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BLOW-UP DIFFUSION IN SINGULAR ELLIPTIC PROBLEMS

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Abstract. In this paper, we study a class of nonlinear singular elliptic problems involving a diffusion operator with blow-up behaviour as the solution approaches a finite threshold value. The combination of the singular lower-order term, the blow-up diffusion, and low-regularity data gives rise to significant analytical difficulties. Using the framework of renormalized solutions, we establish the existence of solutions under suitable assumptions on the nonlinear operator and the singular term. We also derive some regularity properties of the obtained solutions. These results contribute to the study of singular elliptic problems with blow-up diffusion and low-regularity data.

Keywords. Singular elliptic equation, Renormalized solutions, Blow-up diffusion, Existence and regularity of solutions.
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1. Introduction

Given Ω a bounded open set of $\mathbb{R}^N$ ($N \geq 2$). This work is devoted to the study of the existence and regularity of renormalized solutions to the following nonlinear singular elliptic problem with blowing-up. diffusion:

$$\begin{cases} -\operatorname{div}\big(\Gamma(x,u,Du)\big) = \dfrac{f}{u^\theta} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

A Carathéodory function defines the nonlinear operator

$$\Gamma : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N. \tag{1.2}$$

For a structural constraint, we require the usual assumptions on ellipticity of the diffusion operator Γ (with degeneration in u), precisely We assume that there exists a continuous positive function $g : (-\infty, m) \longrightarrow \mathbb{R}^+$ satisfying

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$$g^{p-1}(s) \geq \lambda > 0 \quad \text{for all } s < m, \qquad \lim_{s \to m^-} g(s) = +\infty, \tag{1.3}$$

together with

$$\int_0^m g(s)\, ds < +\infty. \tag{1.4}$$

such that

$$\Gamma(x, s, \xi) \cdot \xi \geq g(s)^{p-1} |\xi|^p, \qquad \Gamma(x, s, 0) = 0, \tag{1.5}$$

and the monotonicity assumption

$$\big(\Gamma(x, s, \xi) - \Gamma(x, s, \xi')\big) \cdot (\xi - \xi') \geq 0,$$

as well as

$$|\Gamma(x, s, \xi)| \leq \beta \left(L(x) + g(s)^{p-1} |\xi|^{p-1} \right), \tag{1.6}$$

for a.e. $x \in \Omega$, for every $s \in \mathbb{R}$ and $\xi, \xi' \in \mathbb{R}^N$, L is a nonnegative function in $L^{p'}(\Omega)$ and $\beta > 0$.

$$f \in L^1(\Omega), \quad f \not\equiv 0, \quad f \geq 0 \text{ a.e. in } \Omega. \tag{1.7}$$

and θ is a positive constant.

Remark 1.1. Studying the problem 1.1 under assumptions (1.3)- (1.4) becomes complicated; however, if we replace them with $\int_0^m g(s)ds = +\infty$, which forces the solution to avoid the value m.

Equations of type (1.1) are used in turbulence models derived from the Navier-Stokes equations that describe the motion of a fluid in a region. The Rynolds stress tensor

$$R = \nu_{turb}(x, .) \left(\nabla \bar{\omega} + \bar{\omega}^T \right) + qI,$$

see [12] and [Sect. 4, [13]] $\nu_{turb}(x, .)$ still remains indetermined, for some detailed description of $k - \varepsilon$-models, we refer to the monograph [Chapter 4, p. 51, [13]]. Problems 1.1 without blow-up (namely without assumptions (1.4) and (1.5), have been extensively studied in the literature; more specifically, results regarding existence, uniqueness, and regularity without assuming the conditions (1.4) and (1.5) can be found in [2, 3, 4, 6, 7, 11, 14, 16, 17, 21]. Moreover, in the non-singular case, it has been studied in [19, 18].

Remark 1.2. The condition $\lim_{s \to m^-} g(s) = +\infty$ implies that the operator exhibits an explosive behavior as s approaches the level m. This creates a natural barrier effect, preventing admissible solutions from reaching this value.

We aim to analyze equations (1.1), which combine a singular behavior near zero with the behavior of solutions on the set $\{x \in \Omega : u = m\}$, which may be non-negligible and taking into account the L^1-datum.

We first establish the existence of a positive renormalized solution under the above structural assumptions. The proof relies on truncation techniques, monotonicity arguments, compactness methods, and suitable a priori estimates adapted

to the singular nature of the problem.
The first challenge arises from the assumption that $f \in L^1(\Omega)$. To overcome this, we work within the framework of renormalized solutions adapted to our equation. This concept was introduced by DiPerna and Lions [10] in their study of the Boltzmann equation; see also Lions [20] for several applications to fluid mechanics models. For elliptic problems, we refer the reader to [1, 6, 14, 15], and for parabolic equations to [5, 9, 7].
The second challenge stems from the behavior of the function $\Gamma(x, s, \xi)$ as $s \to m^-$. This causes difficulties in defining the meaning of a on the set $\{x \in \Omega : u(x) = m\}$.

In what follows, we present the adapted renormalized solution to Problem (1.1).

