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ADAPTIVE NUMERICAL QUENCHING FOR A NONLOCAL NEUMANN PROBLEM WITH NONHOMOGENEOUS FLUX

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Abstract. We analyze an adaptive approximation of quenching for a nonlocal parabolic problem with a nonhomogeneous Neumann-type flux. The flux is treated as a nonnegative exterior input, while the singular source is controlled by a mesh-scaled barrier time step. The discrete scheme preserves the admissible range, quenches in finite accumulated time, and satisfies upper and lower estimates for the numerical quenching time. We also prove convergence before quenching and convergence of fixed-level detection times. The numerical tests verify the invariant bound, the monotone influence of the flux intensity, and the second-order spatial behavior of the detected time.

Keywords. Nonlocal diffusion, Neumann flux, quenching, adaptive time step, numerical quenching time
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1. Introduction

Let $\Omega \subset \mathbb{R}$ be a bounded interval. We consider the nonlocal parabolic problem

$$\begin{cases} u_t(x,t) = \int_\Omega J(x-y)\Big(u(y,t)-u(x,t)\Big)\,dy & \\ \qquad\qquad + \Phi(x,t) + \lambda a(x)(1-u(x,t))^{-p}, & x\in\Omega,\ t>0, \\ u(x,0)=u_0(x), & x\in\Omega. \end{cases} \tag{1.1}$$

Here $p > 0$, $\lambda > 0$, J is a nonnegative symmetric kernel, a is a bounded weight which is positive on the active reaction zone, Φ is a prescribed nonnegative nonhomogeneous Neumann flux, and $0 \le u_0 < 1$.

The integral term is the nonlocal Neumann diffusion operator. It accounts for weighted exchanges inside Ω and does not prescribe exterior Dirichlet values. The term $\Phi(x,t)$ is a nonnegative input produced by exterior or boundary interactions. The singular reaction $\lambda a(x)(1-u)^{-p}$ acts at the finite level $u = 1$ and forces the loss of regularity when the distance $1-u$ tends to zero.

The numerical difficulty is not only the approximation of the integral operator. The update must preserve the singular constraint $u < 1$ at every accepted step,

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while the nonhomogeneous flux accelerates the approach to the critical level. The present analysis is built around the discrete barrier $1 - U_i^n$ and around a time step whose mesh dependence is entirely carried by an h^2 factor.

For $0 < t < T_q$, a quenching solution satisfies

$$\sup_{x\in\Omega} u(x,t) < 1, \qquad \lim_{t\uparrow T_q} \|u(\cdot,t)\|_\infty = 1.$$

Thus the amplitude remains bounded, but the reaction term becomes unbounded at the finite threshold.

The nonlocal diffusion operator is

$$(\mathcal{L}_J v)(x) = \int_\Omega J(x-y)\Big(v(y) - v(x)\Big)\,dy.$$

Since the exchange is restricted to Ω, this operator corresponds to a nonlocal Neumann interaction rather than to an exterior Dirichlet prescription. A nonhomogeneous contribution may be induced by exterior data through

$$\Phi(x,t) = \int_{\mathcal{E}} K_N(x,y)\psi(y,t)\,d\mu(y),$$

where $\mathcal{E}$ is an exterior or boundary interaction region, $K_N \geq 0$ is an interaction kernel, and $\psi \geq 0$ is an imposed exterior state. The analysis only uses the sign and boundedness of the resulting function Φ.

Nonlocal diffusion with integrable kernels appears in convolution models, volume-constrained formulations, dispersal equations, and long-range interaction approximations. Analytical and numerical tools for such operators include integral diffusion, nonlocal boundary interactions, comparison arguments, invariant regions and singular-time estimates; see [1, 2, 5, 6, 12]. Singular parabolic problems and numerical singular-time approximations have been studied for quenching, blow-up, adaptive time stepping and nonlocal reaction-diffusion equations; see [9, 11, 8, 7, 3, 4, 10]. The present paper keeps the kernel-based nonlocal Neumann structure but focuses on a finite singular level produced by the reaction $(1-u)^{-p}$.

The flux Φ changes the quenching process. Since $\Phi \geq 0$, it increases the right-hand side and may reduce the time needed to approach 1. A fixed time step can cross the singular level when $1 - U_i^n$ becomes small. We therefore use an adaptive step of the form

$$\Delta t_n = 5h^2\Lambda^n,$$

where Λ^n is a dimensionless barrier limiter depending on the nodal distances $1 - U_i^n$ and on the current right-hand side. This choice keeps the time scale tied to h^2 and prevents an accepted step from overshooting the level 1.

The main contribution is a discrete quenching analysis in which the nonhomogeneous flux is included without losing the invariant constraint. More precisely:

(i) A nonlocal Neumann quenching model with nonhomogeneous flux and weighted singular reaction is formulated.

(ii) An h^2-scaled adaptive explicit scheme is constructed, together with an implicit-loss variant for comparison.

(iii) Positivity and the strict invariant constraint $0 \leq U_i^n < 1$ are proved under the barrier time-step rule.

(iv) Comparison with respect to the nonhomogeneous flux is established before quenching.
(v) Finite-time numerical quenching, an upper bound, a monotone-maximum lower bound, convergence before quenching, and convergence of fixed-level times are proved.
(vi) Numerical tests use the reference flux $\rho = 0$, the mesh sequence $I = 16, \dots, 512$, and perturbations $\varepsilon = 0$ and $\varepsilon = 10^{-k}$, $1 \le k \le 4$.

The analysis is organized around the barrier $1 - U_i^n$. This separates the present quenching study from a nonlocal blow-up analysis with nonhomogeneous flux. The central quantities are the invariant interval $[0,1)$, the minimum nodal distance to 1, and the fixed-level times associated with $1-\delta$.

