Gulf Journal of Mathematics
Vol 9, Issue 2 (2020) 40-45

A NOTE ON THE AUTOCOVARIANCE OF p-SERIES LINEAR PROCESS

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Abstract. In this note, we provide tight boundaries for the autocovariance function of a stochastic linear process with p-series coefficients.

1. Introduction

Linear processes are a large class of time series that includes autoregressive moving average (ARMA) models. A linear process is defined by the following equation

$$x_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}, \tag{1.1}$$

where $w_t \sim wn(0, \sigma^2)$ is white noise and $\sum_{-\infty}^{\infty} |\psi_j| < \infty$. Linear processes are widely used in science and engineering to model various stochastic phenomena. Despite their simple definition linear processes possess rich mathematical structure. A good introduction to the topic can be found in [**?**, **?**, **?**].

Linear processes are popular due to their amenable mathematical properties. One of the main properties of a linear process is stationarity. It implies that the autocovariance function of the time series depends only on the lag between the time points. In the case of causal AR(1) models, we further know the exact analytical expression for the autocovariance function. Recall that AR(1) process can be described by the difference equation $x_t = \phi x_{t-1} + w_{t-j}$, where $w_t \sim wn(0, \sigma^2)$. The difference equation can be rewritten in the form of a linear process as follows

$$x_t = \sum_{j=0}^{\infty} \phi^j w_{t-j}, \tag{1.2}$$

where $\phi < 1$. It follows from Equation **??** that the autocovariance function of the AR(1) process is given by $\gamma(h) = \frac{\sigma_w^2 \phi^h}{1-\phi^2}$, where h is the lag between time instances. However, in general, the exact expression for the autocovariance function of a linear time series is not available. Our goal is to provide an analytical estimate of the autocovariance function when the coefficients of the linear process are given by a p-series.

Date: Received: Oct 10, 2020; Accepted: Dec 20, 2020.
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2010 *Mathematics Subject Classification.* Primary 62M10; Secondary 91B84.
Key words and phrases. linear process, time series, arma, autocovariance, acf.

The autocovariance function is one of the main tools in studying time series. Unfortunately, in most cases, the exact formula of the autocovariance function does not exist. In this note, we obtain an analytical estimate of the autocovariance function in a special case of a linear process whose coefficients are p-series. In particular, we consider a linear process of the form

$$y_t = \sum_{j=0}^{\infty} \frac{1}{(j+1)^2} w_{t-j}, \tag{1.3}$$

where $w_t \sim wn(0, \sigma^2)$. We obtain tight upper and lower bounds on the autocovariance function of the time series y_t. We show via numerical experiments that the proposed bounds are effective. Although a general analytical formula for the theoretical autocovariance of a linear process does not exist there exists a number of algorithms for calculating the autocovariance in special cases. An exact method for calculating the autocovariance of an autoregressive process was proposed by Pagano [?]. A more general approach to calculate the autocovariance for ARMA models was proposed by McLeod in [?]. His approach was based on an earlier procedure described in [?]. Going one step further Ansley proposed an algorithm for calculating the autocovariance for a vector ARMA process [?].

The paper is structured as follows. In Section 2, we develop the analytical bounds on the autocovariance function and present the numerical experiments to test the efficiency of the proposed bounds. Section 3 concludes the paper with a short summary.

2. Autocovariance estimate

In this section, we present our main result. We develop analytic upper and lower bounds for the autocovariance function values. Then we test the analytic formulas via numerical experiments. The results of the numerical experiments are largely consistent with the theoretical expectations.