Definition 1.3. A measurable function u defined on Ω is a renormalized solution of problem (1.1) if

$$T_k(u) \in W_0^{1,p}(\Omega), \quad \frac{f}{u^\theta} \in L^1_{\text{loc}}(\Omega), \tag{1.8}$$

$$0 < u \leq m, \quad \text{a.e. in } \Omega, \tag{1.9}$$

$$\Gamma(x, T_m^k(u), DT_m^k(u))\chi_{\{u<m\}} \in (L^{p'}(\Omega))^N, \tag{1.10}$$

$$\lim_{s\to+\infty} \int\limits_{\{-s-1\leq u(x)\leq -s\}} \Gamma(x, u, Du)Du dx = 0, \tag{1.11}$$

$$\lim_{\delta\to 0}\frac{1}{\delta} \int\limits_{\{m-2\delta\leq u(x)\leq m-\delta\}} \Gamma(x, u, Du)Du dx = \int\limits_{\{u(x)=m\}} \frac{f}{u^\theta} dx, \tag{1.12}$$

and, for every function S in $W^{1,\infty}(\mathbb{R})$ such that $supp(S)$ is compact and $S(m) = 0$, u satisfies

$$\int\limits_{\Omega} \Gamma(x, u, Du)D(S(u)\varphi)dx = \int\limits_{\Omega} \frac{f}{u^\theta} S(u)\varphi dx, \tag{1.13}$$

for every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Remark 1.4. Conditions (1.8) and (1.10) to show that all term in (1.13) are well defined. The assumption (1.12), has been established in [8].

Let

$$T_\varepsilon^k(s) = \begin{cases} -k, & \text{if } s \leq -k, \\ r, & \text{if } -k \leq r \leq \varepsilon, \\ \varepsilon, & \text{if } \quad r \geq \varepsilon, \end{cases}$$

for For any $k, \varepsilon > 0$, and
For a fixed $l \geq 1$,

$$\sigma_l(s) = T_1(r - T_l(r)), \; \phi_l(s) = 1 - |\sigma_l(s)|,$$

for all $s \in \mathbb{R}$.
The rest of the paper is structured as follows. In Sect 2, we establish the existence of renormalized solutions to Problem 1.1 using Schauder's fixed-point theorem, truncation, and monotonicity arguments. We then present regularity results in

Theorems 2.5–2.6, depending on the parameter θ.

2. Main finding

In what follows, we will present our first main result regarding the existence of renormalized solutions.

Theorem 2.1. *Under assumptions (1.2)-(1.7), there existe a renormalized solution u of problem (1.1).*

Proof of Theorem 2.1: The proof is divided into three main steps.

2.1. Step 1: Regularization and apriori estimate.

Fix $n \geq 1$, let

$$g_n(r) = g(T^n_{m-1/n}(r)),$$

$$\Gamma^n(x, s, \xi) = \Gamma(x, T^n_{m-1/n}(s), \xi),$$

$$f_n \in L^\infty(\Omega), \text{ such that } f_n \to f \text{ strongly in } L^1(\Omega), \text{ as } n \text{ tends to } +\infty, \tag{2.1}$$

Let us now consider the following regularized problem

$$\begin{cases} -\text{div}\,(\Gamma^n(x, u^n, Du^n)) = \frac{f_n}{(u^n + \frac{1}{n})^\theta} \text{ in } \Omega, \\ u^n = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.2}$$

Proof. Let $n \in \mathbb{N}$ be a fixed and let v be a function in $L^p\Omega)$, we know the following non singular problem

$$\begin{cases} -\text{div}\,(\Gamma^n(x, w, Dw)) = \frac{f_n}{(|v| + \frac{1}{n})^\theta} \text{ in } \Omega, \\ u^n = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.3}$$

The problem (2.3) has a unique solution, which we invite the reader to see ([17]). Define

$$G : L^p(\Omega) \to L^p(\Omega),$$

and let $w = G(v)$ to be the unique solution of (2.3). Taking w as a test function, we obtain

$$\lambda \int_\Omega |Dw|^p \leq \int_\Omega \Gamma^n(x, w, Dw)Dwdx = \int_\Omega \frac{f_n\, w}{\left(|v| + \frac{1}{n}\right)^\theta} \leq n^{\theta+1} \int_\Omega |w|.$$

Using the Sobolev inequality for the left-hand side together with Hölder's inequality for the right-hand side, we obtain

$$\left(\int_\Omega |w|^{p^*}\right)^{\frac{p}{p^*}} \leq Cn^{\theta+1} \left(\int_\Omega |w|^{p^*}\right)^{\frac{1}{p^*}},$$

for some constant C independent of v. This implies

$$\|w\|_{L^{p^*}(\Omega)} \leq \left(Cn^{\theta+1}\right)^{\frac{1}{p-1}} = C_n,$$

so that the ball of $L^{p^*}(\Omega)$ of radius C_n is invariant for G. Using the Sobolev embedding, it is easy to prove that G is continuous and compact. Both operators are continuous and compact on $L^{p^*}(\Omega)$. Therefore, by Schauder's fixed point theorem, there exists $u_n \in W_0^{1,p}(\Omega)$ such that $u_n = S(u_n)$. In other words, u_n is a weak solution of

$$\begin{cases} -\operatorname{div}\left(\Gamma^n(x, u^n, Du^n)\right) = \dfrac{f_n}{\left(|u^n| + \frac{1}{n}\right)^\theta} & \text{in } \Omega, \\ u^n = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Using the test function $u^{n-} = \min\{u^n, 0\}$, we obtain $u^n \geq 0$. Since the right-hand side of (2.4) belongs to $L^\infty(\Omega)$, and proceeding as in [22], we deduce that u^n is bounded in $L^\infty(\Omega)$ (although its $L^\infty(\Omega)$-norm may depend on n). □

Lemma 2.2. *Let the assumptions of Theorem 2.1 be satisfied. Consequently, $\{u^n\}$ forms an increasing sequence, and every element of the sequence remains strictly positive in Ω, and for every $\omega \subset\subset \Omega$ there exists $c_\omega > 0$ (independent of n) such that*

$$u^n(x) \geq c_\omega > 0, \qquad \forall x \in \Omega, \quad \forall n \in \mathbb{N}. \tag{2.5}$$

Moreover, there exists a pointwise limit $u \geq c_\omega$ of the sequence u^n.