The paper is organized as follows. Section 2 states the assumptions and records continuous comparison estimates. Section 3 introduces the quadrature approximation of the nonlocal Neumann operator. Section 4 defines the adaptive schemes. Section 5 proves positivity, the strict upper bound, and comparison with respect to the flux. Section 6 proves numerical quenching and estimates the quenching time. Section 7 proves convergence before quenching and fixed-level convergence. Section 8 presents the numerical tests. Section 9 summarizes the numerical validation checks. Section 10 concludes the paper.

2. Assumptions and continuous estimates

We work on $\Omega = (-1, 1)$ in the numerical part. The estimates in this section only require that Ω is a bounded interval.

Assumption 2.1. The kernel J satisfies

$$J \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \qquad J \ge 0, \qquad J(z) = J(-z), \qquad \int_{\mathbb{R}} J(z)\, dz = 1.$$

Assumption 2.2. The coefficients satisfy

$$0 < a_* \le a(x) \le a^*, \qquad 0 \le \Phi(x,t) \le \Phi^*,$$

for all $x \in \overline{\Omega}$ and $t \ge 0$. The initial datum satisfies

$$0 \le u_0(x) \le 1 - \eta_0 \qquad (x \in \overline{\Omega})$$

for some $\eta_0 \in (0,1)$.

Definition 2.1. A classical solution of (1.1) quenches at $T_q < \infty$ if

$$0 \le u(x,t) < 1 \quad (x \in \Omega,\ 0 < t < T_q),$$
$$\lim_{t \uparrow T_q} \|u(\cdot,t)\|_\infty = 1.$$

Remark 2.1. The continuous estimates used below are applied on time intervals on which the exact solution remains separated from the singular level. On such intervals, the singular source is locally Lipschitz in the unknown, and the nonlocal diffusion term is bounded and Lipschitz in the uniform norm under the stated assumptions on the kernel. Hence the usual semilinear evolution argument gives local existence and uniqueness before quenching. The comparison estimates below are used only on such subcritical intervals.

The following comparison estimate explains why the singular level is reached in finite time under a positive weight. It is not used as a replacement for the discrete proof, but it gives the continuous scale of the quenching time.

Lemma 2.1. *Let u solve (1.1) with $0 \le u < 1$. Set*

$$m(t) = \min_{x \in \overline{\Omega}} u(x,t).$$

At every differentiability point of m,

$$m'(t) \ge \lambda a_*(1 - m(t))^{-p}.$$

Consequently,

$$T_q \le \frac{(1 - m(0))^{p+1}}{\lambda a_*(p+1)}.$$

Proof. Let $x_t \in \overline{\Omega}$ satisfy $m(t) = u(x_t,t)$. Then

$$u(y,t) - u(x_t,t) \ge 0 \qquad (y \in \Omega).$$

Since $J \ge 0$,

$$(\mathcal{L}_J u)(x_t,t) = \int_\Omega J(x_t - y)(u(y,t) - u(x_t,t))\,dy \ge 0.$$

Using $a(x_t) \ge a_*$ and $\Phi(x_t,t) \ge 0$,

$$m'(t) = u_t(x_t,t) \ge \lambda a_*(1 - m(t))^{-p}.$$

Put $d(t) = 1 - m(t)$. Then

$$d'(t) \le -\lambda a_* d(t)^{-p}.$$

Multiplying by $d(t)^p$ gives

$$\frac{d}{dt} d(t)^{p+1} = (p+1)d(t)^p d'(t) \le -\lambda a_*(p+1).$$

Thus $d(t)$ reaches zero no later than

$$\frac{d(0)^{p+1}}{\lambda a_*(p+1)}.$$

□

Lemma 2.2. *Let u and v be two solutions of (1.1) with fluxes Φ and Ψ, respectively. Assume that $u_0 \le v_0$, $\Phi \le \Psi$, and both solutions remain in $[0, 1-\eta]$ on $[0,T]$. Then*

$$u(x,t) \le v(x,t) \qquad (x \in \Omega,\ 0 \le t \le T).$$

Proof. Set $w = v - u$. On $[0, 1-\eta]$, the map $s \mapsto (1-s)^{-p}$ is increasing and Lipschitz. Hence, at a negative minimum point of w,

$$(\mathcal{L}_J w)(x,t) \ge 0,$$

$$\lambda a(x)\Big((1-v)^{-p} - (1-u)^{-p}\Big) \ge -L_\eta |w(x,t)|,$$

where $L_\eta = \lambda a^* p \eta^{-p-1}$. Let $\widetilde{w} = e^{-L_\eta t} w$. If $\widetilde{w}$ had a first negative minimum at (x_0,t_0), then

$$\widetilde{w}_t(x_0,t_0) \le 0, \qquad (\mathcal{L}_J \widetilde{w})(x_0,t_0) \ge 0.$$

The equation for $\widetilde{w}$ gives

$$\widetilde{w}_t - (\mathcal{L}_J \widetilde{w}) \geq e^{-L_\eta t}(\Psi - \Phi) \geq 0,$$

which contradicts the strict first negative minimum condition. Therefore $\widetilde{w} \geq 0$ on $\Omega \times [0, T]$, and hence $u \leq v$. □

3. Spatial quadrature of the nonlocal Neumann operator

Let $I \geq 1$, $h = 2/I$, and

$$x_i = -1 + ih, \qquad 0 \leq i \leq I.$$

The trapezoidal weights on $[-1, 1]$ are

$$\omega_0 = \omega_I = \frac{h}{2}, \qquad \omega_j = h \quad (1 \leq j \leq I-1).$$

Define

$$A_{ij} = \omega_j J(x_i - x_j), \qquad d_i = \sum_{j=0}^{I} A_{ij}.$$

The discrete nonlocal operator is

$$(L_h U)_i = \sum_{j=0}^{I} A_{ij}(U_j - U_i).$$

We assume throughout that

$$0 \leq d_i \leq d^* \qquad (0 \leq i \leq I),$$

where d^* is independent of i and h. This is automatic for bounded kernels and the above quadrature weights when h is sufficiently small.

