\infty} \|x - \sum_{n=0}^{m} \alpha_n b_n\| = 0$.

Theorem 2.6. *Let $1 \leq p < \infty$ and define the sequence $c^{(k)} \in \ell_p(\widehat{F}(r,s))$ for every fixed $k \in \mathbb{N}$ by*

$$(c^{(k)})_n = \begin{cases} 0 & (n < k) \\ \frac{1}{r}\left(\frac{-s}{r}\right)^{k-n} \frac{f_{n+1}^2}{f_k f_{k+1}} & (n \geq k) \end{cases} \quad ; (n \in \mathbb{N}). \tag{2.4}$$

Then, the sequence $(c^{(k)})_{k=0}^{\infty}$ is a basis for the space $\ell_p(\widehat{F}(r,s))$ and every $x \in \ell_p(\widehat{F}(r,s))$ has a unique representation of the form

$$x = \sum_k \widehat{F}_k(r,s)(x)c^{(k)}. \tag{2.5}$$

Proof. Let us assume that $1 \leq p < \infty$. Then, it is not hard to verify of the relation $\widehat{F}(r,s)(c^{(k)}) = e^{(k)} \in \ell_p$ $(k \in \mathbb{N})$ by using (2.4), and hence $c^{(k)} \in \ell_p(\widehat{F}(r,s))$ for all $k \in \mathbb{N}$.

Also, let us take any given sequence $x \in \ell_p(\widehat{F}(r,s))$. For all non-negative integer m, we put

$$x^{(m)} = \sum_{k=0}^{m} \widehat{F}_k(r,s)(x)c^{(k)}.$$

Then, we have that

$$\widehat{F}(r,s)(x^{(m)}) = \sum_{k=0}^{m} \widehat{F}_k(r,s)(x)\widehat{F}(r,s)(c^{(k)}) = \sum_{k=0}^{m} \widehat{F}_k(r,s)(x)e^{(k)}$$

and hence

$$\widehat{F}_n(r,s)(x-x^{(m)}) = \begin{cases} 0, & (0 \le n \le m) \\ \widehat{F}_n(r,s)(x), & (n > m) \end{cases} \quad ; (n, m \in \mathbb{N}).$$

Now, for any given $\varepsilon > 0$ there is a non-negative integer m_0 such that

$$\sum_{n=m_0+1}^{\infty} \left|\widehat{F}_n(r,s)(x)\right|^p \le \left(\frac{\varepsilon}{2}\right)^p.$$

Therefore, we have for every $m \ge m_0$ that

$$\begin{aligned} \left\|x - x^{(m)}\right\|_{\ell_p(\widehat{F}(r,s))} &= \left(\sum_{n=m+1}^{\infty} \left|\widehat{F}_n(r,s)(x)\right|^p\right)^{1/p} \\ &\le \left(\sum_{n=m_0+1}^{\infty} \left|\widehat{F}_n(r,s)(x)\right|^p\right)^{1/p} \\ &\le \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

which shows that $\lim_{m\to\infty} \left\|x - x^{(m)}\right\|_{\ell_p(\widehat{F}(r,s))} = 0$ and hence x is represented as in (2.5).

Eventually, we must show the uniqueness of the representation (2.5) of $x \in \ell_p(\widehat{F}(r,s))$. To do this, let us assume that $x = \sum_k \mu_k(x) c^{(k)}$. Since the linear transformation T defined from $\ell_p(\widehat{F}(r,s))$ to ℓ_p, in the proof of Theorem 3.2, is continuous, we observe that

$$\widehat{F}_n(r,s)(x) = \sum_k \mu_k(x) \widehat{F}_n(r,s)(c^{(k)}) = \sum_k \mu_k(x) \delta_{nk} = \mu_n(x); \quad (n \in \mathbb{N}).$$

Therefore, the newly calculated equalities indicates that the representation (2.5) of $x \in \ell_p(\widehat{F}(r,s))$ is unique. This last step concludes the proof. □

3. The α-, β- and γ-Duals of the space $\ell_p(\widehat{F}(r,s))$

In this section, after we recall the α-, β- and γ-duals of the an arbitrary sequence space, we compute the α-, β- and γ-duals of the newly defined sequence space $\ell_p(\widehat{F}(r,s))$ of nonabsolute type.