Proof. Since $\theta > 0$ and $0 \leq f_n \leq f_{n+1}$, we subtract the weak formulations satisfied by u^n and u^{n+1}. For every $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geq 0$, we obtain

$$\begin{aligned} &\int_\Omega \Big(\Gamma^n(x, u^n, Du^n) - \Gamma^{n+1}(x, u^{n+1}, Du^{n+1})\Big) \cdot \nabla\varphi \, dx \\ &= \int_\Omega \left(\frac{f_n}{\left(u^n + \frac{1}{n}\right)^\theta} - \frac{f_{n+1}}{\left(u^{n+1} + \frac{1}{n+1}\right)^\theta} \right) \varphi \, dx \\ &\leq \int_\Omega f_{n+1} \left[\frac{\left(u^{n+1} + \frac{1}{n+1}\right)^\theta - \left(u^n + \frac{1}{n+1}\right)^\theta}{\left(u^n + \frac{1}{n}\right)^\theta \left(u^{n+1} + \frac{1}{n+1}\right)^\theta} \right] \varphi \, dx. \end{aligned} \tag{2.6}$$

Choosing $\varphi = (u^n - u^{n+1})_+$, and using the monotonicity assumption (1.5), we obtain

$$\lambda \int_\Omega |\nabla(u^n - u^{n+1})_+|^p \, dx \leq \int_\Omega f_{n+1} \left[\frac{\left(u^{n+1} + \frac{1}{n+1}\right)^\theta - \left(u^n + \frac{1}{n+1}\right)^\theta}{\left(u^n + \frac{1}{n}\right)^\theta \left(u^{n+1} + \frac{1}{n+1}\right)^\theta} \right] (u^n - u^{n+1})_+ \, dx.$$

Since $\theta \geq 0$ and $f_{n+1} \geq 0$, we have

$$\left[\left(u^{n+1} + \frac{1}{n+1}\right)^\theta - \left(u^n + \frac{1}{n+1}\right)^\theta \right] (u^n - u^{n+1})_+ \leq 0.$$

Therefore,

$$\lambda \int_\Omega |\nabla(u^n - u^{n+1})_+|^p \, dx \leq 0,$$

which yields

$$(u^n - u^{n+1})_+ = 0 \quad \text{a.e. in } \Omega.$$

Hence,

$$u^n \leq u^{n+1} \quad \text{a.e. in } \Omega,$$

for every $n \in \mathbb{N}$.

We first note that $u^1 \in L^\infty(\Omega)$. Hence, there exists a positive constant c_0, depending only on Ω and N, such that

$$\|u^1\|_{L^\infty(\Omega)} \leq c\,\|f_1\|_{L^\infty(\Omega)} \leq c_0.$$

Consequently,

$$-\operatorname{div}\big(\Gamma(x,u^1,\nabla u^1)\big) = \frac{f_1}{(u^1+1)^\theta} \geq \frac{f_1}{(c_0+1)^\theta} \geq 0 \quad \text{in } \Omega.$$

Since

$$\frac{f_1}{(c_0+1)^\theta} \not\equiv 0,$$

the strong maximum principle (see [22]) implies that

$$u^1 > 0 \quad \text{in } \Omega.$$

Moreover, since (u_n) is nondecreasing, we have

$$u_n \geq u^1 \quad \text{for every } n \in \mathbb{N}.$$

Therefore, estimate (2.5) follows for every u^n, with the same constant c_ω, which is independent of n.

By pluging the test function $T_k(u^n)$ in (2.2), we obtain

$$\int_\Omega \Gamma^n(x,u^n,Du^n)DT_k(u^n)dx = \int_\Omega \frac{f_n}{(u^n+\frac{1}{n})^\theta}T_k(u^n)dx.$$

And by (1.3) and (1.5), we get

$$\int_\Omega |DT_k(u^n)|^p\,dx < C, \tag{2.7}$$

for some $C>0$.

Then

$$u^n \to u \quad \text{a.e. in } \Omega, \tag{2.8}$$

for subsequence (u^n). Thus,

$$\begin{cases} T_k(u^n) \to T_k(u) & \text{a.e. in } \Omega \\ T_k(u^n) \to T_k(u) & \text{in } L^p(\Omega). \\ T_k(u^n) \rightharpoonup T_k(u) & \text{in } W_0^{1,p}(\Omega) \end{cases} \tag{2.9}$$

Choosing the function $Z^n = \int_0^{T_k^m(u^n)} g_n(s)ds$ in (2.2), we get

$$\int_\Omega \Gamma^n(x,u^n,Du^n)DZ^n(u^n)dx = \int_\Omega \frac{f_n}{(u^n+\frac{1}{n})^\theta}Z^n(u^n)dx. \tag{2.10}$$

By virtue of assumption (1.5) on Γ^n and assumption (1.3), it follows that

$$\int_\Omega |DZ^n(u^n)|^p\,dx < C. \tag{2.11}$$

So

$$|\Gamma^n(x,T_k^m(u^n),DT_k^m(u^n))| \leq \beta\left[L(x) + |DZ^n(u^n)|^{p-1}\right], \tag{2.12}$$

for all $k>0$.By combining the assumptions on the function L with estimate (2.12), we obtain

$$\Gamma^n(x,T_k^m(u^n),DT_k^m(u^n)) \text{ is bounded in } L^{p'}(\Omega).$$

As a consequence, there exist $\varphi_k \in L^{p'}(\Omega)$ such as

$$\Gamma^n(x, T_k^m(u), DT_k^m(u^n)) \rightharpoonup \varphi_k \text{ weakly in } L^{p'}(\Omega). \tag{2.13}$$

