BLOW-UP OF SOLUTIONS FOR A PARABOLIC KIRCHHOFF TYPE EQUATION WITH LOGARITHMIC NONLINEARITY

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ABSTRACT. This study deals with the parabolic type Kirchhoff equation with logarithmic nonlinearity in a bounded domain. We obtain the finite time blow-up of solutions. This improves and extends some previous studies.

1. INTRODUCTION

In this paper, we investigate the parabolic Kirchhoff type equation with logarithmic nonlinearity

$$\begin{cases}
    u_t - M(\|\nabla u\|^2) \Delta u - \Delta u_t = |u|^{q-2} u \ln |u|, & x \in \Omega, \ t > 0, \\
    u(x, t) = 0, & x \in \partial \Omega, \ t > 0, \\
    u(x, 0) = u_0(x), & x \in \Omega,
\end{cases}$$

(1.1)

here $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $M(s) = 1 + s^\gamma, (\gamma > 0)$ and $2 < q < 2 + \frac{4}{n}$.

Studies of logarithmic nonlinearity have a long history in physics as it occurs naturally in different areas of physics such as supersymmetric field theories, inflation cosmology, nuclear physics and quantum mechanics [1, 5].

When $M \equiv 1$ and $q = 2$, (1.1) become the following semilinear pseudo-parabolic equation

$$u_t - \Delta u - \Delta u_t = u \log |u|. \quad (1.2)$$

Chen and Tian [4] studied the global existence, behavior of vacuum isolation and blow-up of solutions at $+\infty$ of the equation (1.2). Without $\Delta u$, (1.2) become the following semilinear heat equation

$$u_t - \Delta u_t = u \log |u|. \quad (1.3)$$

Chen et al. [3] studied the global existence, decay estimate and blow-up at $+\infty$ of solutions of the equation (1.3).

Peng and Zhou [9] studied the following heat equation

$$u_t - \Delta u_t = |u|^{p-2} u \log |u|. \quad (1.4)$$
They studied the global existence and blow-up of solutions. Also, they discussed the upper bound of blow-up time under suitable conditions.

He et al. [6] studied the following nonlinear pseudo-parabolic equation

$$u_t - \Delta u_t - \text{div} (|\nabla u|^{p-2} \nabla u) = |u|^{p-2} u \log |u|.$$  

They obtained the decay and the finite time blow-up for weak solutions of the equation.

Cao and Liu [2] studied the following nonlinear evolution equation with logarithmic source

$$u_t - \Delta u_t - \text{div} (|\nabla u|^{p-2} \nabla u) - k\Delta u_t = |u|^{p-2} u \log |u|.$$  

They established the existence of global solutions. Moreover, they considered global boundedness and blowing-up at $\infty$.

Li and Han [7] studied the following parabolic type $p$-Kirchhoff initial boundary value problem

$$u_t - (a + b \int_{\Omega} |\nabla u|^p) \Delta_p u = |u|^{q-1} u.$$  

They obtained the global existence and finite time blow up of solutions.

Motivated by the above studies, in this work, we investigate the finite time blow-up of weak solutions for the Eq. (1.1).

This paper is organized as follows: In Section 2, we introduce some lemmas which will be needed later. In Section 3, we prove the blow-up of solutions.

2. Preliminaries

For simplicity, we denote

$$\|u\|_s = \|u\|_{L^s(\Omega)}, \quad \|u\|_{H^1_0(\Omega)} = (\|\nabla u\|^2 + \|u\|^2)^{1\over 2},$$

for $1 < s < \infty$.

