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HILBERT DEPTH OF GOTZMANN IDEALS

AZEEM HAIDER[1] and SARDAR MOHIB ALI KHAN[2] *

Abstract. The Hilbert function of a homogeneous ideal is determined by its associated lexsegment ideal, while the Gotzmann property ensures that the ideal and its lexsegment ideal have the same number of minimal generators. For homogeneous monomial ideals with fewer minimal generators than the number of variables in the polynomial ring, the classes of critical monomial ideals and Gotzmann ideals coincide. In this article, we determine the Hilbert depth of this class of monomial ideals. As an application, we compute the Hilbert depth of the edge ideal of a star graph and propose it as an algebraic graph invariant.

Keywords. Hilbert depth, Edge ideals, Lexsegment ideals, Gotzmann Ideals
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1. Introduction and preliminaries

Let $\Re = \mathcal{F}[x_1, x_2 \ldots, x_n]$ be the polynomial ring with standard grading over a field $\mathcal{F}$. Let $I \subset \Re$ be a monomial ideal. We denote by $G(I)$ the unique set of monomials generators of I, and by $\mathrm{Mon}(\Re)$ the set of monomials in $\Re$. For a finitely generated $\mathbb{Z}^n$-graded $\Re$-module Ω, let Ω_d denote its homogeneous component of degree d. The function $H(\Omega, d) = dim_{\mathcal{F}}(\Omega_d)$ is called the Hilbert function of module Ω, and $H_\Omega(t) = \sum_{d\geq 0} \dim_{\mathcal{F}}(\Omega_d)t^d$ the corresponding Hilbert series. We consider the lexicographic order $<_{lex}$ on $\Re$ with fundamental ordering $x_1 > x_2 > \ldots > x_n$ of the variables. A set of monomials V is called *lexsegment* if, whenever $u \in V$ and v is another monomial of the same degree satisfying $v >_{lex} u$, then $v \in V$. A monomial ideal I is called a *lexsegment ideal* if, for every degree d, the set of all degree d monomials contained in I forms a lexsegment. A *universal lexsegment ideal* is a lexsegment ideal I with the property that for any integer $m \geq 0$, the ideal I remains a lexsegment ideal in $\mathcal{F}[x_1, \ldots, x_{n+m}]$.

Lexsegment ideals play a fundamental role in the study of Hilbert functions. In particular, for every graded ideal I, there exists a unique lexsegment ideal $I^{\mathrm{lex}} = \bigoplus_{d\geq 0} \mathrm{Lex}(I_d)$, where $\mathrm{Lex}(I_d)$ denotes the lexsegment of degree d satisfying $\dim_{\mathcal{F}}(I_d) = \dim_{\mathcal{F}}\left(I_d^{\mathrm{lex}}\right)$. Consequently, I and I^{lex} have the same Hilbert function.

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* Corresponding author

Gotzmann monomial ideals are the ideals for which the $G(I)$ and $G(I^{lex})$ have same cardinality. In recent years, various classifications of Gotzmann ideals have been explored. From a homological perspective, a homogenous ideal I is called Gotzmann if and only if I and I^{lex} have identical graded Betti numbers. It is worth noting that lexsegment ideals attain the largest graded Betti numbers among all homogeneous ideals with the same Hilbert function.
In order to classify Hilbert functions, Murai and Hibi introduced critical numerical functions [14]. Any self map H on $\mathbb{Z}_{\geq 0}$ is considered critical if H is the Hilbert function of a corresponding lexsegment ideal of I of ring $\Re$ for with $|G(I)| \leq n$. Furthermore, for the categorization of ideals possessing a critical Hilbert function and Gotzmann ideals, canonical critical ideals are defined as follows.

For any positive integer $t \leq n$, if $m_1, m_2, \ldots, m_t \in \mathrm{Mon}(\Re)$ with each m_i belongs to $\digamma[x_i, x_{i+1}, \ldots, x_n]$, we define an ideal

$$I_{(m_1,m_2,\ldots,m_t)} = (x_1m_1, x_2m_1m_2, \ldots, m_1m_2\ldots m_{t-1}x_{t-1}, m_1m_2\ldots m_t). \tag{1.1}$$

A *canonical critical* is a monomial ideal I [13] with $I = I_{(m_1,m_2,\ldots,m_t)}$, for some monomials $m_1, m_2, \ldots, m_t$. Using these two definitions the following theorem classify the Gotzmann ideals with $G(I) \leq n$.