Using $T_{2m}^+(u^n) - T_m^+(u^n)$ in (2.2) gives

$$\int_\Omega \Gamma^n(x, u^n, Du^n).D\left(T_{2m}^+(u^n) - T_m^+(u^n)\right) dx = \int_\Omega \frac{f_n}{(u^n + \frac{1}{n})^\theta}\left(T_{2m}^+(u^n) - T_m^+(u^n)\right) dx,$$

thanks to (1.5), we have

$$g(m - \frac{1}{n})^{p-1} \int_\Omega \left|D\left(T_{2m}^+(u^n) - T_m^+(u^n)\right)\right|^p dx \le C, \tag{2.14}$$

Passing to the limit as $n \to +\infty$ in (2.14), we deduce that

$$\begin{array}{rcll} T_{2m}^+(u) - T_m^+(u) & = & 0 & a.e. \text{ in } \Omega, \\ u & \le & m. & \end{array} \tag{2.15}$$

□

Now let $v^n = \int_0^{u^n} g_n(s)\, ds$ and by the test $T_k(v^n)$ in (2.2), leads to

$$\int_\Omega \Gamma^n(x, u^n, Du^n) \cdot DT_k(v^n)\, dx \le C.$$

Then, by assumptions (1.3) and (1.5), we deduce that

$$\int_\Omega |DT_k(v^n)|^p\, dx \le C. \tag{2.16}$$

Based on the above analysis and from (2.16), we obtain the following convergences:

$$v^n \to v \text{ a.e. in } \Omega, \tag{2.17}$$

and

$$T_k(v^n) \rightharpoonup T_k(v) \text{ in } W_0^{1,p}(\Omega).$$

By selecting $\sigma_k(v^n)$ as a test function in (2.2), we infer

$$\lim_{n\to\infty} \int_\Omega \Gamma^n(x, u^n, Du^n) D(\sigma_k(v^n))\, dx = \int_\Omega \frac{f_n}{(u^n + \frac{1}{n})^\theta} \sigma_k(v) dx.$$

Since $f \in L^1(\Omega)$, by Lebesgue's theorem it follows that

$$\lim_{k\to\infty} \lim_{n\to\infty} \int_{\{n \le |v^n| \le n+1\}} \Gamma^n(x, u^n, Du^n).Dv^n dx = 0. \tag{2.18}$$

2.2. **Step 2: Monotonicity trick.**

Lemma 2.3. *For any $k \ge 0$, we have*

$$\lim_{n\to\infty} \int_\Omega \left[\frac{\Gamma^n(x, T_k^m(u^n), DT_k^m(u^n))}{g_n(u^n)^{p-1}} - \frac{\Gamma^n(x, T_k^m(u^n), DT_k^m(u))}{g_n(u^n)^{p-1}}\right] \times (DT_k^m(u^n) - DT_k^m(u))\, dx = 0. \tag{2.19}$$

Proof. For a fixed $k \geq 0$, equality (2.19) can be decomposed as follows:

$$\int_\Omega \left[\frac{\Gamma^n(x,T_k^m(u^n),DT_k^m(u^n))}{g_n(u^n)^{p-1}} - \frac{\Gamma^n(x,T_k^m(u^n),DT_k^m(u))}{g_n(u^n)^{p-1}}\right] \times \left(DT_k^m(u^n) - DT_k^m(u)\right) dx = I_1^n + I_2^n + I_3^n, \tag{2.20}$$

where

$$I_1^n = \int_\Omega \frac{\Gamma^n(x,T_k^m(u^n),DT_k^m(u^n))}{g_n(u^n)^{p-1}} DT_k^m(u^n)dx,$$

$$I_2^n = -\int_\Omega \frac{\Gamma^n(x,T_k^m(u^n),DT_k^m(u^n))}{g_n(u^n)^{p-1}} DT_k^m(u)dx,$$

$$I_3^n = -\int_\Omega \frac{\Gamma^n(x,T_k^m(u^n),DT_k^m(u))}{g_n(u^n)^{p-1}} \left(DT_k^m(u^n) - DT_k^m(u)\right) dx.$$

In the sequel, we consider the limit as $n \to +\infty$ in (2.20).

Limit of I_1^n. We choose $\phi_l(v^n)\int_0^{T_k^m(u)} \frac{1}{g(s)^{p-1}}ds$ as a test function in (2.2) to obtain

$$\begin{gathered}\int_\Omega \phi_l(v^n)\Gamma^n(x,u^n,Du^n)\frac{DT_k^m(u)}{g(u)^{p-1}}dx + \\ \int_\Omega \Gamma^n(x,u^n,Du^n)D\phi_l(v^n).\left(\int_0^{T_k^m(u)} \frac{1}{g(s)^{p-1}}ds\right)dx \\ = \int_\Omega \frac{f_n}{(u^n+\frac{1}{n})^\theta}\phi_l(v^n)\int_0^{T_k^m(u)}\frac{1}{g(s)^{p-1}}dsdx.\end{gathered} \tag{2.21}$$

Since ϕ_l has compact support, we have, for sufficiently large n

$$|\Gamma^n(x,u^n,Du^n)\phi_l(v^n)| \leq \beta\left[L(x) + |DT_{l+1}(v^n)|^{p-1}\right]. \tag{2.22}$$

From (2.22) and (2.16), we deduce that

$$\Gamma^n(x,T_{(l+1)/\lambda}(u^n),DT_{(l+1)/\lambda}(u^n))\phi_l(v^n) \text{ is bounded in } L^{p'(x)}(\Omega), \tag{2.23}$$

for every large n.

Using estimate (2.23), we extract another subsequence, still indexed by l, for which

$$\Gamma^n(x,T_{(l+1)/\lambda}(u^n),DT_{(l+1)/\lambda}(u^n))\phi_l(v^n) \rightharpoonup \psi_l \text{ weakly in } L^{p'}(\Omega), \tag{2.24}$$

as n tends to $+\infty$.