Lemma 3.1. *Let $U = (U_0, \ldots, U_I)$. If $U_k = \min_i U_i$, then*

$$(L_h U)_k \geq 0.$$

If $U_\ell = \max_i U_i$, then

$$(L_h U)_\ell \leq 0.$$

Moreover, for every i,

$$|(L_h U)_i| \leq 2d^* \|U\|_\infty.$$

Proof. If $U_k = \min_i U_i$, then $U_j - U_k \geq 0$ for every j. Since $A_{kj} \geq 0$,

$$(L_h U)_k = \sum_{j=0}^{I} A_{kj}(U_j - U_k) \geq 0.$$

If $U_\ell = \max_i U_i$, then $U_j - U_\ell \leq 0$ for every j. Therefore

$$(L_h U)_\ell \leq 0.$$

For arbitrary i,

$$|(L_h U)_i| \leq \sum_{j=0}^{I} A_{ij}|U_j - U_i| \leq 2d_i \|U\|_\infty \leq 2d^* \|U\|_\infty.$$

□

Lemma 3.2. *For all vectors $U, V \in \mathbb{R}^{I+1}$,*

$$\|L_h U - L_h V\|_\infty \le 2d^* \|U - V\|_\infty .$$

Proof. Set $W = U - V$. Then

$$(L_h U - L_h V)_i = (L_h W)_i.$$

Lemma 3.1 gives

$$|(L_h W)_i| \le 2d^* \|W\|_\infty .$$

Hence the asserted estimate follows. □

The semidiscrete system is

$$\frac{dU_i}{dt} = (L_h U)_i + \Phi_i(t) + \lambda a_i (1 - U_i)^{-p}, \qquad 0 \le i \le I,$$

where

$$a_i = a(x_i), \qquad \Phi_i(t) = \Phi(x_i, t).$$

The spatial discretization preserves the sign of the nonlocal diffusion at minima and maxima. This property is used in the invariant and quenching estimates below.

4. Adaptive schemes

For a vector $U^n = (U_0^n, \ldots, U_I^n)$, define

$$M^n = \max_{0 \le i \le I} U_i^n, \qquad m^n = \min_{0 \le i \le I} U_i^n, \qquad D^n = 1 - M^n.$$

The explicit adaptive approximation is

$$U_i^{n+1} = U_i^n + \Delta t_n F_i^n, \tag{4.1}$$

where

$$F_i^n = (L_h U^n)_i + \Phi_i^n + \lambda a_i (1 - U_i^n)^{-p}.$$

Here $\Phi_i^n = \Phi(x_i, t_n)$. The time step is written as an h^2-scaled barrier limiter:

$$\Delta t_n = \gamma h^2 \Lambda^n, \qquad \Lambda^n = \min\left\{1, \theta \min_{F_i^n > 0} \frac{1 - U_i^n}{F_i^n}\right\}, \qquad 0 < \theta < 1. \tag{4.2}$$

If $F_i^n \le 0$ for all i, we set $\Lambda^n = 1$. Thus every time increment is exactly h^2 multiplied by a dimensionless limiter. The limiter contains no inverse power of h and only reduces the parabolic step when a component is close to the singular level.

The implicit-loss variant used in the numerical comparison is

$$U_i^{n+1} = \frac{U_i^n + \Delta t_n \left(\sum_{j=0}^{I} A_{ij} U_j^n + \Phi_i^n + \lambda a_i (1 - U_i^n)^{-p}\right)}{1 + \Delta t_n d_i}. \tag{4.3}$$

Only the loss part $d_i U_i$ is implicit. The singular reaction is evaluated at level n, and the same h^2-scaled barrier step is used.

Definition 4.1. The explicit adaptive approximation quenches numerically if

$$0 \le U_i^n < 1 \qquad (0 \le i \le I,\ n \ge 0),$$

$$\lim_{n\to\infty} M^n = 1, \qquad T_h^q := \sum_{n=0}^{\infty} \Delta t_n < \infty.$$

The number T_h^q is called the numerical quenching time.

Remark 4.1. The limiter Λ^n is dimensionless. It equals 1 away from the singular level and decreases only when the parabolic step γh^2 would move at least one component too close to 1. If $F_i^n > 0$ and $\gamma h^2 \le 1$, then

$$\Delta t_n F_i^n \le \gamma h^2 \theta (1 - U_i^n) \le \theta(1 - U_i^n),$$

which keeps every component strictly below the singular level after the update.

5. Invariant bounds and flux comparison

The invariant bound is the central estimate of this paper. It replaces the large-amplitude estimate used in a blow-up analysis.

Proposition 5.1. *Assume $0 \le U_i^n < 1$ for all i. If*

$$\Delta t_n d_i \le 1 \qquad (0 \le i \le I),$$

then the explicit update (4.1) satisfies

$$U_i^{n+1} \ge 0 \qquad (0 \le i \le I).$$

Proof. Using the definition of L_h, write

$$U_i^{n+1} = U_i^n (1 - \Delta t_n d_i) + \Delta t_n \sum_{j=0}^{I} A_{ij} U_j^n + \Delta t_n \Phi_i^n + \Delta t_n \lambda a_i (1 - U_i^n)^{-p}.$$

The assumptions give

$$U_i^n (1 - \Delta t_n d_i) \ge 0, \qquad A_{ij} U_j^n \ge 0,$$
$$\Phi_i^n \ge 0, \qquad \lambda a_i (1 - U_i^n)^{-p} \ge 0.$$