The α-, β- and γ-duals of a sequence space X are respectively defined by

$$X^{\alpha} = \left\{ a = (a_k) \in \omega : \sum_{k=0}^{\infty} |a_k x_k| < \infty \text{ for all } x = (x_k) \in X \right\},$$

$$X^{\beta} = \left\{ a = (a_k) \in \omega : \sum_{k=0}^{\infty} a_k x_k \text{ convergent for all } x = (x_k) \in X \right\},$$

and

$$X^{\gamma} = \left\{ a = (a_k) \in \omega : \sum_{k=0}^{\infty} a_k x_k \text{ bounded for all } x = (x_k) \in X \right\},$$

[7].

We assume throughout that $p, q \geq 1$ with $p^{-1} + q^{-1} = 1$ and denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$.

The following known results [56] are fundamental for our investigation.

Lemma 3.1. *$A = (a_{nk}) \in (\ell_p, \ell_1)$ if and only if*

$$\sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} a_{nk} \right| < \infty; \quad 1 < p \leq \infty.$$

Lemma 3.2. *$A = (a_{nk}) \in (\ell_p, c)$ if and only if*

$$\lim_{n \to \infty} a_{nk} \text{ exists for all } k \in \mathbb{N}, \tag{3.1}$$

$$\sup_{n \in \mathbb{N}} \sum_k |a_{nk}|^q < \infty \;;\; 1 < p < \infty. \tag{3.2}$$

Lemma 3.3. *$A = (a_{nk}) \in (\ell_\infty, c)$ if and only if (3.1) holds and*

$$\lim_{n \to \infty} \sum_k |a_{nk}| = \sum_k \left| \lim_{n \to \infty} a_{nk} \right|.$$

Lemma 3.4. *$A = (a_{nk}) \in (\ell_p, \ell_\infty)$ if and only if (3.2) holds with $1 < p \leq \infty$.*

Since the case $p = 1$ can be proved by analogy, we omit the proof of that case and consider only the case $1 < p \leq \infty$ in the proof of Theorems 3.5 and 4.6, respectively.

Theorem 3.5. *The α-dual of the space $\ell_p(\widehat{F}(r, s))$ is the set*

$$\widehat{d_1}(r, s) = \left\{ a = (a_k) \in \omega : \sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} \frac{1}{r} \left(\frac{-s}{r} \right)^{n-k} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n \right|^q < \infty \right\},$$

where $1 < p \leq \infty$.

Proof. Let us assume that $1 < p \leq \infty$. Consider any sequence $a = (a_n) \in \omega$, define the matrix $B = (b_{nk})$ by

$$b_{nk} = \begin{cases} \frac{1}{r} \left(\frac{-s}{r} \right)^{n-k} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n & (0 \leq k \leq n) \\ 0 & (k > n) \end{cases}$$

for all $n, k \in \mathbb{N}$. Additionally, for every $x = (x_n) \in \omega$ we put $y = \widehat{F}x$. Thus, it is obtained by (2.2) that

$$a_n x_n = \sum_{k=0}^{n} \frac{1}{r} \left(\frac{-s}{r} \right)^{n-k} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n y_k = B_n(y); \quad (n \in \mathbb{N}). \tag{3.3}$$

Hence, we observe by (3.3) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in \ell_p(\widehat{F}(r, s))$ iff $By \in \ell_1$ whenever $y \in \ell_p$. Thus, we derive by using Lemma 3.1 that

$$\sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} \frac{1}{r} \left(\frac{-s}{r} \right)^{n-k} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n \right|^q < \infty$$

which results in that $(\ell_p(\widehat{F}(r, s)))^\alpha = \widehat{d_1}(r, s)$. $\square$