Let us introduce the energy functional $J$ and Nehari functional $I$ defined on $H^1_0(\Omega) \setminus \{0\}$ as following

$$J(u) = \frac{1}{2} \|\nabla u\|^2 + \frac{1}{2(\gamma + 1)} \|\nabla u\|^{2(\gamma+1)} - \frac{1}{q} \int_{\Omega} |u|^q \ln |u| \, dx + \frac{1}{q^2} \|u\|^q_q,$$  

and

$$I(u) = \|\nabla u\|^2 + \|\nabla u\|^{2(\gamma+1)} - \int_{\Omega} |u|^q \ln |u| \, dx.$$  

By (2.1) and (2.2), we get

$$J(u) = \frac{1}{q} I(u) + \frac{q-2}{2q} \|\nabla u\|^2 + \frac{q-2\gamma - 2}{2q\gamma + 2q} \|\nabla u\|^{2(\gamma+1)} + \frac{1}{q^2} \|u\|^q_q.$$  

Let

$$\mathcal{N} = \{u \in H^1_0(\Omega) \setminus \{0\} : I(u) = 0\},$$

be the Nehari manifold. Lemma 2.1, indicates $\mathcal{N}$ is not empty. Thus, we may define

$$d = \inf_{u \in \mathcal{N}} J(u).$$  

Lemma 2.1. Let $u \in H_0^1(\Omega) \setminus \{0\}$. Then we possess

(i) $\lim_{\lambda \to 0^+} j(\lambda) = 0$ and $\lim_{\lambda \to +\infty} j(\lambda) = -\infty$;

(ii) there is a unique $\lambda^* > 0$ such that $j'(\lambda^*) = 0$;

(iii) $j(\lambda)$ is increasing on $(0, \lambda^*)$, decreasing on $(\lambda^*, +\infty)$ and attains the maximum at $\lambda^*$;

(iv) $I(\lambda u) > 0$ for $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ for $\lambda^* < \lambda < +\infty$ and $I(\lambda^* u) = 0$.

Proof. For $u \in H_0^1(\Omega) \setminus \{0\}$, by the definition of $j$, we get

$$j(\lambda) = \frac{1}{2} \| \nabla (\lambda u) \|^2 + \frac{1}{2(\gamma + 1)} \| \nabla (\lambda u) \|^{2(\gamma + 1)}$$

$$- \frac{1}{q} \int_{\Omega} |\lambda u|^q \ln |\lambda u| \, dx + \frac{1}{q^2} \| \lambda u \|_q^q$$

$$= \frac{\lambda^2}{2} \| \nabla u \|^2 + \frac{\lambda^{2(\gamma + 1)}}{2(\gamma + 1)} \| \nabla u \|^{2(\gamma + 1)}$$

$$- \frac{\lambda^q}{q} \int_{\Omega} |u|^q \ln |u| \, dx - \frac{\lambda^q}{q} \int_{\Omega} \lambda \| u \|_q^q \ln \lambda \| u \|_q^q + \frac{\lambda^q}{q^2} \| u \|_q^q.$$  \hspace{1cm} (2.5)

It is clear that (i) holds due to $\int_{\Omega} |u|^q \, dx \neq 0$. We have

$$\frac{d}{d\lambda} j(\lambda) = \lambda \| \nabla u \|^2 + \lambda^{2\gamma + 1} \| \nabla u \|^{2(\gamma + 1)} - \lambda^{\gamma - 1} \int_{\Omega} |u|^q \ln |u| \, dx$$

$$- \lambda^{\gamma - 1} \ln \lambda \| u \|_q^q - \frac{\lambda^{\gamma - 1}}{q} \| u \|_q^q + \frac{\lambda^{\gamma - 1}}{q} \| u \|_q^q$$

$$= \lambda \| \nabla u \|^2 + \lambda^{2\gamma + 1} \| \nabla u \|^{2(\gamma + 1)} - \lambda^{\gamma - 1} \int_{\Omega} |u|^q \ln |u| \, dx - \lambda^{\gamma - 1} \ln \lambda \| u \|_q^q$$

$$= \lambda \left( \| \nabla u \|^2 + \lambda^{2\gamma} \| \nabla u \|^{2(\gamma + 1)} - \lambda^{\gamma - 2} \int_{\Omega} |u|^q \ln |u| \, dx - \lambda^{\gamma - 2} \ln \lambda \| u \|_q^q \right).$$