Theorem 1.1. [13] *For a homogeneous ideal I of the ring $\Re$ the following statements are equivalent:*

1. *The Hilbert function of I is a critical Hilbert function*
2. *There exists a linear function ψ on $\Re$ such that $\psi(I)$ is canonical critical ideal*
3. *I is Gotzmann ideal with $|G(I)| \leq n$.*

A decomposition of a module Ω is the finite direct sum of a $\mathbb{Z}^n$-graded $\digamma$-subspaces. In recent years two such decomposition have been discussed intensively, Stanley decomposition and Hilbert decomposition and two invariants are associated with these decompositions namely Stanley depth (sdepth) and Hilbert depth (hdepth), both of which lie between the depth and dimension of a module. Stanley decomposition and sdepth of canonical critical ideals were studied by authors [8].

Methods for calculating the Hilbert depth[15], and the multi-graded Hilbert depth of modules [11], have been developed. However, computing the Hilbert depth is generally challenging, and several articles have been dedicated to computing it for specific classes of ideals and quotient modules [6], [12], [2]. In this work, we aim to establish general bounds for the Hilbert depth of monomial ideals, some of which are analogous to those found in the work of W. Bruns et al. [3]. In particular, we will calculate the hdepth of critical monomial ideals with $|G(I)| = m \leq n$.

During the past few years, significant work have been published to establishing a connections between algebra and graph invariants (see for example [5], [7] and [20]), in particular with the Stanley depth (sdepth) of associated edge ideals [17], [1], [18], [21]. This has led to an intriguing exploration of the sdepth of graphs through their associated edge ideals. In the final section of this article, we discuss various bounds on the hdepth of edge ideals in general and calculate the Hilbert depth of edge ideals that are Gotzmann ideals.

2. Hilbert Depth

Hilbert decomposition of a module Ω is the direct sum of graded $\digamma$-vector subspaces of $\Re$ i.e. $\Omega = \bigoplus_{i\in N} \Re_i(-s_i)$ and the decomposition has a natural $\digamma$-module structure [3]. Clearly the Hilbert decomposition of any module Ω depends only on the Hilbert function of Ω. Hilbert depth of a module Ω is the maximal depth of any Hilbert decomposition of Ω, denoted by *hdepth*. Uliczka [3] proved that hdepth can be calculated by the formula

$$hdepth(\Omega) = \max\{u : (1-t)^u H_\Omega(t) = \sum_{-n}^{\infty} a_n t^n \text{ with } a_n \geq 0, \forall n\}.$$

We will consider an alternative definition of hdepth which can be easily shown equivalent to above and revisit some known results and provide some examples. Consider Δ be the difference operator defined on a sequence $\{a_n\}_{n\geq 1}$ for any fix value of a_0 as,

$$\Delta(a_n) = a_n - a_{n-1} \text{ and } \Delta^i(a_n) = \Delta(\Delta^{i-1}(a_n)),\ i \geq 1.$$

Lemma 2.1. *For a standard graded module* Ω,

$$\text{hdepth}(\Omega) = max\{u \,:\, \Delta^u H(\Omega, d) \geq 0, \forall d\}.$$

Proof. Recall that $H_{\Omega(-1)}(t) = tH_\Omega(t)$ where $H_{\Omega(-1)}(t)$ denotes the shift by -1. So we consider the difference operator Δ as follows,

$$\Delta H_\Omega(t) = H_\Omega(t) - H_{\Omega(-1)}(t) = (1-t)(H_\Omega(t)).$$

□

Lemma 2.2. *Let Ω be an $\Re$-module and $x \in \Omega$ be a regular element of degree 1. Then* $\text{hdepth}(\Omega/x\Omega) = \text{hdepth}(\Omega) - 1$.

Proof. Follows from the fact that $H(\Omega/x\Omega, d) = \Delta H(\Omega, d)$. □

The notion of hdepth is non trivial [3], following Proposition and Examples shows the same in the context of Lemma 2.1.

Proposition 2.3. *Let Ω be a standard graded $\Re$-module. Then*

i) $depth(\Omega) \leq \text{hdepth}(\Omega) \leq \dim(\Omega)$,
ii) $\text{sdepth}(\Omega) \leq \text{hdepth}(\Omega)$.