Theorem 3.6. *Define the sets* $\widehat{d_2}(r,s)$, $\widehat{d_3}(r,s)$ *and* $\widehat{d_4}(r,s)$ *by*

$$\widehat{d_2}(r,s) = \left\{ a = (a_k) \in \omega : \sum_{j=k}^{\infty} \frac{1}{r}\left(\frac{-s}{r}\right)^{j-k} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \text{ exists for all } k \in \mathbb{N} \right\},$$

$$\widehat{d_3}(r,s) = \left\{ a = (a_k) \in \omega : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \frac{1}{r}\left(\frac{-s}{r}\right)^{j-k} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right|^q < \infty \right\},$$

and

$$\begin{aligned}\widehat{d_4}(r,s) &= \left\{ a = (a_k) \in \omega : \lim_{n\to\infty} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \frac{1}{r}\left(\frac{-s}{r}\right)^{j-k} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right| \right\} \\ &= \left\{ a = (a_k) \in \omega : \sum_{k} \left| \sum_{j=k}^{\infty} \frac{1}{r}\left(\frac{-s}{r}\right)^{j-k} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right| < \infty \right\}.\end{aligned}$$

Then $(\ell_p(\widehat{F}(r,s)))^\beta = \widehat{d_2}(r,s) \cap \widehat{d_3}(r,s)$ *and* $(\ell_\infty(\widehat{F}(r,s)))^\beta = \widehat{d_2}(r,s) \cap \widehat{d_4}(r,s)$, *where* $1 < p < \infty$.

Proof. The proof is reached by considering the definition of $\beta-$dual. For this, let us assume that $a = (a_k) \in \omega$ and after then we compute

$$\begin{aligned}\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} a_k \left(\sum_{j=0}^{k} \frac{1}{r}\left(\frac{-s}{r}\right)^{k-j} \frac{f_{k+1}^2}{f_j f_{j+1}} y_j \right) \\ &= \sum_{k=0}^{n} \left(\sum_{j=k}^{n} \frac{1}{r}\left(\frac{-s}{r}\right)^{j-k} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right) y_k \qquad (3.4) \\ &= D_n(y),\end{aligned}$$

where $D = (d_{nk})$ is defined by

$$d_{nk} = \begin{cases} \sum\limits_{j=k}^{n} \frac{1}{r}\left(\frac{-s}{r}\right)^{j-k} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j & (0 \le k \le n) \\ 0 & (k > n) \end{cases} ; \quad n, k \in \mathbb{N}.$$

In that case, we have the right to say, from Lemma 3.2 with (3.4), that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in \ell_p(\widehat{F}(r,s))$ if and only if $Dy \in c$ whenever $y = (y_k) \in \ell_p$. Then, $(a_k) \in (\ell_p(\widehat{F}(r,s)))^\beta$ if and only if $(a_k) \in \widehat{d_2}(r,s)$ and $(a_k) \in \widehat{d_3}(r,s)$ by (3.1) and (3.2), respectively. Hence, $(\ell_p(\widehat{F}(r,s)))^\beta = \widehat{d_2} \cap \widehat{d_3}$. It is clear that one can also prove the case $p = \infty$ by the same technique used in the proof of the case $1 < p < \infty$ with Lemma 3.3 instead of Lemma 3.2. So, we leave the detailed proof to the reader. □

Theorem 3.7. $(\ell_p(\widehat{F}(r,s)))^\gamma = \widehat{d_3}(r,s)$, *where* $1 < p \le \infty$.

Proof. According to the definition of $\gamma-$dual, this proof can be directly reached by using both (3.4) and Lemma 3.4. □

4. Some matrix transformations related to the sequence space $\ell_p(\widehat{F}(r,s))$

In this section, we focus on the characterize the classes $(\ell_p(\widehat{F}(r,s)), X)$, in which $1 \leq p \leq \infty$ and $X \in \{\ell_\infty, \ell_1, c, c_0\}$.

For simplicity in notation, we write

$$\widetilde{a}_{nk} = \sum_{j=k}^{\infty} \frac{1}{r} \left(\frac{-s}{r}\right)^{j-k} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj}$$

for all $k, n \in \mathbb{N}$.

The following lemma is essential for our results.