2.1. Analytic bounds.

Let y_t be the times series given in Equation **??**. For convenience, we rewrite Equation **??** as

$$y_t = \sum_{j=1}^{\infty} \frac{1}{j^2} w_{t-j}, \tag{2.1}$$

where $w_t \sim wn(0, \sigma^2)$. The autocovariance function of y_t is given by

$$\begin{aligned}
\gamma(t+h,t) &= \mathrm{Cov}(x_{t+h}, x_t) \\
&= \mathrm{Cov}(\sum_{k=1}^{\infty} \frac{1}{k^2} w_{t+h-k}, \sum_{j=1}^{\infty} \frac{1}{j^2} w_{t-j}) \\
&= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{k^2} \frac{1}{j^2} \mathrm{Cov}(w_{t+h-k}, w_{t-j}) \\
&= \sum_{j=1}^{\infty} \frac{1}{(j+h)^2} \frac{1}{j^2} \mathrm{Cov}(w_{t-j}, w_{t-j}) \\
&= \sigma_w^2 \sum_{j=1}^{\infty} \frac{1}{j^2 (j+h)^2}.
\end{aligned} \tag{2.2}$$

We get that the autocovariance function for the time series y_t is given by the equation

$$\gamma(h) = \sigma_w^2 \sum_{j=1}^{\infty} \frac{1}{j^2 (j+h)^2}. \tag{2.3}$$

Our primary goal in this section is to determine the upper and lower bounds for $\gamma(h)$ in Equation **??**. To obtain the upper bound we use the following inequality

$$\sum_{j=2}^{\infty} \frac{1}{j^2 (j+h)^2} < \int_1^{\infty} \frac{1}{x^2 (x+h)^2} \, dx \tag{2.4}$$

The inequality in Equation **??** follows from comparing the areas of rectangles of height $\frac{1}{j^2(j+h)^2}$ to the area under the graph of the function $f(x) = \frac{1}{x^2(x+h)^2}$. Similarly, the lower bound is given by the following inequality

$$\sum_{j=1}^{\infty} \frac{1}{j^2 (j+h)^2} > \int_1^{\infty} \frac{1}{x^2 (x+h)^2} \, dx \tag{2.5}$$

Combining the inequalities in Equations **??** and **??** we obtain the following bounds

$$\mathcal{I} < \gamma(h) < \mathcal{I} + \frac{1}{(1+h)^2}, \tag{2.6}$$

where $\mathcal{I} = \int_1^{\infty} \frac{1}{x^2(x+h)^2} \, dx$.

It remains to calculate the value of the integral $\mathcal{I}$. To this end, we decompose the integrand using partial fractions

$$\frac{1}{x^2 (x+h)^2} = -\frac{2}{h^3 x} + \frac{1}{h^2 x^2} + \frac{2}{h^3 (x+h)} + \frac{1}{h^2 (x+h)^2}. \tag{2.7}$$

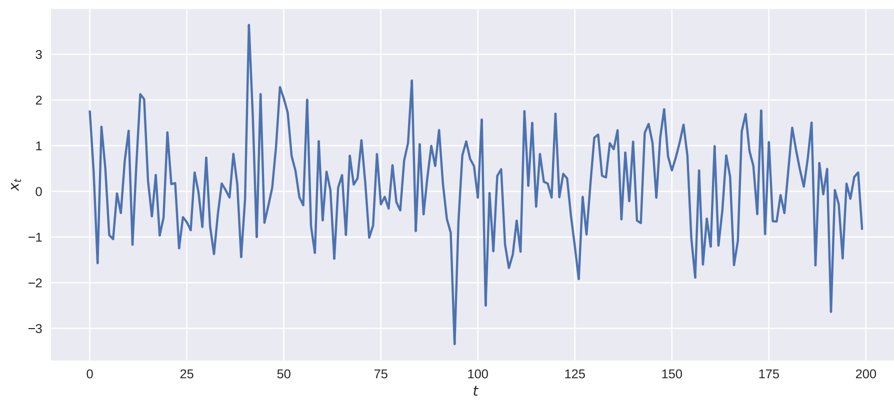

FIGURE 1. The initial portion of the simulated times series.