Proof. i) Let $\Omega \neq 0$ be a finitely generated $\Re$-module of dimension d, then we know by Hilbert theorem $\Delta^d H(\Omega, i) = 0$ for $i \gg 0$. It implies $\Delta^{d+1} H(\Omega, i) < 0$ for some i, because at least the first non-zero value remains unchanged. So we have $\text{hdepth}(\Omega) \leq d$ and $depth(\Omega) \leq \text{hdepth}(\Omega)$ follows from Lemma 2.2.
ii) Consider a Stanley decomposition of the module Ω

$$\mathcal{D}:\ M = \bigoplus_{j=1}^{k} u_j \digamma[Z_j]$$

with $\deg u_j = e_j$ and $\operatorname{sdepth}(\mathcal{D}) = |Z_1|$. It is enough to show that $|Z_1| \leq \operatorname{hdepth}(\Omega)$.

From the above decomposition we can always write the Hilbert Series

$$H_\Omega(t) = \sum_{j=1}^{k} \frac{t^{e_i}}{(1-t)^{|Z_j|}}.$$

By (2.1) we have

$$\Delta^{|Z_1|} H_\Omega(t) = \sum_{j=1}^{k} \frac{t^{e_i}}{(1-t)^{|Z_j|-|Z_1|}}. \tag{2.1}$$

Since $|Z_j| - |Z_1| \geq 0$ for all j and when we write (2.1) in series, each coefficient of t^i represents the value $\Delta^{|Z_1|} H(\Omega, i)$, hence the result follows. □

We give some examples to show that the inequalities in Proposition 2.3 are strict.

Example 2.4. For the ideal $I = (x_1^2, x_1x_2) \subset \digamma[x_1, x_2]$. The $\operatorname{hdepth}(\Re/I) = 0$ but $\dim(\Re/I) = 1$.

Example 2.5. If $I = (x_1^2, x_1x_2, x_1x_3, x_2^2x_3) \subset \digamma[x_1, x_2, x_3]$, then $\operatorname{depth}(\Re/I) = \operatorname{sdepth}(\Re/I) = 0$, but $\operatorname{hdepth}(\Re/I) = 1$.

Corollary 2.6. *Let Ω be a standard graded $\Re$-module and $H(\Omega, d+1) < H(\Omega, d)$ for any $d \geq 0$. Then* $\operatorname{depth}(\Omega) = \operatorname{sdepth}(\Omega) = 0$.

Proof. If $H(\Omega, d+1) < H(\Omega, d)$ for any $d \geq 0$, then $\operatorname{hdepth}(\Omega) = 0$ and the Corollary follows from Proposition 2.3. □

The following corollary is the Hilbert depth analogue of the results obtained for depth and sdepth presented in [9, Lemma 3.6], which follows from Lemma 2.2.

Corollary 2.7. *Let $J \subset I$ be monomial ideals of $\Re$ and let $T = \Re[x_{n+1}]$ be the polynomial ring over $\Re$ in the variable x_{n+1}. Then*

$$\operatorname{hdepth}(IT/JT) = \operatorname{hdepth}(I/J) + 1.$$

In the following Lemma we see the behavior of hdepth along short exact sequence of $\Re$-modules. A similar result for sdepth is evaluated by A. Rauf [16].

Lemma 2.8. *If $0 \to L \to \Omega \to N \to 0$ be a short exact sequence of $\Re$-modules, then* $\operatorname{hdepth}(\Omega) \geq \min\{\operatorname{hdepth}(L), \operatorname{hdepth}(N)\}$.

Proof. It follows from the fact that if $0 \to L \to \Omega \to N \to 0$, then $H(\Omega, d) = H(L, d) + H(N, d)$. □

The following is easy consequence of Lemma 2.8.

Corollary 2.9. *In Lemma 2.8 if* $\operatorname{hdepth}(\Omega) < \operatorname{hdepth}(N)$, *then* $\operatorname{hdepth}(\Omega) \geq \operatorname{hdepth}(L)$.

Lemma 2.10. *For any $I \subset \Re$, we have* $\operatorname{hdepth}(I) \geq \max\{1, n - |G(I)| + 1\}$ *and* $\operatorname{hdepth}(\Re/I) \geq n - |G(I)|$.

Proof. For any monomial ideal I, $\operatorname{sdepth}(I) \geq \max\{1, n-|G(I)|+1\}$ ([9, Proposition 3.4]) and $\operatorname{sdepth}(\Re/I) \geq n-|G(I)|$ ([4, Proposition 1.3]), hence the result follows from Proposition 2.3. $\square$

Lemma 2.11. *If $H(\Omega, i)$ is the first non-zero Hilbert function for any graded module Ω over R, then hdepth$(\Omega) \leq \left\lfloor \frac{H(\Omega,i+1)}{H(\Omega,i)} \right\rfloor$.*

Proof. Since $H(\Omega, j) = 0$ for all $j < i$, so $\Delta^k H(\Omega, i+1) = H(\Omega, i+1) - kH(\Omega, i)$, which shows that $\operatorname{hdepth}(\Omega) \leq \left\lfloor \frac{H(\Omega,i+1)}{H(\Omega,i)} \right\rfloor$. $\square$

3. Hilbert Depth of homogeneous Gotzmann Ideals

In this section we calculate the hdepth of a class of critical monomial ideals (see [8]) having all generators of same degree.