Lemma 4.1. *(see [32], Theorem 4.1]) Let λ be an FK-space, U be a triangle, V be its inverse and μ be arbitrary subset of ω. Then we have $A = (a_{nk}) \in (\lambda_U, \mu)$ if and only if*

$$C^{(n)} = \left(c_{mk}^{(n)}\right) \in (\lambda, c) \quad \textit{for all } n \in \mathbb{N}$$

and

$$C = (c_{nk}) \in (\lambda, \mu),$$

where $c_{mk}^{(n)} = \begin{cases} \sum_{j=k}^{m} a_{nj} v_{jk} & (0 \leq k \leq m) \\ 0 & (k > m) \end{cases}$ *and* $c_{nk} = \sum_{j=k}^{\infty} a_{nj} v_{jk}$ *for all* $k, m, n \in \mathbb{N}$.

Now, we list the following conditions.

$$\sup_{m \in \mathbb{N}} \sum_{k=0}^{m} \left| \sum_{j=k}^{m} \frac{1}{r} \left(\frac{-s}{r}\right)^{j-k} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} \right|^q < \infty \tag{4.1}$$

$$\lim_{m \to \infty} \sum_{j=k}^{m} \frac{1}{r} \left(\frac{-s}{r}\right)^{j-k} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} = \widetilde{a}_{nk} \ ; \forall n, k \in \mathbb{N} \tag{4.2}$$

$$\lim_{m \to \infty} \sum_{k=0}^{m} \left| \sum_{j=k}^{m} \frac{1}{r} \left(\frac{-s}{r}\right)^{j-k} \frac{f_{j+1}^2}{f_k f_{jk+1}} a_{nj} \right| = \sum_k |\widetilde{a}_{nk}| \quad \text{for each } n \in \mathbb{N} \tag{4.3}$$

$$\sup_{n \in \mathbb{N}} \sum_k |\widetilde{a}_{nk}|^q < \infty \tag{4.4}$$

$$\sup_{N \in \mathcal{F}} \sum_k \left| \sum_{n \in \mathbb{N}} \widetilde{a}_{nk} \right|^q < \infty \tag{4.5}$$

$$\lim_{n \to \infty} \widetilde{a}_{nk} = \widetilde{\alpha}_k \ ; k \in \mathbb{N} \tag{4.6}$$

$$\lim_{n \to \infty} \sum_k |\widetilde{a}_{nk}| = \sum_k |\widetilde{\alpha}_k| \tag{4.7}$$

$$\lim_{n \to \infty} \sum_k \widetilde{a}_{nk} = 0 \tag{4.8}$$

$$\sup_{n,k \in \mathbb{N}} |\widetilde{a}_{nk}| < \infty \tag{4.9}$$

$$\sup_{k,m\in\mathbb{N}}\left|\sum_{j=k}^{m}\frac{1}{r}\left(\frac{-s}{r}\right)^{j-k}\frac{f_{j+1}^{2}}{f_k f_{k+1}}a_{nj}\right|<\infty \tag{4.10}$$

$$\sup_{k\in\mathbb{N}}\sum_{n}|\widetilde{a}_{nk}|<\infty \tag{4.11}$$

$$\sup_{N,K\in\mathcal{F}}\left|\sum_{n\in N}\sum_{k\in K}\widetilde{a}_{nk}\right|<\infty. \tag{4.12}$$

Then, by combining Lemma 4.1 with the results in [56], we immediately derive the following results.

Theorem 4.2. *(a) $A=(a_{nk})\in(\ell_1(\widehat{F}(r,s)),\ell_\infty)$ if and only (4.2), (4.9) and (4.10) hold.*

(b) $A=(a_{nk})\in(\ell_1(\widehat{F}(r,s)),c)$ if and only if (4.2), (4.6), (4.9) and (4.10) hold.

(c) $A=(a_{nk})\in(\ell_1(\widehat{F}(r,s)),c_0)$ if and only if (4.2), (4.6) with $\widetilde{\alpha}_k=0$, (4.9) and (4.10) hold.

(d) $A=(a_{nk})\in(\ell_1(\widehat{F}(r,s)),\ell_1)$ if and only (4.2), (4.10) and (4.11) hold.

