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EULER–MARUYAMA SCHEMES FOR G-STOCHASTIC VOLTERRA EQUATIONS

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Abstract. We study Euler–Maruyama approximations for stochastic Volterra integral equations driven by G-Brownian motion. The proposed explicit Volterra scheme discretizes the drift term, the quadratic variation term and the stochastic integral term. Under standard Lipschitz, growth and time-regularity assumptions, we prove uniform moment bounds and strong mean-square convergence under sublinear expectation. The convergence rate reflects the time regularity of the Volterra kernels and recovers the classical order one–half in the root mean square sense in the Lipschitz case. A numerical experiment under volatility uncertainty illustrates the theory.

Keywords. G-expectation, G-Brownian motion, stochastic Volterra equations, Euler–Maruyama method, strong convergence.
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1. Introduction

Stochastic Volterra integral equations provide a natural framework for stochastic systems with memory, since their coefficients may depend on both the observation time and the past integration time. This two-time structure is useful in models with hereditary effects and non-Markovian dynamics, including viscoelasticity, finance, and stochastic control. Classical stochastic Volterra equations have been studied extensively; see [6, 14, 10, 8, 1]. Recent developments on Volterra-type systems and control-related formulations can be found, for example, in [7, 4].

In many applications, the probabilistic model itself is uncertain. A typical case is volatility uncertainty, where the quadratic variation of the driving noise cannot be described by a single deterministic parameter. Peng's theory of nonlinear expectations and G-Brownian motion provides a stochastic calculus for such problems; see [11, 12, 13]. The corresponding sublinear expectation admits a representation as the supremum of expectations over a family of probability measures; see [3, 5]. Equations driven by a G-Brownian motion therefore naturally involve an additional integral with respect to the quadratic variation.

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Stochastic Volterra integral equations driven by G-Brownian motion were recently studied in [17]. The equation considered there, and studied numerically in this paper, is

$$\begin{aligned} X(t) = \varphi(t) + \int_0^t b(t,s,X(s))\,ds + \int_0^t h(t,s,X(s))\,d\langle B\rangle_s \\ + \int_0^t \sigma(t,s,X(s))\,dB_s, \qquad 0 \le t \le T. \end{aligned} \tag{1.1}$$

Here the coefficients depend on both the current time t and the past integration time s. Existence, uniqueness, mean-square continuity, pathwise continuity and comparison results were established in [17], but numerical approximation of (1.1) was not addressed.

The purpose of this paper is to develop and analyze an Euler–Maruyama approximation for (1.1). The proposed scheme is adapted to the Volterra structure and discretizes the Lebesgue integral, the quadratic variation integral and the G-Itô integral simultaneously. We prove strong mean-square convergence under the sublinear expectation $\widehat{\mathbb{E}}$, with an error estimate that reflects the time regularity of the Volterra kernels.

Numerical approximations for stochastic Volterra equations and related integral equations have been studied in the classical probabilistic framework. Zhang [15] proved strong convergence of the Euler–Maruyama method for stochastic Volterra integral equations, while Zhang and Li [16] considered generalized stochastic Volterra equations driven by Lévy noise. Related numerical treatments can also be found in [9]. Numerical methods under the G-framework have been studied in related settings, for instance for G-SDDEs in [2]. The Volterra-memory structure in (1.1), however, leads to a different discretization and error analysis.

Our main contribution is to introduce an explicit Volterra-type Euler–Maruyama scheme for (1.1) and prove its strong mean-square convergence under sublinear expectation. Under Lipschitz continuity, linear growth and Hölder-type regularity assumptions, the grid error is of order $|\pi|^\kappa$, where $\kappa = \min\{2\gamma, 1\}$ and γ is the time-regularity exponent. We also obtain a uniform-in-time estimate for a continuous Volterra interpolation of the scheme. In the time-Lipschitz case, this gives the root-mean-square order $1/2$.

The paper is organized as follows. Section 2 recalls the basic G-expectation tools. Sections 3–5 present the equation, assumptions and numerical scheme. Section 6 proves strong convergence, and Section 7 gives a numerical illustration.

2. Preliminaries on G-expectation

We recall only the notation and estimates needed in the sequel. For a detailed presentation of G-expectation, G-Brownian motion and the corresponding stochastic calculus, we refer to [11, 12, 13].

Let $\Omega = C_0([0,\infty);\mathbb{R})$ be the canonical path space and let $B_t(\omega) = \omega_t$ be the canonical process. Fix $0 < \underline{\sigma} \le \overline{\sigma} < \infty$ and define

$$G(a) = \frac{1}{2}\left(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-\right), \qquad a \in \mathbb{R}.$$

The associated sublinear expectation is denoted by $\widehat{\mathbb{E}}$, and B is a one-dimensional G-Brownian motion under $\widehat{\mathbb{E}}$.

For $p \geq 1$, $L_G^p(\Omega_T)$ denotes the usual G-expectation space, and $M_G^p(0,T)$ denotes the completion of step processes under the norm

$$\|\eta\|_{M_G^p} = \left(\widehat{\mathbb{E}}\left[\int_0^T |\eta(t)|^p \, dt\right]\right)^{1/p}.$$

For $\eta \in M_G^2(0,T)$, the G-Itô integral $\int_0^t \eta_s \, dB_s$ is well defined, while for $\xi \in M_G^1(0,T)$ the quadratic variation integral $\int_0^t \xi_s \, d\langle B\rangle_s$ is well defined.