Let $\varphi(\lambda) = \lambda^{-1} j'(\lambda)$, thus we obtain

$$\varphi(\lambda) = \lambda^{-1} j'(\lambda)$$

$$= \lambda^{-1} \lambda \left( \| \nabla u \|^2 + \lambda^{2\gamma} \| \nabla u \|^{2(\gamma + 1)} - \lambda^{\gamma - 2} \left( \int_{\Omega} |u|^q \ln |u| \, dx - \ln \lambda \| u \|_q^q \right) \right)$$

$$= \| \nabla u \|^2 + \lambda^{2\gamma} \| \nabla u \|^{2(\gamma + 1)} - \lambda^{\gamma - 2} \int_{\Omega} |u|^q \ln |u| \, dx - \lambda^{\gamma - 2} \ln \lambda \| u \|_q^q.$$
which implies that there exists a \( \lambda^* > 0 \) such that \( \varphi'(\lambda) > 0 \) on \((0, \lambda^*)\), \( \varphi'(\lambda) < 0 \) on \((\lambda^*, +\infty)\) and \( \varphi'(\lambda) = 0 \). So, \( \varphi(\lambda) \) is increasing on \((0, \lambda^*)\), decreasing on \((\lambda^*, +\infty)\). Since \( \lim_{\lambda \to 0^+} \varphi(\lambda) = \| \nabla u \|^2 > 0 \), \( \lim_{\lambda \to +\infty} \varphi(\lambda) = -\infty \), there exists a unique \( \lambda^* > 0 \) such that \( \varphi(\lambda^*) = 0 \), i.e., \( j'(\lambda^*) = 0 \). So, \((ii)\) holds. Then, \( j'(\lambda) = \lambda \varphi(\lambda) \) is positive on \((0, \lambda^*)\), negative on \((\lambda^*, +\infty)\). Thus, \( j(\lambda) \) is increasing on \((0, \lambda^*)\), decreasing on \((\lambda^*, +\infty)\) and attains the maximum at \( \lambda^* \). So, \((iii)\) holds. From (2.2), we obtain

\[
I(\lambda u) = \| \nabla (\lambda u) \|^2 + \| \nabla (\lambda u) \|^{2(\gamma + 1)} - \int_{\Omega} |\lambda u|^q \ln |\lambda u| \, dx
\]

\[
= \lambda^2 \| \nabla u \|^2 + \lambda^{2(\gamma + 1)} \| \nabla u \|^{2(\gamma + 1)} - \lambda^q \int_{\Omega} |u|^q \ln |u| \, dx - \lambda^q \int_{\Omega} |u|^q \ln \lambda \, dx
\]

\[
= \lambda^2 \| \nabla u \|^2 + \lambda^{2(\gamma + 1)} \| \nabla u \|^{2(\gamma + 1)} - \lambda^q \int_{\Omega} |u|^q \ln |u| \, dx - \lambda^q \ln \lambda \|u\|^q_q
\]

\[
= \lambda \left( \lambda \| \nabla u \|^2 + \lambda^{2\gamma + 1} \| \nabla u \|^{2(\gamma + 1)} - \lambda^{q-1} \right) \left( \int_{\Omega} |u|^q \ln |u| \, dx - \ln \lambda \|u\|^q_q \right)
\]

\[
= \lambda j'(\lambda).
\]

Thus, \( I(\lambda u) > 0 \) for \( 0 < \lambda < \lambda^* \), \( I(\lambda u) < 0 \) for \( \lambda^* < \lambda < +\infty \) and \( I(\lambda^* u) = 0 \). So, \((iv)\) holds. Therefore, the proof is completed. \( \square \)

**Lemma 2.2.** \( d \) defined by (2.4) is positive and there exists a positive function \( u \in \mathcal{N} \) such that \( J(u) = d \).