Theorem 4.3. *Let $1<p<\infty$. Then, we have*

(a) $A=(a_{nk})\in(\ell_p(\widehat{F}(r,s)),\ell_\infty)$ if and only if (4.1), (4.2) and (4.4) hold.

(b) $A=(a_{nk})\in(\ell_p(\widehat{F}(r,s)),c)$ if and only if (4.1), (4.2), (4.4) and (4.6) hold.

(c) $A=(a_{nk})\in(\ell_p(\widehat{F}(r,s)),c_0)$ if and only if (4.1), (4.2), (4.4) and (4.6) with $\widetilde{\alpha}_k=0$ hold.

(d) $A=(a_{nk})\in(\ell_p(\widehat{F}(r,s)),\ell_1)$ if and only if (4.1), (4.2) and (4.5) hold.

Theorem 4.4. *(a) $A=(a_{nk})\in(\ell_\infty(\widehat{F}(r,s)),\ell_\infty)$ if and only (4.2), (4.3) and (4.4) with $q=1$ hold.*

(b) $A=(a_{nk})\in(\ell_\infty(\widehat{F}(r,s)),c)$ if and only (4.2), (4.3), (4.6) and (4.7) hold.

(c) $A=(a_{nk})\in(\ell_\infty(\widehat{F}(r,s)),c_0)$ if and only (4.2), (4.3) and (4.8) hold.

(d) $A=(a_{nk})\in(\ell_\infty(\widehat{F}(r,s)),\ell_1)$ if and only (4.2), (4.3) and (4.12) hold.

5. Some geometric properties of the space $\ell_p(\widehat{F}(r,s))$ $(1<p<\infty)$

In this section, we study some geometric properties of the space $\ell_p(\widehat{F}(r,s))$ for $1<p<\infty$.

For these properties, one can see [21, 22, 26, 27, 33, 34, 41, 43, 44, 53].

A Banach space X is said to have the *Banach-Saks property* if every bounded sequence (x_n) in X admits subsequence (z_n) such that the sequence $\{t_k(z)\}$ is convergent in the norm in X [43], where

$$t_k(z)=\frac{1}{k+1}(z_0+z_1+...+z_k);\quad(k\in\mathbb{N}). \tag{5.1}$$

A Banach space X is said to have the *weak Banach-Saks property* whenever given any weakly null sequence $(x_n)\subset X$ and there exists a subsequence (z_n) of (x_n) such that the sequence $\{t_k(z)\}$ strongly convergent to zero.

In [43], García-Falset introduce the following coefficient:

$$R(X) = \sup\left\{\liminf_{n\to\infty} \|x_n - x\| : (x_n) \subset B(X),\ x_n \xrightarrow{w} 0,\, x \in B(X)\right\},$$

where $B(X)$ denotes the unit ball of X.

Remark 5.1. A Banach space X with $R(X) < 2$ has the weak fixed point property [27].

Let $1 < p < \infty$. A Banach space is said to have the *Banach-Saks type* p or property $(BS)_p$, if every weakly null sequence (x_k) has a subsequence (x_{k_j}) such that for some $C > 0$,

$$\left\|\sum_{j=0}^{n} x_{k_j}\right\| < C(n+1)^{1/p}, \tag{5.3}$$

for all $n \in \mathbb{N}$ (see [34]).

For a normed linear space E, Gurarii's modulus of convexity is defined by

$$\beta_{(E)}(\varepsilon) = \inf\left\{1 - \inf_{0\le\alpha\le1} \|\alpha x + (1-\alpha)y\| : x, y \in S(E), \|x-y\| = \varepsilon\right\},$$

where $S(E)$ denotes the unit sphere of E and $0 \le \varepsilon \le 2$ [29, 47]. Now, we give some geometric properties of the space $\ell_p(\widehat{F}(r,s))$, where $1 < p < \infty$.