Assuming that $\max(k,m) \leq \frac{l}{\lambda}$, it follows that

$$\begin{aligned}&\Gamma^n(x,T_{(l+1)/\lambda}(u^n),DT_{(l+1)/\lambda}(u^n))\phi_l(v^n)\ \chi_{\{-k<u^n<m\}} \\ &\quad = \phi_l(v^n)\Gamma^n(x,T_k^m(u^n),DT_k^m(u^n))\ \chi_{\{-k<u^n<m\}}\end{aligned}$$

$a.e.$ in Ω. Using the covergences (2.24), (2.17), and (2.24) and letting n tends to $+\infty$, we have for

$$\psi_l DT_k^m(u) = \phi_l(v)\varphi_k DT_k^m(u) \ a.e. \text{in } \Omega. \tag{2.25}$$

Letting now n tends to $+\infty$ and l tends to $+\infty$. The first term in (2.21) yeilds

$$\lim_{l\to+\infty}\lim_{n\to+\infty}\int_\Omega \phi_l(v^n)\Gamma^n(x,u^n,Du^n)\frac{DT_k^m(u)}{g(u)^{p-1}}dx=\int_\Omega \varphi_k\frac{DT_k^m(u)}{g(u)^{p-1}}dx. \tag{2.26}$$

The second term of (2.21)

$$\left|\Gamma^n(x,u^n,Du^n)D\phi_l(v^n).\left(\int_0^{T_k^m(u)}\frac{1}{g(s)^{p-1}}ds\right)\right|\leq\frac{\max(m,k)}{\lambda}\left|\Gamma^n(x,u^n,Du^n)Dv^n\right|.$$

Since (2.18), we deduce that

$$\lim_{l\to+\infty}\lim_{n\to+\infty}\int_\Omega \Gamma^n(x,u^n,Du^n)D\phi_l(v^n).\left(\int_0^{T_k^m(u)}\frac{1}{g(s)^{p-1}}ds\right)dx=0. \tag{2.27}$$

Due to (2.26) and (2.27) , we have

$$\int_\Omega \varphi_k\frac{DT_k^m(u)}{g(u)^{p-1}}dx=\int_\Omega\frac{f}{u^\theta}\left(\int_0^{T_k^m(u)}\frac{1}{g(s)^{p-1}}ds\right)dx. \tag{2.28}$$

Take $\int_0^{T_k^m(u^n)}\frac{1}{g_n(s)^{p-1}}ds$ as a test function in (2.2), we get

$$\int_\Omega \Gamma^n(x,u^n,Du^n)\frac{DT_k^m(u^n)}{g_n(u^n)^{p-1}}dx=\int_\Omega\frac{f_n}{(u^n+\frac{1}{n})^\theta}\int_0^{T_k^m(u^n)}\frac{1}{g_n(s)^{p-1}}dsdx. \tag{2.29}$$

Taking the limit as $n\to+\infty$ in (2.29) and using (2.28), we deduce that

$$\lim_{n\to+\infty}I_1^n=\int_\Omega\varphi_k\frac{DT_k^m(u)}{g(u)^{p-1}}dx. \tag{2.30}$$

Limit of I_2^n. By the assumption of g_n, we remark

$$\frac{1}{g_n(u^n)^{p-1}}\to\frac{1}{g(u)^{p-1}}\ a.e.\text{ in }\Omega, \tag{2.31}$$

as n tends to $+\infty$. Since (2.9), (2.13), and (2.31), we have

$$\lim_{n\to+\infty}I_2^n=-\int_\Omega\varphi_k\frac{DT_k^m(u)}{g(u)^{p-1}}dx. \tag{2.32}$$

Limit of I_3^n. We notice (1.2), (1.3), and (2.9), we show

$$\frac{\Gamma^n(x,T_k^m(u^n),DT_k^m(u))}{g_n(u^n)^{p-1}}\to\frac{\Gamma(x,T_k^m(u),DT_k^m(u))}{g(u)^{p-1}}\ a.e.\text{ in }\Omega, \tag{2.33}$$

as n tends to $+\infty$, and

$$\left|\frac{\Gamma^n(x,T_k^m(u^n),DT_k^m(u))}{g_n(u^n)^{p-1}}\right|\leq\frac{1}{\lambda}\left[\beta\left[L(x)+|DT_k^m(u^n)|^{p-1}\right]\right]\ a.e.\text{ in }\Omega, \tag{2.34}$$

uniformly with respect to n. by (2.33), and (2.34), we deduce

$$\frac{\Gamma^n(x,T_k^m(u^n),DT_k^m(u))}{g_n(u^n)^{p-1}}\to\frac{\Gamma(x,T_k^m(u),DT_k^m(u))}{g(u)^{p-1}}\text{ weakly in }L^{p'}(\Omega)\ , \tag{2.35}$$

as n tends to $+\infty$.

From (2.9), we conclude

$$DT_k^m(u^n) - DT_k^m(u) \to 0 \text{ weakly in } L^p(\Omega). \tag{2.36}$$

Due to (2.35), and (2.36) imply that

$$\lim_{n\to+\infty} I_3^n = 0. \tag{2.37}$$

Combining (2.20) with (2.30)–(2.37), we establish (2.19).

□

Lemma 2.4. *For fixed $k \geq 0$, one has*

$$\varphi_k = \Gamma(x, T_k^m(u), DT_k^m(u)) \ a.e.in\ \{x \in \Omega\ ;\ u(x) < m\}. \tag{2.38}$$

And

$$\begin{aligned} &\frac{\Gamma^n(x, T_k^m(u^n), DT_k^m(u^n))}{g_n(u^n)^{p-1}} DT_k^m(u^n) \\ &\to \frac{\Gamma(x, T_k^m(u), DT_k^m(u))}{g(u)^{p-1}} DT_k^m(u) \ \text{ weakly in } L^1(\Omega), \end{aligned} \tag{2.39}$$

when $n \to +\infty$.