Each term in the representation of U_i^{n+1} is nonnegative. Therefore $U_i^{n+1} \ge 0$. $\square$

Proposition 5.2. *Assume $0 \le U_i^n < 1$ for all i, let Δt_n be given by (4.2), and suppose $\gamma h^2 \le 1$. Then*

$$U_i^{n+1} < 1 \qquad (0 \le i \le I).$$

More precisely, if $F_i^n > 0$, then

$$1 - U_i^{n+1} \ge (1 - \theta)(1 - U_i^n).$$

Proof. If $F_i^n \le 0$, then

$$U_i^{n+1} = U_i^n + \Delta t_n F_i^n \le U_i^n < 1.$$

If $F_i^n > 0$, the time-step definition gives

$$\Delta t_n F_i^n \le \gamma h^2 \theta (1 - U_i^n) \le \theta (1 - U_i^n).$$

Therefore

$$U_i^{n+1} \le U_i^n + \theta(1 - U_i^n) = 1 - (1 - \theta)(1 - U_i^n) < 1.$$

Rearranging the last inequality gives

$$1 - U_i^{n+1} \geq (1-\theta)(1-U_i^n).$$

□

Corollary 5.1. *Assume $0 \leq U_i^0 < 1$ for all i, $\Phi_i^n \geq 0$, $a_i > 0$, and*

$$\gamma h^2 d^* \leq 1, \qquad \gamma h^2 \leq 1.$$

Then the explicit adaptive solution satisfies

$$0 \leq U_i^n < 1 \qquad (0 \leq i \leq I,\ n \geq 0).$$

Proof. The assertion is true at $n = 0$ by the initial condition. Suppose $0 \leq U_i^n < 1$ for all i. Since

$$\Delta t_n = \gamma h^2 \Lambda^n \leq \gamma h^2, \qquad \gamma h^2 d_i \leq \gamma h^2 d^* \leq 1,$$

Proposition 5.1 gives $U_i^{n+1} \geq 0$. Proposition 5.2 gives $U_i^{n+1} < 1$. Induction gives the result for all n. □

Proposition 5.3. *Assume $0 \leq U_i^n < 1$ and $\gamma h^2 \leq 1$. The implicit-loss update (4.3), with the same h^2-scaled barrier step (4.2) computed from the explicit right-hand side, satisfies*

$$0 \leq U_i^{n+1} < 1.$$

Proof. The numerator in (4.3) is nonnegative because $A_{ij} \geq 0$, $U_j^n \geq 0$, $\Phi_i^n \geq 0$, and $a_i > 0$. The denominator is $1 + \Delta t_n d_i > 0$. Hence $U_i^{n+1} \geq 0$. The explicit numerator before division satisfies

$$U_i^n + \Delta t_n \left(\sum_{j=0}^{I} A_{ij} U_j^n - d_i U_i^n + \Phi_i^n + \lambda a_i (1 - U_i^n)^{-p} \right) < 1$$

by Proposition 5.2. Since the implicit-loss formula subtracts the loss through the positive denominator, it cannot increase the value above the corresponding explicit-loss update. Thus $U_i^{n+1} < 1$. □

Proposition 5.4. *Let U^n and V^n be two explicit adaptive solutions computed on the same grid with the same time levels, the same kernel, the same weight, and the same initial datum. Assume*

$$\Phi_i^n \leq \Psi_i^n \qquad (0 \leq i \leq I,\ n \geq 0).$$

Assume also that both solutions remain in $[0, 1-\eta]$ for $0 \leq n \leq N$. If

$$\Delta t_n d^* \leq 1 \qquad (0 \leq n < N),$$

then

$$U_i^n \leq V_i^n \qquad (0 \leq i \leq I,\ 0 \leq n \leq N).$$

Proof. Set $f(s) = (1-s)^{-p}$. On $[0, 1-\eta]$, the function f is increasing.

Step 1. Monotone form of the update. The explicit update can be written as

$$\mathcal{S}_i(W, \Phi^n) = W_i(1 - \Delta t_n d_i) + \Delta t_n \sum_{j=0}^{I} A_{ij} W_j + \Delta t_n \Phi_i^n + \Delta t_n \lambda a_i f(W_i).$$

Since $\Delta t_n d_i \le \Delta t_n d^* \le 1$, $A_{ij} \ge 0$, $a_i \ge 0$, and f is increasing, the map $W \mapsto \mathcal{S}(W, \Phi^n)$ is componentwise nondecreasing on $[0, 1-\eta]^{I+1}$.

Step 2. Induction. At $n = 0$, $U_i^0 = V_i^0$ for every i. Assume $U_i^n \le V_i^n$ for every i. Since $\Phi_i^n \le \Psi_i^n$,

$$U_i^{n+1} = \mathcal{S}_i(U^n, \Phi^n) \le \mathcal{S}_i(V^n, \Psi^n) = V_i^{n+1}.$$

Induction gives $U_i^n \le V_i^n$ for $0 \le n \le N$. □

6. Finite-time numerical quenching

We now prove that the adaptive trajectory approaches the singular level in finite accumulated time. The proof uses the minimum to force approach to the singular level and the barrier step to sum the final time increments.