Using the partial fraction decomposition we calculate the integral

$$
\begin{aligned}
\int_1^\infty \frac{1}{x^2(x+h)^2}\,dx &= \int_1^\infty -\frac{2}{h^3 x} + \frac{1}{h^2x^2} + \frac{2}{h^3(x+h)} + \frac{1}{h^2(x+h)^2}\,dx \\
&= \left[-\frac{2\ln|x|}{h^3} - \frac{1}{h^2 x} + \frac{2\ln|x+h|}{h^3} - \frac{1}{h^2(x+h)} \right]\Bigg|_1^\infty \\
&= \left[\frac{2}{h^3}\ln\left|\frac{x+h}{x}\right| - \frac{1}{h^2x} - \frac{1}{h^2(x+h)} \right]\Bigg|_1^\infty \\
&= -\frac{2}{h^3}\ln(1+h) + \frac{1}{h^2} + \frac{1}{h^2(1+h)}.
\end{aligned} \tag{2.8}
$$

We summarize the above calcualtions in the following theorem.

Theorem 2.1. *Let* $y_t = \sum_{j=1}^{\infty} \frac{1}{j^2} w_{t-j}$, *where* $w_t \sim wn(0,\sigma^2)$ *be a linear time series process. Then the autocovariance function of* y_t *is bounded by the following double inequality*

$$\mathcal{I} < \gamma(h) < \mathcal{I} + \frac{1}{(1+h)^2},$$

where $\mathcal{I} = -\frac{2}{h^3}\ln(1+h) + \frac{1}{h^2} + \frac{1}{h^2(1+h)}$.

2.2. **Numerical experiment.** To test the theoretical findings above we designed a simulated time series. We compared the theoretical upper and lower bounds to the sample autocovariance function values. The time series was simulated based on Equation **??**, where $w_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$ and $t = 1, 2, ..., 5000$. The initial portion of the time series graph is presented in Figure **??**.

The simulated time series was used to calculate the sample autocovariance function values. The initial autocovariance values are provided in Figure **??** together with the theoretical upper and lower bounds. As shown, in Figure **??**, the results of the numerical experiments match the theoretical expectations. The small deviation of the autocovariance at the lag $h = 7$ is due the sampling variation.

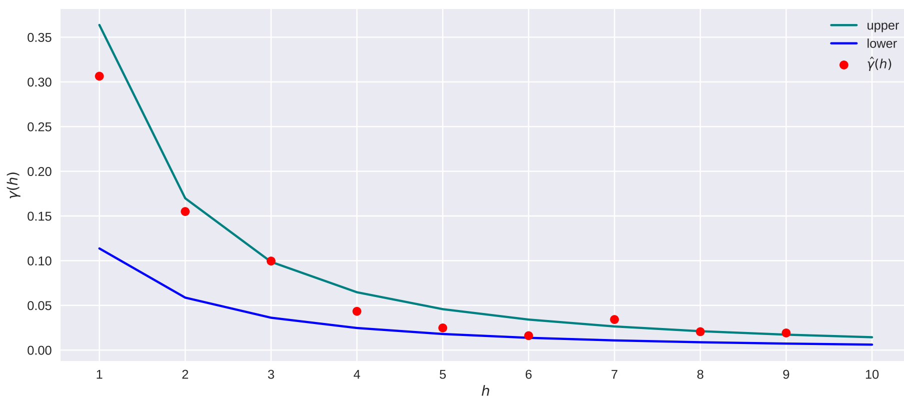

Figure 2. The theoretical upper and lower bounds together with sample autocovariance function values.

3. Conclusion

Linear processes are widely used in many areas of science and engineering. A range of stochastic processes can be described using a linear model [?, ?, ?]. Autocovariance function is an important tool in the study of times series. Unfortunately, in most cases, there does not exist a closed form formula for autocovariance function. In this paper, we develop analytic upper and lower bounds for a linear process based on p-series. Numerical experiments support our theoretical expectations. We believe that the approach outlined in this paper can be further utilized in other time series cases.

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