**Proof.** Let \( \{u_k\}_{k}^\infty \subset \mathcal{N} \) be a minimizing sequence for \( J \), which means that

\[
\lim_{k \to \infty} J(u_k) = d. \tag{2.6}
\]

It is easy to show that \( \{|u_k|\}_{k} \subset \mathcal{N} \) is also a minimizing sequence for \( J \) due to \( |u_k| \in \mathcal{N} \) and \( J(|u_k|) = J(u_k) \). For this reason, we assume that \( u_k > 0 \) a.e. \( \Omega \) for all \( k \in N \).

On the other hand, we have already observed that \( J \) is coercive on \( \mathcal{N} \) which implies that \( \{u_k\}_{k}^\infty \) is bounded in \( H_0^1(\Omega) \). Let \( \mu > 0 \) be sufficiently small such that \( q + \mu < \frac{2m}{n-2} \). Since \( H_0^1(\Omega) \hookrightarrow L^{q+\mu}(\Omega) \) is compact, there exists a function \( u \) and a subsequence of \( \{u_k\}_{k}^\infty \), still denote by \( \{u_k\}_{k}^\infty \), such that

\[ u_k \to u \text{ weakly in } H_0^1(\Omega), \]

\[ u_k \to u \text{ strongly in } L^{q+\mu}(\Omega), \]

\[ u_k(x) \to u(x) \text{ a.e. in } \Omega. \]

Hence, \( u \geq 0 \) a.e. in \( \Omega \). Firstly, we prove \( u \neq 0 \). Using the dominated convergence theorem, we get

\[
\int_{\Omega} |u|^q \ln |u| \, dx = \lim_{k \to \infty} \int_{\Omega} |u_k|^q \ln |u_k| \, dx, \tag{2.7}
\]

and

\[
\int_{\Omega} |u|^q \, dx = \lim_{k \to \infty} \int_{\Omega} |u_k|^q \, dx. \tag{2.8}
\]
By the weak lower semicontinuity of $H^1_0(\Omega)$, we get
\begin{equation}
\|\nabla u\|^2 \leq \liminf_{k \to \infty} \|\nabla u_k\|^2, \tag{2.9}
\end{equation}
\begin{equation}
\|\nabla u\|^{2(\gamma+1)} \leq \lim_{k \to \infty} \|\nabla u_k\|^{2(\gamma+1)}.
\end{equation}
Using (2.1), (2.6), (2.7), (2.8) and (2.9), we have
\begin{equation}
J(u) = \frac{1}{2} \|\nabla u\|^2 + \frac{1}{2(\gamma+1)} \|\nabla u\|^{2(\gamma+1)} - \frac{1}{q} \int_\Omega |u|^q \ln |u| \, dx + \frac{1}{q^2} \|u\|_q^q
\leq \liminf_{k \to \infty} \frac{1}{2} \|\nabla u_k\|^2 + \lim_{k \to \infty} \frac{1}{2(\gamma+1)} \|\nabla u_k\|^{2(\gamma+1)}
- \lim_{k \to \infty} \frac{1}{q} \int_\Omega |u_k|^q \ln |u_k| \, dx + \lim_{k \to \infty} \frac{1}{q^2} \|u_k\|_q^q
= \liminf_{k \to \infty} \left( \frac{1}{2} \|\nabla u_k\|^2 + \frac{1}{2(\gamma+1)} \|\nabla u_k\|^{2(\gamma+1)} \right)
- \liminf_{k \to \infty} \left( \frac{1}{q} \int_\Omega |u_k|^q \ln |u_k| \, dx + \frac{1}{q^2} \|u_k\|_q^q \right)
= \liminf_{k \to \infty} I(u_k) = d. \tag{2.10}
\end{equation}
Using (2.2), (2.7) and (2.9), we have
\begin{equation}
I(u) = \|\nabla u\|^2 + \|\nabla u\|^{2(\gamma+1)} - \int_\Omega |u|^q \ln |u| \, dx
\leq \liminf_{k \to \infty} \|\nabla u_k\|^2 - \lim_{k \to \infty} \|\nabla u_k\|^{2(\gamma+1)} - \lim_{k \to \infty} \int_\Omega |u_k|^q \ln |u_k| \, dx
= \liminf_{k \to \infty} \left( \|\nabla u_k\|^2 - \|\nabla u_k\|^{2(\gamma+1)} - \int_\Omega |u_k|^q \ln |u_k| \, dx \right)
= \liminf_{k \to \infty} I(u_k) = 0. \tag{2.11}
\end{equation}
Since $u_k \in \mathcal{N}$, we have $I(u_k) = 0$. So, by using the fact $x^{-\mu} \ln x \leq (e\mu)^{-1}$ for $x \geq 1$ and the Sobolev embedding inequality, we obtain
\begin{equation}
\|\nabla u_k\|^2 + \|\nabla u_k\|^{2\delta+2} = \int_\Omega |u_k|^q \ln |u_k| \, dx
\leq (e\mu)^{-1} \int_\Omega |u_k|^{q+\mu} \, dx
\leq (e\mu)^{-1} \|u_k\|_q^{q+\mu}
\leq C \|\nabla u_k\|_2^{q+\mu},
\end{equation}
for some positive constant $C$. This implies that
\begin{equation}
\int_\Omega |u_k|^q \ln |u_k| \, dx = \|\nabla u_k\|^2 + \|\nabla u_k\|^{2(\gamma+1)} \geq C.
\end{equation}
Combining this inequality with (2.7), we obtain
\begin{equation}
\int_\Omega |u|^q \ln |u| \, dx \geq C.
\end{equation}
Thus, we have \( u \in H_0^1(\Omega) \setminus \{0\} \).