Theorem 3.1. *If $I \subset \Re$ is a critical monomial ideal having all generators of same degree s and $|G(I)| = m \leq n$, then* $\operatorname{hdepth}(I) = n - m + \lceil \frac{m}{2} \rceil$.

Proof. Let I be a critical monomial ideal generated by m monomials having the same degree s. Then by [14] the Hilbert series of I is,

$$H_I(t) = t^s \sum_{i=1}^{m} \frac{1}{(1-t)^{n+1-i}}$$

$$\Delta^{n-m+\lceil \frac{m}{2} \rceil} H_I(t) = t^s \left[\sum_{i=1}^{\lceil \frac{m}{2} \rceil} \frac{1}{(1-t)^{m-\lceil \frac{m}{2} \rceil+1-i}} + \sum_{j=\lceil \frac{m}{2} \rceil+1}^{m} (1-t)^{\lceil \frac{m}{2} \rceil+j-m-1} \right]$$

We observe the coefficient of t^{d+s}, when d is even we have,

$$\Delta^{n-m+\lceil \frac{m}{2} \rceil} H(I, d+s) = \sum_{i=1}^{\lceil \frac{m}{2} \rceil} \binom{m-\lceil \frac{m}{2} \rceil - i + d}{d} + \sum_{j=\lceil \frac{m}{2} \rceil+1}^{m} \binom{\lceil \frac{m}{2} \rceil + j - m - 1}{d} \geq 0$$

and if d is odd, then

$$\Delta^{n-m+\lceil \frac{m}{2} \rceil} H(I, d+s) = \sum_{i=1}^{\lceil \frac{m}{2} \rceil} \binom{m-\lceil \frac{m}{2} \rceil - i + d}{d} - \sum_{j=\lceil \frac{m}{2} \rceil+1}^{m} \binom{\lceil \frac{m}{2} \rceil + j - m - 1}{d} \geq 0,$$

since for each index j there is a unique index i corresponding to j such that $m - \lceil \frac{m}{2} \rceil - i \geq \lceil \frac{m}{2} \rceil + j - m - 1$ which shows that $\operatorname{hdepth}(I) \geq n - m + \lceil \frac{m}{2} \rceil$. But the coefficient of t^{1+s} in $\Delta^{n-m+\lceil \frac{m}{2} \rceil+1} H_I(t)$ is,

$$\Delta^{n-m+\lceil \frac{m}{2} \rceil+1} H(I, 1+s) = \sum_{i=1}^{\lceil \frac{m}{2} \rceil - 1} \left(m - \lceil \frac{m}{2} \rceil - i\right) - \sum_{j=\lceil \frac{m}{2} \rceil}^{m} \left(\lceil \frac{m}{2} \rceil - m + j\right)$$

If $m = 2k$ is even, then $\lceil \frac{m}{2} \rceil = k$ and

$$\Delta^{n-m+\lceil \frac{m}{2} \rceil+1} H(I, 1+s) = \sum_{i=1}^{k-1} (2k - k - i) - \sum_{j=k}^{2k} (k - 2k + j)$$

$$= \frac{k(k-1)}{2} - \frac{k(k+1)}{2} < 0$$

and if $m = 2k+1$, then $\lceil \frac{m}{2} \rceil = k + \lceil \frac{1}{2} \rceil = k+1$ and hence

$$\Delta^{n-m+\lceil \frac{m}{2} \rceil+1} H(I, 1+s) = \sum_{i=1}^{k} (k-i) - \sum_{j=k+1}^{2k+1} (-k+j)$$

$$= \frac{k(k-1)}{2} - \frac{(k+1)(k+2)}{2} < 0.$$

This implies that $\operatorname{hdepth}(I) = n - m + \lceil \frac{m}{2} \rceil$. □

Corollary 3.2. *Let $I = (x_1, x_2 \dots, x_n) \subset \Re$ be the maximal ideal of $\Re$. Then* $\operatorname{hdepth}(I) = \lceil \frac{n}{2} \rceil$.