Lemma 6.1. *Under the hypotheses of Corollary 5.1,*

$$m^{n+1} \ge m^n + \Delta t_n \lambda a_* (1 - m^n)^{-p}.$$

Proof. For each fixed i, write the update as

$$U_i^{n+1} = U_i^n(1 - \Delta t_n d_i) + \Delta t_n \sum_{j=0}^{I} A_{ij} U_j^n + \Delta t_n \Phi_i^n + \Delta t_n \lambda a_i (1 - U_i^n)^{-p}.$$

By Corollary 5.1, the positivity condition gives

$$1 - \Delta t_n d_i \ge 0.$$

Since $U_j^n \ge m^n$ for every j,

$$U_i^n(1 - \Delta t_n d_i) + \Delta t_n \sum_{j=0}^{I} A_{ij} U_j^n \ge m^n(1 - \Delta t_n d_i) + \Delta t_n d_i m^n = m^n.$$

Since $U_i^n \ge m^n$ and $0 \le U_i^n < 1$,

$$(1 - U_i^n)^{-p} \ge (1 - m^n)^{-p}.$$

Using $a_i \ge a_*$ and $\Phi_i^n \ge 0$,

$$U_i^{n+1} \ge m^n + \Delta t_n \lambda a_* (1 - m^n)^{-p} \qquad (0 \le i \le I).$$

Taking the minimum over i gives the assertion. □

Lemma 6.2. *Let $0 < \delta < 1$. If $M^n \le 1 - \delta$, then there exists a constant $c_\delta > 0$, independent of n, such that*

$$\Delta t_n \ge c_\delta h^2.$$

Proof. If $M^n \le 1 - \delta$, then

$$0 \le U_i^n \le 1 - \delta \qquad (0 \le i \le I).$$

The right-hand side is bounded above by

$$F_i^n \le 2d^* + \Phi^* + \lambda a^* \delta^{-p} = B_\delta.$$

If $F_i^n > 0$, then

$$\frac{1 - U_i^n}{F_i^n} \ge \frac{\delta}{B_\delta}.$$

From (4.2),

$$\Lambda^n \geq \min\left\{1, \theta \frac{\delta}{B_\delta}\right\}.$$

Thus

$$\Delta t_n = \gamma h^2 \Lambda^n \geq c_\delta h^2, \qquad c_\delta = \gamma \min\left\{1, \theta \frac{\delta}{B_\delta}\right\}.$$

The constant c_δ is independent of n and h. $\square$

Lemma 6.3. *The adaptive solution satisfies*

$$\lim_{n\to\infty} M^n = 1.$$

Proof. Suppose that the conclusion is false. Then there are $\delta > 0$ and infinitely many indices for which

$$M^n \leq 1 - \delta.$$

On every time level satisfying $M^n \leq 1 - \delta$, Lemma 6.2 gives

$$\Delta t_n \geq c_\delta h^2.$$

Lemma 6.1 gives

$$m^{n+1} - m^n \geq c_\delta h^2 \lambda a_* (1 - m^n)^{-p} \geq c_\delta h^2 \lambda a_*.$$

Therefore m^n increases by at least a fixed positive amount on such levels. The last inequality cannot hold on infinitely many levels while $m^n \leq M^n \leq 1 - \delta$. Hence, for every $\delta > 0$, $M^n > 1 - \delta$ for all sufficiently large n. Since $M^n < 1$ by Corollary 5.1, we obtain $M^n \to 1$. $\square$

Theorem 6.1. *Under the hypotheses of Corollary 5.1, the adaptive approximation quenches numerically in finite accumulated time:*

$$T_h^q = \sum_{n=0}^{\infty} \Delta t_n < \infty.$$

Proof. Step 1. Approach to the singular level. Lemma 6.3 gives

$$M^n \to 1.$$

It remains to prove that the accumulated time is finite.

Step 2. Summability of the time increments. Let k_n be such that $m^n = U_{k_n}^n$, and set

$$d_n = 1 - m^n, \qquad r_n = \Delta t_n \lambda a_* d_n^{-p}.$$

At the minimum node, Lemma 3.1, $\Phi_i^n \geq 0$ and $a_i \geq a_*$ give

$$F_{k_n}^n \geq \lambda a_* d_n^{-p} > 0.$$

The barrier rule therefore yields

$$0 \leq r_n \leq \Delta t_n F_{k_n}^n \leq \theta d_n.$$

By Lemma 6.1,

$$d_{n+1} \leq d_n - r_n.$$

For $q(d) = d^{p+1}$ and $0 \le r \le \theta d$,

$$q(d) - q(d-r) = (p+1)\int_{d-r}^{d} s^p\, ds \ge (p+1)(1-\theta)^p d^p r.$$

Thus

$$d_n^{p+1} - d_{n+1}^{p+1} \ge (p+1)(1-\theta)^p \lambda a_* \Delta t_n.$$

Summing from 0 to $N-1$ gives

$$(p+1)(1-\theta)^p \lambda a_* \sum_{n=0}^{N-1} \Delta t_n \le d_0^{p+1}.$$

Letting $N \to \infty$ proves $T_h^q < \infty$. □

Proposition 6.1. *Let $m^0 = \min_i U_i^0$. Assume that $\gamma h^2 \le 1$. The numerical quenching time satisfies*

$$T_h^q \le C_q (1-m^0)^{p+1}, \qquad C_q = \frac{1}{(p+1)(1-\theta)^p \lambda a_*}.$$

The constant C_q is independent of h.

Proof. The summation estimate in the proof of Theorem 6.1 gives, with $d_0 = 1 - m^0$,

$$(p+1)(1-\theta)^p \lambda a_* \sum_{n=0}^{N-1} \Delta t_n \le d_0^{p+1} \qquad (N \ge 1).$$