If \( I(u_k) < 0 \), by Lemma 2.1, there exists a \( \lambda^* \) such that \( I(\lambda^*u) = 0 \) and \( 0 < \lambda^* < 1 \). Thus, \( \lambda^*u \in \mathcal{N} \). By (2.3), (2.4), (2.8) and (2.9), we have

\[
d \leq J(\lambda^*u)
\]

\[
= \frac{1}{q} I(\lambda^*u) + \frac{q - 2}{2q} \| \nabla (\lambda^*u) \|^2 + \frac{q - 2\gamma - 2}{2q\gamma + 2q} \| \nabla (\lambda^*u) \|^{2(\gamma+1)} + \frac{1}{q^2} \| \lambda^*u \|^q
\]

\[
= \frac{q - 2}{2q} \| \nabla (\lambda^*u) \|^2 + \frac{q - 2\gamma - 2}{2q\gamma + 2q} \| \nabla (\lambda^*u) \|^{2(\gamma+1)} + \frac{1}{q^2} \| \lambda^*u \|^q
\]

\[
= (\lambda^*)^2 \left( \frac{q - 2}{2q} \right) \| \nabla u \|^2 + (\lambda^*)^2(\gamma+1) \left( \frac{q - 2\gamma - 2}{2q\gamma + 2q} \right) \| \nabla u \|^{2(\gamma+1)}
\]

\[
+ (\lambda^*)^2 \left( \frac{1}{q^2} \right) \| u \|^q
\]

\[
\leq (\lambda^*)^2 \liminf_{k \to \infty} \left[ \frac{q - 2}{2q} \| \nabla u \|^2 + \frac{q - 2\gamma - 2}{2q\gamma + 2q} \| \nabla u \|^{2(\gamma+1)} + \frac{1}{q^2} \| u \|^q \right]
\]

\[
\leq (\lambda^*)^2 \liminf_{k \to \infty} J(u_k)
\]

\[
= (\lambda^*)^2d,
\]

which indicates \( \lambda^* \geq 1 \) by \( d > 0 \). It contradicts \( 0 < \lambda^* < 1 \). Then, by (2.11), we have \( I(u) = 0 \). Therefore, \( u \in \mathcal{N} \). By (2.6), we have \( J(u) \geq d \). By (2.10), we have \( J(u) \leq d \). So, \( J(u) = d \).