Definition 3.3. We call an ideal $I \subset \Re$ *harmonic* if $\operatorname{sdepth}(I) = \operatorname{hdepth}(I)$.

Clearly every harmonic ideal satisfies the Stanley's inequality i.e. $\operatorname{depth}(I) \leq \operatorname{sdepth}(I)$. Following example shows that not every ideal is harmonic.

Example 3.4. If $I = (x_1, x_2^2, x_3^3, x_4^4) \subset F[x_1, x_2, x_3, x_4]$, then $\operatorname{sdepth}(I) = 2$ but $\operatorname{hdepth}(I) = 3$.

Corollary 3.5. *Every canonical critical monomial ideal $I \subset \Re$ having the same degree of generators is harmonic. In particular maximal ideal of $\Re$ is harmonic.*

Proof. If I is a canonical critical monomial ideal having the same degree of generators, then by [8] we have $I = (x_1u, x_2u, \dots, x_mu)$ where $u \in \Re$ is a monomial and $\gcd(G(I)) = u$. By using the [4, Proposition 1.3] we have $\operatorname{sdepth}(I) = \operatorname{sdepth}(x_1, x_2, \dots, x_m) = n - m + \lceil \frac{m}{2} \rceil$ and by Theorem 3.1 we have $\operatorname{sdepth}(I) = \operatorname{hdepth}(I)$. □

4. Hilbert Depth of Edge Ideals

For any graph $G = (V, E)$ consider $V(G) = \{x_1, x_2, \dots, x_n\}$ with $|V(G)| = n$ and $|E(G)| = m$. The edge ideal $I(G)$ of the graph G is defined as

$$I(G) = \langle x_i x_j : \{x_i x_j\} \in E(G) \rangle \subset \Re.$$

Proposition 4.1. *For any edge ideal $I(G) \subset \Re$, corresponding to the graph G, we have $\operatorname{hdepth}(I(G)) \leq \left\lfloor n - \frac{1}{2m} \sum_{i=1}^{n} a_i(a_i - 1) \right\rfloor$ if G contains no triangles, where $a_i = \deg(v_i)$ for $v_i \in V(G)$.*

Proof. Clearly the first nonzero Hilbert function is $H(I(G), 2) = m$. For $H(\Omega, 3)$, each generator corresponds to n values in $H(\Omega, 3)$, so there are at most nm values in $H(\Omega, 3)$ with connection of edges in G are repeated. If $\deg(v_i) = a_i$ for a vertex $v_i \in V(G)$, then there are $\frac{1}{2} a_i(a_i - 1)$ edge connections at v_i of G. Therefore, the total number of such connections in G are $\frac{1}{2} \sum_{i=1}^{n} a_i(a_i - 1)$ and G contains no triangles, so $H(\Omega,3) = nm - \frac{1}{2}\sum_{i=1}^{n} a_i(a_i-1) + k$.

By Lemma 2.11, we have,

$$\operatorname{hdepth}(I(G)) \leq \left\lfloor \frac{H(\Omega,3)}{H(\Omega,2)} \right\rfloor = \left\lfloor \frac{nm - \frac{1}{2}\sum_{i=1}^{n} a_i(a_i-1)}{m} \right\rfloor = \left\lfloor n - \frac{1}{2m}\sum_{i=1}^{n} a_i(a_i-1) \right\rfloor.$$

□

We finalize the section with the calculation of hdepth for Gotzmann ideals and this can be consider an algebraic invariant for graphs. We have following characterization of Gotzmann ideals.

Theorem 4.2. [10] *The edge ideal $I(G)$ is Gotzmann if and only if G is a star.*

Theorem 4.3. *Let G be a star graph with n vertices, then hdepth$(I(G)) = \lfloor \frac{n}{2} \rfloor + 1$.*

Proof. The edge ideal associated to star graph is: $I(G) = (x_1x_2, x_1x_3, \dots, x_1x_n)$ which is a canonical critical monomial ideal [8] with generators of same degree and the number of generators, $m = n-1 < n$, hence by Theorem 3.1 we have; $\operatorname{hdepth}(I(G)) = n - (n-1) + \lceil \frac{n-1}{2} \rceil = 1 + \lfloor \frac{n}{2} \rfloor$. □

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[1] Department of Mathematics, College of Science, Jazan University, P.O. Box: 114, Jazan 45142, Kingdom of Saudi Arabia.
Email address: aahaider@jazanu.edu.sa

[2] Department of Mathematics, College of Science, Mathematics and Technology, KEAN University, New Jersey, USA.
Email address: sakhan@kean.edu