Letting $N \to \infty$ gives the stated bound. □

Proposition 6.2. *Let $M^0 = \max_i U_i^0$. Assume in addition that the discrete maximum is nondecreasing:*

$$M^{n+1} \ge M^n \qquad (n \ge 0).$$

Then

$$T_h^q \ge \frac{(1-M^0)^{p+1}}{(p+1)(\lambda a^* + \Phi^* + 2d^*)}.$$

Proof. Put

$$B = \lambda a^* + \Phi^* + 2d^*.$$

For every node and every time level,

$$(L_h U^n)_i \le 2d^*, \qquad \Phi_i^n \le \Phi^*,$$

and, since $1 - U_i^n \ge 1 - M^n$,

$$\lambda a_i (1-U_i^n)^{-p} \le \lambda a^* (1-M^n)^{-p}.$$

Therefore

$$M^{n+1} - M^n \le B \Delta t_n (1-M^n)^{-p}.$$

Set $D^n = 1 - M^n$. The monotonicity assumption gives $0 \le D^{n+1} \le D^n$. Hence

$$\Delta t_n \ge \frac{(D^n)^p (D^n - D^{n+1})}{B}.$$

Since

$$(D^n)^p (D^n - D^{n+1}) \ge \frac{(D^n)^{p+1} - (D^{n+1})^{p+1}}{p+1},$$

we obtain

$$\Delta t_n \geq \frac{(D^n)^{p+1} - (D^{n+1})^{p+1}}{(p+1)B}.$$

Summing from 0 to $N-1$ yields

$$\sum_{n=0}^{N-1} \Delta t_n \geq \frac{(D^0)^{p+1} - (D^N)^{p+1}}{(p+1)B}.$$

Letting $N \to \infty$ and using $D^N \to 0$ gives the asserted estimate. $\square$

7. Convergence before quenching and level times

The singular source is not globally Lipschitz on $[0,1)$. The convergence proof is therefore restricted to intervals where the exact solution remains separated from the level 1.

Theorem 7.1. *Let $T < T_q$. Assume that the exact solution of (1.1) satisfies*

$$\max_{0\leq t\leq T} \|u(\cdot,t)\|_\infty \leq 1 - \eta_T$$

for some $\eta_T > 0$. Assume that u, J, a, and Φ are sufficiently smooth for the trapezoidal consistency estimate to be second order. Then the explicit adaptive scheme satisfies

$$\max_{t_n\leq T} \max_i |U_i^n - u(x_i,t_n)| \leq C_T h^2.$$

Proof. Step 1. Separation from the singular level. On $[0, 1-\eta_T/2]$, the singular function is Lipschitz:

$$|(1-r)^{-p} - (1-s)^{-p}| \leq p(\eta_T/2)^{-p-1}|r-s|.$$

Define N_T as the largest index with $t_n \leq T$ and introduce the bootstrap set

$$\mathcal{N}_h = \left\{0 \leq n \leq N_T : \left\|U^k - u(\cdot,t_k)\right\|_\infty \leq \frac{\eta_T}{2} \text{ for } 0 \leq k \leq n\right\}.$$

For h small, $0 \in \mathcal{N}_h$.

Step 2. Error equation. Let

$$e_i^n = U_i^n - u(x_i,t_n).$$

On the bootstrap interval the exact and numerical values remain in $[0, 1-\eta_T/2]$. The trapezoidal approximation of $\mathcal{L}_J u$ is $O(h^2)$, and the forward Euler local residual is $O(\Delta t_n)$. Since $\Delta t_n \leq \gamma h^2$,

$$|\tau_i^n| \leq C_T h^2.$$

Subtracting the exact nodal equation from the numerical scheme gives

$$\begin{aligned} e_i^{n+1} &= e_i^n + \Delta t_n (L_h e^n)_i \\ &\quad + \Delta t_n \lambda a_i \left((1-U_i^n)^{-p} - (1-u(x_i,t_n))^{-p}\right) + \Delta t_n \tau_i^n. \end{aligned}$$

Lemma 3.2 and the Lipschitz bound for the singular source imply

$$\left\|e^{n+1}\right\|_\infty \leq (1 + C_T\Delta t_n)\left\|e^n\right\|_\infty + C_T \Delta t_n h^2.$$

Step 3. Closing the bootstrap. The discrete Gronwall inequality gives

$$\max_{0\le n\le N_T} \|e^n\|_\infty \le C_T h^2.$$

For h small enough, $C_T h^2 < \eta_T/2$, so the bootstrap set equals $\{0,\dots,N_T\}$. Hence

$$\max_{t_n\le T}\max_i |U_i^n - u(x_i,t_n)| \le C_T h^2.$$

□

Definition 7.1. For $0<\delta<1$, define the exact and numerical level times by

$$T(\delta) = \inf\{t>0 : \|u(\cdot,t)\|_\infty \ge 1-\delta\},$$

$$T_h(\delta) = \inf\{t_n>0 : M^n \ge 1-\delta\}.$$

Theorem 7.2. *Assume that the exact crossing of the level $1-\delta$ is nondegenerate, namely there exist $\alpha>0$ and $r>0$ such that*

$$\|u(\cdot,t)\|_\infty \le 1-\delta-\alpha|t-T(\delta)| \quad \textit{for } T(\delta)-r\le t<T(\delta),$$

$$\|u(\cdot,t)\|_\infty \ge 1-\delta+\alpha|t-T(\delta)| \quad \textit{for } T(\delta)<t\le T(\delta)+r.$$

Then

$$T_h(\delta)\to T(\delta) \qquad \textit{as } h\to 0.$$

Consequently, a diagonal choice $\delta_h\downarrow 0$ gives

$$T_h(\delta_h)\to T_q.$$

Proof. Let $\varepsilon\in(0,r)$. The nondegenerate crossing gives

$$\|u(\cdot,T(\delta)-\varepsilon)\|_\infty < 1-\delta,$$

$$\|u(\cdot,T(\delta)+\varepsilon)\|_\infty > 1-\delta.$$

Theorem 7.1 gives uniform nodal convergence on $[0,T(\delta)+\varepsilon]$. Hence, for h small,