\[\square\]

3. Blow up

As in ([8]), we introduce the following sets:

\[
\mathcal{W}_1 = \{ u \in H_0^1(\Omega) \setminus \{0\} : J(u) < d \}, \quad \mathcal{W}_2 = \{ u \in H_0^1(\Omega) \setminus \{0\} : J(u) = d \},
\]

\[
\mathcal{W}^+_1 = \{ u \in \mathcal{W}_1 : I(u) > 0 \}, \quad \mathcal{W}^+_2 = \{ u \in \mathcal{W}_2 : I(u) > 0 \},
\]

\[
\mathcal{W}^-_1 = \{ u \in \mathcal{W}_1 : I(u) < 0 \}, \quad \mathcal{W}^-_2 = \{ u \in \mathcal{W}_2 : I(u) < 0 \},
\]

\[
\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2, \quad \mathcal{W}^+ = \mathcal{W}^+_1 \cup \mathcal{W}^+_2, \quad \mathcal{W}^- = \mathcal{W}^-_1 \cup \mathcal{W}^-_2.
\]

**Definition 3.1.** (Maximal Existence Time). Let \( u(t) \) be weak solutions of problem (1.1). We define the maximal existence time \( T_{\text{max}} \) as follows

\[
T_{\text{max}} = \sup \{ T > 0 : u(t) \text{ exists on } [0,T] \}.
\]

Then

(i) If \( T_{\text{max}} < +\infty \), we say that \( u(t) \) blows up in finite time and \( T_{\text{max}} \) is the blow-up time;

(ii) If \( T_{\text{max}} = +\infty \), we say that \( u(t) \) is global.

**Theorem 3.2.** Assume that \( u_0 \in \mathcal{W}^- \), and let \( u(t) \) be a unique local weak solution to the problem (1.1). Then \( u(t) \) blows up in finite time, that is, there exists \( T_* > 0 \)
such that
\[ \lim_{t \to T_\ast} \|u(t)\|_{H_0^1(\Omega)}^2 = \infty. \]

Proof. Since \( u_0 \in W_1^- \), we obtain a unique local solution \( u(t) \) of problem (1.1) satisfying the energy inequality
\[ \int_0^t \|u_s(s)\|_{H_0^1(\Omega)}^2 \, ds + J(u(t)) \leq J(u_0) < d, \quad 0 \leq t \leq T_{\text{max}}, \]
where \( T_{\text{max}} \) is maximal existence time of solution \( u(t) \).

Next, we prove that \( u_0 \in W_1^- \) for \( t \in [0, T_{\text{max}}] \). We assume \( u(t) \) leaves \( W_1^+ \) at time \( t = t_0 \), then there exists a sequence \( \{t_n\} \), such that \( I(u(t_n)) \leq 0 \) as \( t_n \to t_0^- \).

By the lower semicontinuity of \( H_0^1(\Omega) \), we have
\[ I(u(t_0)) \leq \liminf_{n \to \infty} I(u(t_n)) \leq 0. \]

We have \( I(u(t_0)) = 0 \) according to \( u(t_0) \notin W_1^+ \). However, by variational definition of \( d \) and energy inequality, this leads to a contradiction
\[ d = \inf_{u \in \mathcal{N}} J(u) \leq J(u(t_0)) < d. \]

So, \( u(t) \in W_1^- \) for \( t \in [0, T_{\text{max}}] \). Now, we prove that \( u(t) \) blows up at a finite time. By contradiction, we assume that \( u(t) \) is global. We contract a function \( \Gamma : [0, \infty) \to \mathbb{R}^+ \), and
\[ \Gamma(t) = \int_0^t \|u(s)\|_{H_0^1(\Omega)}^2 \, ds. \]