Proof. Let $k \geq 0$ be fixed. From (2.8) and (2.35), we obtain

$$\lim_{n\to+\infty} \int_\Omega \frac{\Gamma^n(x, T_k^m(u^n), DT_k^m(u^n))}{g_n(u^n)^{p-1}} DT_k^m(u^n)\, dx = \int_\Omega \frac{\varphi_k}{g(u)^{p-1}} DT_k^m(u)\, dx.$$

Since (2.31) and (1.3), we have for every ψ

$$\begin{aligned} 0 \leq \lim_{n\to+\infty} &\int_\Omega \left[\frac{\Gamma^n(x, T_k^m(u^n), DT_k^m(u^n))}{g_n(u^n)^{p-1}} - \frac{\Gamma^n(x, T_k^m(u^n), \psi)}{g_n(u^n)^{p-1}}\right] [DT_k^m(u^n) - \psi]\, dx \\ &= \lim_{n\to+\infty} \int_\Omega \frac{\Gamma^n(x, T_k^m(u^n), DT_k^m(u^n))}{g_n(u^n)^{p-1}} [DT_k^m(u^n) - \psi]\, dx \\ &\quad - \lim_{n\to+\infty} \int_\Omega \frac{\Gamma^n(x, T_k^m(u^n), \psi)}{g_n(u^n)^{p-1}} [DT_k^m(u^n) - \psi]\, dx \\ &= \int_\Omega \left(\frac{\varphi_k}{g(u)^{p-1}} - \frac{\Gamma(x, T_k^m(u), \psi)}{g(u)^{p-1}}\right) [DT_k^m(u^n) - \psi]\, dx. \end{aligned} \tag{2.40}$$

By Minty's trick, we conclude that for any

$$\frac{\varphi_k}{g(u)^{p-1}} = \frac{\Gamma(x, T_k^m(u), DT_k^m(u))}{g(u)^{p-1}} \ a.e.\text{in } \Omega. \tag{2.41}$$

Since (2.41) and (2.15), we deduce (2.38). To establish (2.39), we use the monotonicity property of the operator a along with equality (2.19), which leads to

$$\left[\frac{\Gamma^n(x, T_k^m(u^n), DT_k^m(u^n))}{g_n(u^n)^{p-1}} - \frac{\Gamma^n(x, T_k^m(u^n), DT_k^m(u^n))}{g_n(u^n)^{p-1}}\right] \times \\ [DT_k^m(u^n) - DT_k^m(u^n)] \to 0$$

strongly in $L^1(\Omega)$ as n tends to $+\infty$. From (2.9), (2.13), (2.35), and (2.38), we conclude when $n \to +\infty$

$$\frac{\Gamma^n(x, T_k^m(u^n), DT_k^m(u^n))}{g_n(u^n)^{p-1}} DT_k^m(u) \to \frac{\Gamma(x, T_k^m(u), DT_k^m(u))}{g(u)^{p-1}} DT_k^m(u) \text{ weakly in } L^1(\Omega), \tag{2.42}$$

and

$$\frac{\Gamma^n(x, T_k^m(u^n), DT_k^m(u))}{g_n(u^n)^{p-1}} DT_k^m(u^n) \to \frac{\Gamma(x, T_k^m(u), DT_k^m(u))}{g(u)^{p-1}} DT_k^m(u) \text{ weakly in } L^1(\Omega), \tag{2.43}$$

and

$$\frac{\Gamma^n(x, T_k^m(u^n), DT_k^m(u))}{g_n(u^n)^{p-1}} DT_k^m(u) \to \frac{\Gamma(x, T_k^m(u), DT_k^m(u))}{g(u)^{p-1}} DT_k^m(u) \text{ weakly in } L^1(\Omega). \tag{2.44}$$

By the convergences (2.42), (2.43), and (2.44), we obtain that for any $k \geq 0$

$$\frac{\Gamma^n(x, T_k^m(u^n), DT_k^m(u^n))}{g_n(u^n)^{p-1}} DT_k^m(u^n) \to \frac{\Gamma(x, T_k^m(u), DT_k^m(u))}{g(u)^{p-1}} DT_k^m(u) \text{ weakly in } L^1(\Omega),$$

as n tends to $+\infty$.

2.3. **Step 3: End of the proof.** Taking $T^{s+1}(u^n) - T^s(u^n)$ as a test function in (2.2) gives

$$\int_\Omega \Gamma^n(x, u^n, Du^n).D\left(T^{s+1}(u^n) - T^s(u^n)\right) dx = \int_\Omega \frac{f_n}{(u^n + \frac{1}{n})^\theta} \left(T^{s+1}(u^n) - T^s(u^n)\right) dx. \tag{2.45}$$

Since $supp\,(T^{s+1}(.) - T^s(.)) \subset [-(s+1), s]$, we obtain

$$\int_{\{-1-s \leq |u^n| \leq -s\}} \Gamma^n(x, u^n, Du^n).Du^n dx \tag{2.46}$$

$$\begin{aligned}
&= \int_\Omega \Gamma^n(x, u^n, Du^n) D\left(T^{s+1}(u^n) - T^s(u^n)\right) dx \\
&= \int_\Omega \frac{\Gamma^n(x, u^n, Du^n)}{g_n(u^n)^{p-1}} D\left(T^{s+1}(u^n) - T^s(u^n)\right) g_n(T^{s+1}(u^n))^{p-1} dx \\
&= \int_\Omega \frac{\Gamma^n(x, T^{s+1}(u^n), DT^{s+1}(u^n))}{g_n(T^{s+1}(u^n))^{p-1}} D\left(T^{s+1}(u^n)\right) g_n(T^{s+1}(u^n))^{p-1} \qquad (2.47) \\
&\quad - \int_\Omega \frac{\Gamma^n(x, T^s(u^n), DT^s(u^n))}{g_n(u^n)^{p-1}} D\left(T^s(u^n)\right) g_n(T^{s+1}(u^n))^{p-1}
\end{aligned}$$