Theorem 5.2. *The space $\ell_p(\widehat{F}(r,s))$ has the Banach-Saks type p, where $1 < p < \infty$.*

Proof. Let (ε_n) be a sequence of positive numbers such that $\sum \varepsilon_n \le 1/2$, and (x_n) be a weakly null sequence in $B(\ell_p(\widehat{F}(r,s)))$. Set $z_0 = x_0 = 0$ and $z_1 = x_{n_1} = x_1$. Then, there exists $k_1 \in \mathbb{N}$ such that

$$\left\|\sum_{i=k_1+1}^{\infty} z_1(i)e^{(i)}\right\|_{\ell_p(\widehat{F}(r,s))} < \varepsilon_1.$$

Since (x_n) is weakly null sequence implies $x_n \to 0$ coordinatewise, there is an $n_2 \in \mathbb{N}$ such that

$$\left\|\sum_{i=0}^{k_1} x_n(i)e^{(i)}\right\|_{\ell_p(\widehat{F}(r,s))} < \varepsilon_1$$

for $n \ge n_2$. Set $z_2 = x_{n_2}$. Then, there exists an $k_2 > k_1$ such that

$$\left\|\sum_{i=k_2+1}^{\infty} z_2(i)e^{(i)}\right\|_{\ell_p(\widehat{F}(r,s))} < \varepsilon_2.$$

Again using the fact that $x_n \to 0$ coordinatewise, there exists an $n_3 \ge n_2$ such that

$$\left\|\sum_{i=0}^{k_2} x_n(i)e^{(i)}\right\|_{\ell_p(\widehat{F}(r,s))} < \varepsilon_2$$

for $n \ge n_3$.

If we continue this process, we can find two increasing subsequences (k_i) and (n_i) such that

$$\left\| \sum_{i=0}^{k_j} x_n(i) e^{(i)} \right\|_{\ell_p(\widehat{F}(r,s))} < \varepsilon_j$$

for each $n \geq n_{j+1}$ and

$$\left\| \sum_{i=k_j+1}^{\infty} z_j(i) e^{(i)} \right\|_{\ell_p(\widehat{F}(r,s))} < \varepsilon_j,$$

where $z_j = x_{n_j}$. Hence,

$$\begin{aligned} \left\| \sum_{j=0}^{n} z_j \right\|_{\ell_p(\widehat{F}(r,s))} &= \left\| \sum_{j=0}^{n} \left(\sum_{i=0}^{k_{j-1}} z_j(i) e^{(i)} + \sum_{i=k_{j-1}+1}^{k_j} z_j(i) e^{(i)} + \sum_{i=k_j+1}^{\infty} z_j(i) e^{(i)} \right) \right\|_{\ell_p(\widehat{F}(r,s))} \\ &\leq \left\| \sum_{j=0}^{n} \left(\sum_{i=k_{j-1}+1}^{k_j} z_j(i) e^{(i)} \right) \right\|_{\ell_p(\widehat{F}(r,s))} + 2 \sum_{j=0}^{n} \varepsilon_j. \end{aligned}$$

On the other hand, it can be seen that $\|x\|_{\ell_p(\widehat{F}(r,s))} < 1$. Therefore, we have

$$\begin{aligned} \left\| \sum_{j=0}^{n} \left(\sum_{i=k_{j-1}+1}^{k_j} z_j(i) e^{(i)} \right) \right\|_{\ell_p(\widehat{F}(r,s))}^p &= \sum_{j=0}^{n} \sum_{i=k_{j-1}+1}^{k_j} \left| r \frac{f_i}{f_{i+1}} z_j(i) + s \frac{f_{i+1}}{f_i} z_j(i-1) \right|^p \\ &\leq \sum_{j=0}^{n} \sum_{i=0}^{\infty} \left| r \frac{f_i}{f_{i+1}} z_j(i) + s \frac{f_{i+1}}{f_i} z_j(i-1) \right|^p \\ &\leq (n+1). \end{aligned}$$

Hence, we obtain

$$\left\| \sum_{j=0}^{n} \left(\sum_{i=k_{j-1}+1}^{k_j} z_j(i) e^{(i)} \right) \right\| \leq (n+1)^{1/p}.$$

By using the fact that $1 \leq (n+1)^{1/p}$ for all $n \in \mathbb{N}$ and $1 < p < \infty$, we have

$$\left\| \sum_{j=0}^{n} z_j \right\|_{\ell_p(\widehat{F}(r,s))} \leq (n+1)^{1/p} + 1 \leq 2(n+1)^{1/p}$$

which means that $\ell_p(\widehat{F}(r,s))$ has the Banach-Saks type p. □

Remark 5.3. Since $\ell_p(\widehat{F}(r,s))$ is linearly isomorphic to ℓ_p, we have $R(\ell_p(\widehat{F}(r,s))) = R(\ell_p) = 2^{1/p}$.