Then, a direct calculation gives
\[ \Gamma'(t) = \|u(s)\|_{H_0^1(\Omega)}^2 - \|u_0\|_{H_0^1(\Omega)}^2 \]
\[ = 2 \int_0^t \int_\Omega (u_s u + \nabla u_s \nabla u) \, dx ds. \quad (3.2) \]

By (2.2) and (3.2), we get
\[ \Gamma''(t) = 2 \int_\Omega (u_s u + \nabla u_s \nabla u) \, dx \]
\[ = 2 \int_\Omega u (u_s - \Delta u_s) \, dx \]
\[ = 2 \int_\Omega |u|^q \ln |u| - 2 \int_\Omega M(\|\nabla u\|^2) u \Delta u \]
\[ = 2 \int_\Omega |u|^q \ln |u| - 2 \int_\Omega (1 + \|\nabla u\|^{2\gamma})(\nabla u)^2 \]
\[ = -2I(u). \quad (3.3) \]

By (3.3) and \( I(u) < 0 \), we have \( \Gamma''(t) > 0 \), so
\[ \Gamma'(t) > \Gamma'(0) = \|u_0\|_{H_0^1(\Omega)}^2 > 0. \quad (3.4) \]
Thanks to the Hölder’s inequality and combining (3.3), we obtain
\[
\frac{1}{4} (\Gamma'(t) - \Gamma'(0))^2 = \frac{1}{4} \left( \int_0^t \Gamma''(s) \, ds \right)^2 
\]
\[
= \left( \int_0^t \int_\Omega (u_s u + \nabla u_s \nabla u) \, dx \, ds \right)^2
\]
\[
\leq \int_0^t \|u(s)\|^2_{H_0^1(\Omega)} \, ds \int_0^t \|u_s\|^2_{H_0^1(\Omega)} \, ds.
\] (3.5)

From (2.3) and (3.3) that it follows
\[
\Gamma''(t) = -2I(u)
\]
\[
= -2qJ(u) + (q - 2) \|\nabla u\|^2 + \frac{q - 2\gamma - 2}{\gamma + 1} \|\nabla u\|^{2(\gamma + 1)} + \frac{2}{q} \|u\|^q
\]
\[
\geq -2qJ(u_0) + 2q \int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} \, ds
\]
\[
+ (q - 2) \|\nabla u\|^2 + \frac{q - 2\gamma - 2}{\gamma + 1} \|\nabla u\|^{2(\gamma + 1)} + \frac{2}{q} \|u\|^q.
\] (3.6)

Since \(u(t) \in W_1^\gamma\), \(t \in [0, T]\), so \(I(u) < 0\). By Lemma 2.1, there exists a \(\lambda^* \in (0, 1)\) such that \(I(\lambda^* u(t)) = 0\). Thus, by the definition of \(d\), it follows that
\[
d = \inf_{u \in \mathcal{N}} J(u) \leq J(\lambda^* u(t))
\]
\[
\leq \frac{q - 2}{2q} \|\nabla u\|^2 + \frac{q - 2\gamma - 2}{2q\gamma + 2q} \|\nabla u\|^{2(\gamma + 1)} + \frac{1}{q^2} \|u\|^q.
\] (3.7)

Combining (3.6) and (3.7), we have
\[
\Gamma''(t) \geq -2qJ(u_0) + 2q \int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} \, ds
\]
\[
+ (q - 2) \|\nabla u\|^2 + \frac{q - 2\gamma - 2}{\gamma + 1} \|\nabla u\|^{2(\gamma + 1)} + \frac{2}{q} \|u\|^q
\]
\[
= -2qJ(u_0) + 2q \int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} \, ds
\]
\[
+ 2q \left[ \frac{q - 2}{2q} \|\nabla u\|^2 + \frac{q - 2\gamma - 2}{2q\gamma + 2q} \|\nabla u\|^{2(\gamma + 1)} + \frac{1}{q^2} \|u\|^q \right]
\]
\[
\geq 2q (d - J(u_0)) + 2q \int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} \, ds.
\] (3.8)