We deduce from (2.8) and (2.39) that

$$\begin{aligned}
&\lim_{n\to+\infty}\int_{\{-1-s\leq\ u^n\leq -s\}}\Gamma^n(x,u^n,Du^n).Du^n dx\\
&=\int_\Omega \Gamma^n(x,u^n,Du^n)D\left(T^{s+1}\left(u^n\right)-T^s\left(u^n\right)\right)dx\\
&=\int_\Omega \frac{\Gamma(x,T^{s+1}\left(u\right),DT^{s+1}\left(u\right))}{g_n(T^{s+1}\left(u^n\right))^{p-1}}D\left(T^{s+1}\left(u\right)\right)g_n(T^{s+1}\left(u\right))^{p-1}\\
&-\int_\Omega \frac{\Gamma(x,T^s\left(u\right),DT^s\left(u\right))}{g(u)^{p-1}}D\left(T^s\left(u\right)\right)g_n(T^{s+1}\left(u\right))^{p-1} \qquad (2.48)\\
&=\int_{\{-1-s\leq u\leq -s\}}\Gamma(x,u,Du)Dudx.
\end{aligned}$$

Passing to the limit as $s\to+\infty$ in (2.45) and using estimates (2.47) and (2.48), we conclude that u satisfies (1.11).

Choosing $\phi_l(v^n)\frac{1}{\delta}(T^+_{m-\delta}(u)-T^+_{m-2\delta}(u))$ as a test function in (2.2), we have

$$\begin{aligned}
&\frac{1}{\delta}\int_\Omega \Gamma^n(x,u^n,Du^n)D(\phi_l(v^n)\left(T^+_{m-\delta}(u)-T^+_{m-2\delta}(u)\right))dx\\
&=\frac{1}{\delta}\int_\Omega \frac{f_n}{(u^n+\frac{1}{n})^\theta}\phi_l(v^n)(T^+_{m-\delta}(u)-T^+_{m-2\delta}(u))dx. \qquad (2.49)
\end{aligned}$$

Since $\operatorname{supp}(\phi_l)\subset[-(l+1),\,l+1]$, we obtain

$$\begin{aligned}
&\frac{1}{\delta}\int_\Omega \phi_l(v^n)\Gamma^n(x,u^n,Du^n)D(T^+_{m-\delta}(u)-T^+_{m-2\delta}(u))dx=\\
&\frac{1}{\delta}\int_\Omega \phi_l(v^n)\Gamma^n(x,T_{(l+1)/\lambda}\left(u^n\right),DT_{(l+1)/\lambda}\left(u^n\right))D(T^+_{m-\delta}(u)-T^+_{m-2\delta}(u))dx. \qquad (2.50)
\end{aligned}$$

In addition, by applying the same arguments as above, we deduce that

$$\begin{aligned}
&\lim_{l\to+\infty}\lim_{n\to+\infty}\frac{1}{\delta}\int_\Omega \phi_l(v^n)\Gamma^n(x,u^n,Du^n)D(T^+_{m-\delta}(u)-T^+_{m-2\delta}(u))dx\\
&=\frac{1}{\delta}\int_{\{m-2\delta\leq u\leq m-\delta\}}\Gamma(x,u,Du)Dudx. \qquad (2.51)
\end{aligned}$$

Passing to the limit as $\delta\to 0$ in (2.49) and applying estimates (2.50) and (2.51), we conclude that u satisfies (1.12).

Let $S\in W^{1,\infty}(\mathbb{R})$ be a function with compact support satisfying $S(m)=0$, and let $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. Choosing $S(u)\,h_l(v^n)\,\varphi$ as a test function in (2.2), we obtain

$$\begin{aligned}
&\int_\Omega \phi_l(v^n)\Gamma^n(x,u^n,Du^n)D\left(S(u)\varphi\right)dx+\int_\Omega S(u)\varphi\Gamma^n(x,u^n,Du^n)D\phi_l(v^n)dx \qquad (2.52)\\
&=\int_\Omega \frac{f_n}{(u^n+\frac{1}{n})^\theta}S(u)\phi_l(v^n)\varphi dx.
\end{aligned}$$

Taking the limits as $n\to+\infty$ and $l\to+\infty$ in (2.52).

Limit of first term in (2.52)
Given that $\operatorname{supp}(h_l) \subset [-(l+1),\, l+1]$, it follows that

$$\Gamma^n(x, T_{(l+1)/\lambda}(u^n), DT_{(l+1)/\lambda}(u^n))\phi_l(v^n)) = \phi_l(v^n)\Gamma^n(x, u^n, Du^n) \; a.e. \text{in } \Omega.$$

From (2.15), (2.24), (2.25) and (2.38), we get

$$\begin{aligned} &\lim_{l\to+\infty}\lim_{n\to+\infty} \int_\Omega \phi_l(v^n)\Gamma^n(x, u^n, Du^n) D\left(S(u)\varphi\right) dx \\ = &\lim_{l\to+\infty} \int_\Omega \phi_l(v)\Gamma(x, T_k^m(u^n), DT_k^m(u^n)) D\left(S(u)\varphi\right) dx \\ = &\int_\Omega \Gamma(x, u, Du) D\left(S(u)\varphi\right) dx. \end{aligned}$$

Limit of second term in (2.52)
In view of (2.18), we deduce that

$$\lim_{l\to+\infty}\lim_{n\to+\infty} \int_\Omega S(u)\varphi\Gamma^n(x, u^n, Du^n) D\phi_l(v^n) dx = 0.$$

Limit of the Right-Hand Side of (2.52)
From (2.1) and (2.17)

$$\lim_{l\to+\infty}\lim_{n\to+\infty} \int_\Omega \frac{f_n}{(u^n + \frac{1}{n})^\theta} S(u)\phi_l(v^n)\varphi dx = \int_\Omega \frac{f}{u^\theta} S(u)\varphi dx.$$

Then, u satisfies (1.13). □

In what follows, we will present the results regarding regularity stated in Theorems 2.5 and Theorem2.6.