$$M^n < 1-\delta \quad \text{for all } t_n\le T(\delta)-\varepsilon,$$

and there exists a level index with

$$M^n \ge 1-\delta \quad \text{for some } t_n \le T(\delta)+\varepsilon.$$

Therefore

$$T(\delta)-\varepsilon \le T_h(\delta) \le T(\delta)+\varepsilon$$

for h small. Since ε is arbitrary,

$$T_h(\delta)\to T(\delta).$$

Since $T(\delta)\uparrow T_q$ as $\delta\downarrow 0$, choose $\delta_h\downarrow 0$ slowly enough that the fixed-level convergence applies along the mesh sequence. Then

$$T_h(\delta_h)\to T_q.$$

□

8. Implementation and numerical tests

The numerical tests use a smooth compactly supported kernel,

$$J(z) = \frac{3}{4}(1 - z^2)\mathbf{1}_{\{|z|<1\}},$$

which is nonnegative, symmetric and normalized on $\mathbb{R}$. The initial datum, the spatial weight and the prescribed flux are

$$U_i^0 = 0.18 + 0.04 \cos\left(\frac{\pi x_i}{2}\right),$$

$$a(x) = 1 + 0.35\cos(\pi x), \qquad \Phi_\rho(x) = \rho\left(1 + 0.25\cos(\pi x)\right),$$

with $\lambda = 0.65$ and $p = 2$. The parameter $\rho \geq 0$ measures the intensity of the nonhomogeneous flux. The value $\rho = 0$ is kept in the tables as the homogeneous-flux reference case, so that the effect of the exterior input is measured against a fixed baseline.

The explicit scheme is (4.1), and the implicit-loss scheme is (4.3). In all computations the adaptive step is kept in the h^2-scaled form

$$\Delta t_n = 5h^2\Lambda^n, \qquad \Lambda^n = \min\left\{1, 0.82 \min_{F_i^n>0} \frac{1 - U_i^n}{F_i^n}\right\}.$$

When $F_i^n \leq 0$ for every node, we set $\Lambda^n = 1$. Hence the mesh dependence is entirely carried by the factor h^2, while Λ^n is a dimensionless limiter depending on the distance from the singular level. The computation stops when

$$1 - M^n \leq 10^{-6}$$

or when

$$\Delta t_n \leq 10^{-16}.$$

The detected quenching time and the Richardson indicator are

$$T_h^q = \sum_{k=0}^{n-1} \Delta t_k,$$

$$s_x = \frac{\log\left((T_{4h} - T_{2h})/(T_{2h} - T_h)\right)}{\log 2},$$

whenever the quotient is positive.

Table 1. Explicit adaptive scheme for $\rho = 0.10$.

I	T_h^q	iterations	CPU (s)	s_x
16	0.18493289	169	0.0061	–
32	0.17678235	663	0.0211	–
64	0.17482156	2552	0.0901	2.055
128	0.17433601	9759	0.3732	2.014
256	0.17421491	37204	1.8914	2.003
512	0.17418466	141443	21.0491	2.001

TABLE 2. Implicit-loss adaptive scheme for $\rho = 0.10$.

I	T_h^q	iterations	CPU (s)	s_x
16	0.18798583	169	0.0063	–
32	0.17750141	663	0.0250	–
64	0.17499849	2552	0.0973	2.067
128	0.17438007	9760	0.4311	2.017
256	0.17422592	37204	2.2498	2.004
512	0.17418741	141444	21.4191	2.001

Tables 1 and 2 use the same mesh sequence. The detected time stabilizes near 0.17418 as the grid is refined. The two schemes give close values on the finest meshes, which shows that the loss term introduced in the implicit variant does not change the detected time at the reported accuracy. The Richardson indicator reaches the second-order range on the refined levels: $s_x = 2.001$ at $I = 512$ for both schemes. The CPU times are reported only to document the computational scale of the tests; the convergence assessment is based on the detected times and on s_x.

TABLE 3. Effect of the nonhomogeneous flux intensity for $I = 256$.

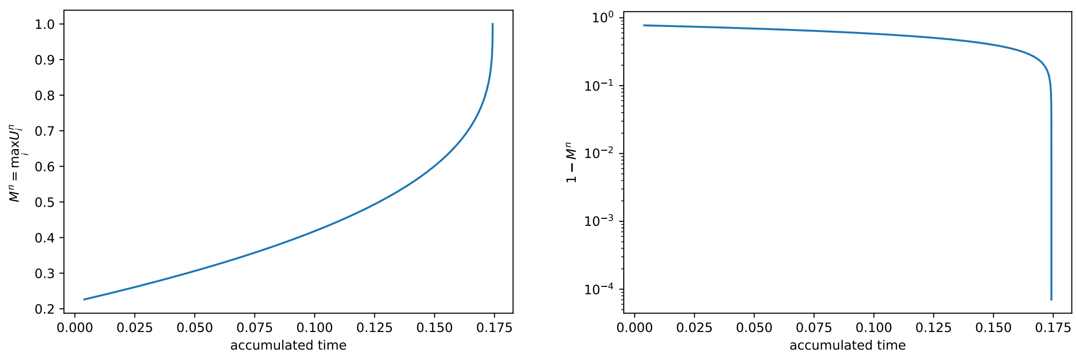
ρ	T_h^q	iterations	CPU (s)
0.00	0.18338090	37204	1.9788
0.05	0.17865835	37204	2.0388
0.10	0.17421491	37204	1.9913
0.15	0.17002465	37204	2.0462
0.20	0.16606486	37204	2.1056
0.30	0.15875936	37204	2.0589
0.40	0.15216493	37204	2.0966

Table 3 shows a monotone decrease of the detected quenching time as ρ increases from the reference value $\rho = 0$. This agrees with the comparison result: a larger nonnegative flux increases the reaction-diffusion balance toward the singular level and therefore reduces the detected quenching time.

TABLE 4. Perturbation of the flux amplitude around $\rho = 0.10$ for $I = 256$.