Using (3.1), (3.5) and (3.8), we get
\[
\Gamma(t)\Gamma''(t) \geq 2q \int_0^t \|u(s)\|^2_{H_0^1(\Omega)} \, ds \int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} \, ds + 2q (d - J(u_0)) \Gamma(t)
\]
\[
\geq 2q (d - J(u_0)) \Gamma(t) + \frac{q}{2} (\Gamma'(t) - \Gamma'(0))^2.
\] (3.9)

Now, fix a \(t_0 > 0\), then from (3.4), we get
\[
\Gamma(t) \geq \Gamma(t_0) \geq \|u_0\|^2_{H_0^1(\Omega)} t_0 > 0, \text{ for all } t \geq t_0.
\] (3.10)
Hence, by (3.9) and (3.10), we get
\[
\Gamma(t)\Gamma''(t) - \frac{q}{2} (\Gamma'(t) - \Gamma'(0))^2 \geq 2q (d - J(u_0)) \|u_0\|_{H_0^1(\Omega)}^2 t_0 > 0, \text{ for all } t \geq t_0.
\] (3.11)

Choose \( T > t_0 \) sufficiently large and let
\[
\psi(t) = \Gamma(t) + (T - t) \|u_0\|_{H_0^1(\Omega)}^2 t_0, \text{ for all } t \in [0, T].
\]

Then \( \psi(t) \geq \Gamma(t) > 0, \psi'(t) = \Gamma'(t) - \Gamma'(0) \) and \( \psi''(t) = \Gamma''(t) > 0 \), so (3.11) implies
\[
\psi(t)\psi''(t) - \frac{q}{2} \psi'(t)^2 \geq \Gamma(t)\Gamma''(t) - \frac{q}{2} (\Gamma'(t) - \Gamma'(0))^2
\geq 2q (d - J(u_0)) \|u_0\|_{H_0^1(\Omega)}^2 t_0 > 0, \text{ for all } t \in [t_0, T].
\] (3.12)

Let \( y(t) = \psi(t)^{-\frac{2q}{2}} \). Thus,
\[
y'(t) = -\frac{q - 2}{2} \psi(t)^{-\frac{2q}{2}} \psi'(t).
\] (3.13)

From (3.12) and (3.13), we get
\[
y''(t) = \frac{q(q - 2)}{4} \psi(t)^{-\frac{2q+2}{2}} \psi'(t)^2 - \frac{q - 2}{2} \psi(t)^{-\frac{2q}{2}} \psi''(t)
= \frac{q - 2}{2} \psi(t)^{-\frac{2q+2}{2}} \left[ \frac{q}{2} \psi'(t)^2 - \psi(t) \psi''(t) \right] < 0, \text{ for all } t \in [t_0, T].
\]

This shows that, for any sufficiently large \( T > t_0 \), \( y(t) \) is a concave function in \([t_0, T]\). Since \( y(t_0) > 0 \) and \( y'(t_0) < 0 \), there exists a finite time \( T_* \) such that
\[
\lim_{t \to T_*^-} y(t) = 0,
\]
which implies
\[
\lim_{t \to T_*^-} \psi(t) = \infty.
\]

Hence, we get
\[
\lim_{t \to T_*^-} \int_0^t \|u(s)\|_{H_0^1(\Omega)}^2 ds = \infty,
\]
i.e.,
\[
\lim_{t \to T_*^-} \|u(s)\|_{H_0^1(\Omega)}^2 = \infty.
\]

This contradicts with \( u(t) \) being a global solution. \( \square \)
References


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