Theorem 2.5. *Let $1 < p < N$, $\theta > 1$ and let u^n be the solution of problem (1.1), hence $T_k^{\frac{\theta+p-1}{p}}(u^n) \in W_0^{1,p}(\Omega)$.*

Proof. We restrict our analysis to the case $1 < p < N$, since the arguments for the range $p \geq N$ follow with only straightforward modifications. Choosing the truncation $T_k(u_n)^\theta$ as a test function in the weak formulation (2.2), we obtain

$$\begin{aligned} \theta \int_\Omega |DT_k(u^n)|^p \, T_k(u^n)^{\theta-1} dx &\leq \int_\Omega \Gamma^n(x, u^n, Du^n)\, DT_k(u^n) dx \\ &= \int_\Omega \frac{f_n\, T_k(u^n)^\theta}{\left(T_k(u^n) + \frac{1}{n}\right)^\theta} dx \\ &\leq \int_\Omega f_n dx \\ &\leq \int_\Omega f dx \\ &= \|f\|_{L^1(\Omega)}. \end{aligned}$$

Therefore

$$\left\| D\left(T_k(u^n)^{\frac{\theta+p-1}{p}}\right)\right\|_{L^p(\Omega)} \leq C \, \|f\|_{L^1(\Omega)}^{\frac{1}{p}}$$

and the first part of the thesis is proved. Recalling the assumption on the exponent s we deduce that

$$\|T_k(u^n)\|_{L^s(\Omega)} = \left(\int_\Omega T_k(u^n)^{\frac{p^*(\theta+p-1)}{p}} dx\right)^{\frac{1}{s}} = \left\|T_k(u^n)^{\frac{\theta+p-1}{p}}\right\|_{L^{p^*}(\Omega)}^{\frac{p^*}{s}}$$
$$\leq C\left\|D\left(T_k(u^n)^{\frac{\theta+p-1}{p}}\right)\right\|_{L^p(\Omega)}^{\frac{p^*}{s}} \leq C\,\|f\|_{L^1(\Omega)}^{\frac{p^*}{ps}},$$

The previous estimate ensures that the family $\{T_k(u^n)\}$ is uniformly bounded in $L^s(\Omega)$.

It remains to derive the local Sobolev regularity of the truncations. This can be obtained by taking the test function $T_k(u^n)\varphi^2$, where $\varphi \in C_c^\infty(\Omega)$, in (2.2), which yields a uniform estimate in $W^{1,p}_{\text{loc}}(\Omega)$.

Alternatively, this conclusion follows directly from the first part of the theorem. Indeed, the sequence $\left\{T_k(u^n)^{\frac{\theta+p-1}{p}}\right\}$ is uniformly bounded in $W_0^{1,p}(\Omega)$. Moreover, since the truncations $\{T_k(u^n)\}$ remain uniformly positive on every compact subset of Ω, these two properties immediately imply that $\{T_k(u^n)\}$ is uniformly bounded in $W^{1,p}_{\text{loc}}(\Omega)$. □

Theorem 2.6. *Let $1 < p < N$ and let $0 < \theta < 1$. Assume that $f \in L^m(\Omega)$ with*

$$m = \frac{Np}{N(p-1)+p+\theta(N-p)} = \left(\frac{p^*}{1-\theta}\right)', \qquad p^* = \frac{Np}{N-p}.$$

Then problem (1.1) has a solution $u \in W_0^{1,p}(\Omega)$.

If $p = N$, the same result holds assuming that $f \in L^m(\Omega)$ for some $m > 1$. In the case $p > N$, the result is true if $f \in L^1(\Omega)$.

Proof. We only consider the case $1 < p < N$. The other cases are simpler and can be proved in a similar way with trivial changes. We have that

$$\int_\Omega |DT_k(u^n)|^p dx = \int_\Omega \frac{f_n T_k(u^n)}{(u^n+\frac{1}{n})^\theta} dx \leq \int_\Omega f_n T_k(u^n)^{1-\theta} dx \leq \|f\|_{L^m(\Omega)}\left(\int_\Omega T_k(u^n)^{(1-\theta)m'} dx\right)^{\frac{1}{m'}}. \tag{2.53}$$

Since $(1-\theta)m' = p^*$, then by Sobolev inequality, we get

$$\left(\int_\Omega |T_k(u^n)|^{p^*} dx\right)^{\frac{p}{p^*}} \leq C\int_\Omega |DT_k(u^n)|^p\, dx \leq C\,\|f\|_{L^m(\Omega)}\left(\int_\Omega |T_k(u^n)|^{p^*} dx\right)^{\frac{1}{m'}}. \tag{2.54}$$

Since $\frac{p}{p^*} > \frac{1}{m'}$, by (2.54), we get that

$$\|T_k(u^n)\|_{L^{p^*}(\Omega)} \leq \left(C\,\|f\|_{L^m(\Omega)}\right)^{\frac{p^*}{p-\frac{p^*}{m'}}}.$$

From this, exploiting (2.53), we deduce that $\{T_k(u^n)\}$ is uniformly bounded in $W_0^{1,p}(\Omega)$. □

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