From Remarks (5.1) and (5.3), we have:

Theorem 5.4. *The space $\ell_p(\widehat{F}(r,s))$ has the weak fixed point property, where $1 < p < \infty$.*

Theorem 5.5. *Let $1 \leq p < \infty$. The modulus of convexity for the space $\ell_p(\widehat{F}(r,s))$ is*

$$\beta_{\ell_p(\widehat{F}(r,s))}(\varepsilon) \leq 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{1/p},$$

where $0 \leq \varepsilon \leq 2$.

Proof. For $x \in \ell_p(\widehat{F}(r,s))$, we have

$$\|x\|_{\ell_p(\widehat{F}(r,s))} = \|\widehat{F}(r,s)x\|_p = \left(\sum_n |\widehat{F}_n(r,s)|^p\right)^{1/p}.$$

Let $0 \leq \varepsilon \leq 2$ and consider the following sequences:

$$u = \left(\widehat{F}^{-1}(r,s)(1-(\varepsilon/2)^p)^{1/p}, \widehat{F}^{-1}(r,s)(\varepsilon/2), 0, 0, ...\right)$$

and

$$v = \left(\widehat{F}^{-1}(r,s)(1-(\varepsilon/2)^p)^{1/p}, \widehat{F}^{-1}(r,s)(-\varepsilon/2), 0, 0, ...\right),$$

where $\widehat{F}^{-1}(r,s)$ is the inverse of the matrix $\widehat{F}(r,s)$. Clearly, the $\widehat{F}(r,s)$-transforms of u and v are as follows:

$$\widehat{F}(r,s)u = \left((1-(\varepsilon/2)^p)^{1/p}, \varepsilon/2, 0, 0, ...\right)$$

and

$$\widehat{F}(r,s)v = \left((1-(\varepsilon/2)^p)^{1/p}, -\varepsilon/2, 0, 0, ...\right).$$

Then, $\|\widehat{F}(r,s)u\|_p = \|u\|_{\ell_p(\widehat{F}(r,s))} = 1$ and $\|\widehat{F}(r,s)v\|_p = \|v\|_{\ell_p(\widehat{F}(r,s))} = 1$, that is, $u, v \in S(\ell_p(\widehat{F}(r,s)))$. Also, we have $\|\widehat{F}(r,s)u - \widehat{F}(r,s)v\|_p = \|u-v\|_{\ell_p(\widehat{F}(r,s))} = \varepsilon$. For $0 \leq \alpha \leq 1$, we have

$$\begin{aligned}\|\alpha u + (1-\alpha)v\|^p_{\ell_p(\widehat{F}(r,s))} &= \|\alpha\widehat{F}(r,s)u + (1-\alpha)\widehat{F}(r,s)v\|^p_p \\ &= 1 - \left(\frac{\varepsilon}{2}\right)^p + |2\alpha-1|^p\left(\frac{\varepsilon}{2}\right)^p.\end{aligned}$$

Hence, we obtain

$$\inf_{0\leq\alpha\leq1} \|\alpha u + (1-\alpha)v\|_{\ell_p(\widehat{F}(r,s))} = \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{1/p}.$$

Consequently, for $u, v \in S(\ell_p(\widehat{F}(r,s)))$ with $\|\widehat{F}(r,s)u - \widehat{F}(r,s)v\|_p = \|u-v\|_{\ell_p(\widehat{F}(r,s))} = \varepsilon$, we have

$$\beta_{\ell_p(\widehat{F}(r,s))}(\varepsilon) \leq 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{1/p},$$

where $1 \leq p < \infty$. Thus the proof is completed. $\square$

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