ε	$T_h^q((1+\varepsilon)\rho)$	$\lvert T_h^q((1+\varepsilon)\rho) - T_h^q(\rho)\rvert$	iterations
0	0.17421491	0.000×10^0	37204
10^{-1}	0.17335738	8.575×10^{-4}	37204
10^{-2}	0.17412871	8.621×10^{-5}	37204
10^{-3}	0.17420629	8.625×10^{-6}	37204
10^{-4}	0.17421405	8.626×10^{-7}	37204

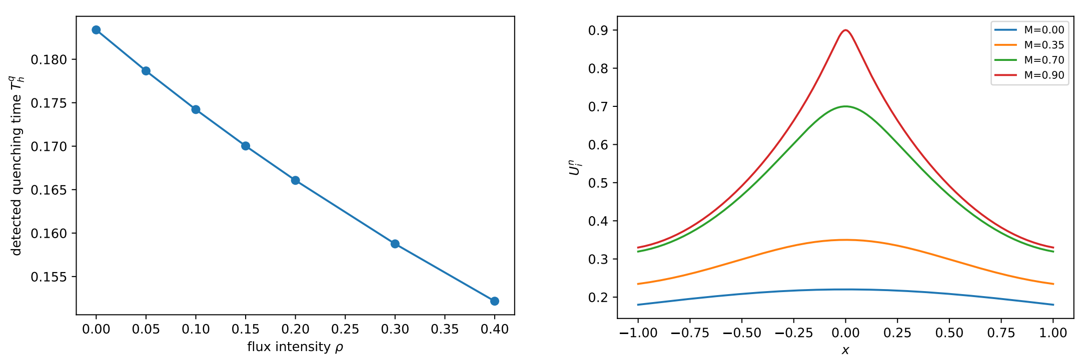
Table 4 includes the baseline $\varepsilon = 0$. The time variation is approximately proportional to the perturbation amplitude over the reported range. This supports the stability of the computed quenching time with respect to small multiplicative changes in the flux coefficient.



(A) Growth of the maximum. (B) Distance to the singular level.

FIGURE 1. Approach to the quenching level for $I = 256$ and $\rho = 0.10$. The left panel shows $M^n = \max_i U_i^n$; the right panel shows $1 - M^n$ on a logarithmic scale.

Figure 1 confirms the barrier behavior of the scheme. The maximum approaches 1 from below, while the logarithmic plot of $1 - M^n$ shows the rapid decay of the distance to the singular level near the detected time.



(A) Effect of the flux intensity. (B) Profiles at successive levels.

FIGURE 2. Flux dependence and solution profiles. The detected quenching time decreases as ρ increases, while the profile remains below 1 and reaches its maximum near the strongest weighted source.

Figure 2 shows the effect of the nonhomogeneous flux and the profile shape near the detected levels. Increasing ρ shifts the detected time downward. The displayed profiles remain below 1 and locate their largest values in the region where the weighted singular source and the flux contribution are strongest.

9. Numerical validation

This section summarizes the numerical checks that support the estimates proved above. The purpose is not to replace the proofs, but to verify that the implemented scheme exhibits the invariant, monotonicity and refinement properties established in the analysis.

Invariant bound. For every mesh used in Tables 1 and 2, all accepted iterates satisfy

$$0 \le U_i^n < 1$$

until one of the stopping criteria is reached. No component crosses the singular level. This confirms the practical effect of the barrier time-step rule and agrees with the invariant estimate proved in Proposition 5.2.

Influence of the nonhomogeneous flux. Table 3 gives the strict ordering

$$T_h^q(0.00) > T_h^q(0.05) > T_h^q(0.10) > T_h^q(0.15) > T_h^q(0.20) > T_h^q(0.30) > T_h^q(0.40).$$

Thus increasing the nonnegative flux decreases the detected quenching time. This behavior is consistent with the comparison result, since a larger flux accelerates the approach to the singular level on intervals where the compared solutions remain separated from one.

Mesh refinement. For the explicit scheme,

$$|T_{16}^q - T_{32}^q| = 8.15053 \times 10^{-3}, \qquad |T_{256}^q - T_{512}^q| = 3.02567 \times 10^{-5}.$$

A similar decrease is observed for the implicit-loss approximation. Together with the reported Richardson indicators, this confirms that the detected quenching time stabilizes under mesh refinement in the expected second-order range.

Perturbation of the flux amplitude. The perturbation test includes the reference case $\varepsilon = 0$. The time variation decreases with the perturbation size:

$$|T_h^q((1+10^{-1})\rho) - T_h^q(\rho)| = 8.575 \times 10^{-4},$$
$$|T_h^q((1+10^{-4})\rho) - T_h^q(\rho)| = 8.626 \times 10^{-7}.$$

This supports the stability of the detected quenching time under small multiplicative perturbations of the flux amplitude.

10. Conclusion

We developed and analyzed an adaptive approximation for a nonlocal Neumann quenching problem with nonhomogeneous flux. The analysis is based on the barrier $1 - U_i^n$ and on the invariant interval $[0, 1)$. The time-step rule prevents the numerical solution from crossing the singular level and gives finite accumulated numerical quenching time.

The proofs establish positivity, strict preservation of the singular constraint, flux comparison, finite-time numerical quenching, an upper bound and a monotone-maximum lower bound for T_h^q, convergence before quenching, and convergence of fixed-level detection times. The numerical tests show that the nonhomogeneous flux reduces the quenching time, that the computed time stabilizes under mesh refinement, and that small perturbations of the flux amplitude produce small changes in the detected time.

Three questions remain separate from the present analysis. The localization of the quenching set under the competition between $a(x)$ and $\Phi(x,t)$, the continuity of the singular time with respect to flux perturbations, and IMEX or Crank–Nicolson discretizations with the same h^2-scaled barrier step each require a distinct set of estimates.

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