The quadratic variation of B satisfies, quasi-surely,

$$\underline{\sigma}^2(t-s) \leq \langle B\rangle_t - \langle B\rangle_s \leq \overline{\sigma}^2(t-s), \qquad 0 \leq s \leq t \leq T. \tag{2.1}$$

Hence, in estimates, we use $d\langle B\rangle_t \leq \overline{\sigma}^2 \, dt$ in the usual quasi-sure sense. Moreover, the following G-Itô estimate holds for every $\eta \in M_G^2(0,T)$:

$$\widehat{\mathbb{E}}\left[\left|\int_0^t \eta_s \, dB_s\right|^2\right] \leq \overline{\sigma}^2 \widehat{\mathbb{E}}\left[\int_0^t |\eta_s|^2 \, ds\right], \qquad 0 \leq t \leq T. \tag{2.2}$$

We shall also use the representation theorem for G-expectation: there exists a weakly compact family $\mathcal{P}$ of probability measures such that

$$\widehat{\mathbb{E}}[\xi] = \sup_{P \in \mathcal{P}} E^P[\xi]$$

for all admissible random variables ξ; see [3, 5]. A property is said to hold quasi-surely if it holds outside a polar set.

Throughout the paper, C denotes a generic positive constant which may change from line to line, but never depends on the mesh size $|\pi|$.

3. The G-SVIE and assumptions

Set

$$\Delta_T = \{(t,s) \in [0,T]^2 : 0 \leq s \leq t \leq T\}.$$

We consider the G-stochastic Volterra integral equation

$$\begin{aligned} X(t) = \varphi(t) + \int_0^t b(t,s,X(s)) \, ds + \int_0^t h(t,s,X(s)) \, d\langle B\rangle_s \\ + \int_0^t \sigma(t,s,X(s)) \, dB_s, \qquad 0 \leq t \leq T. \end{aligned} \tag{3.1}$$

To keep the presentation focused on numerical analysis, we use deterministic kernels and avoid technicalities related to random kernels under capacity.

Assumption 3.1. The following conditions hold.

(A1) The free term φ is an adapted process such that

$$\varphi(t) \in L_G^2(\Omega_t), \qquad 0 \leq t \leq T,$$

$$\sup_{0 \leq t \leq T} \widehat{\mathbb{E}}\big[|\varphi(t)|^2\big] < \infty,$$

and $\varphi \in M_G^2(0,T)$.

(A2) The coefficients $b, h, \sigma : \Delta_T \times \mathbb{R} \to \mathbb{R}$ are deterministic continuous functions such that, for each $t \in [0,T]$ and $x \in \mathbb{R}$,

$$b(t,\cdot,x),\ h(t,\cdot,x),\ \sigma(t,\cdot,x) \in M_G^2(0,t).$$

(A3) There exists a constant $L > 0$ such that, for all $(t,s) \in \Delta_T$ and $x, y \in \mathbb{R}$,

$$\begin{aligned}&|b(t,s,x) - b(t,s,y)| + |h(t,s,x) - h(t,s,y)| + |\sigma(t,s,x) - \sigma(t,s,y)| \\ &\quad \leq L|x-y|.\end{aligned}$$

(A4) There exists a constant $L > 0$ such that, for all $(t,s) \in \Delta_T$ and $x \in \mathbb{R}$,

$$|b(t,s,x)|^2 + |h(t,s,x)|^2 + |\sigma(t,s,x)|^2 \leq L^2(1+|x|^2).$$

(A5) There exists a continuous increasing function $\rho : [0,\infty) \to [0,\infty)$, with $\rho(0) = 0$, such that, for every $\ell = b, h, \sigma$,

$$|\ell(t,s,x) - \ell(t',s,x)| \leq \rho(|t-t'|)(1+|x|),$$

for all $0 \leq s \leq t \leq T$, $0 \leq s \leq t' \leq T$, and $x \in \mathbb{R}$.

Remark 3.2. Assumption 3.1(A5) controls the dependence of the Volterra kernels on the first time variable. The factor $1+|x|$ is compatible with linear growth and natural for kernels depending on $t-s$. For instance,

$$b(t,s,x) = 0.2(t-s)x, \qquad h(t,s,x) = 0.1(1+t-s)x, \qquad \sigma(t,s,x) = 0.3e^{-(t-s)}x$$

satisfy Assumption 3.1(A5). The contraction argument relies on Lipschitz continuity in the state variable, while Assumption 3.1(A5) is used for well-definedness and time regularity of the Volterra terms.

Proposition 3.3. *Under Assumption 3.1, equation (3.1) admits a unique solution $X \in M_G^2(0,T)$. Moreover,*

$$\sup_{0\leq t\leq T} \widehat{\mathbb{E}}\big[|X(t)|^2\big] < \infty.$$

Proof. The proof follows the fixed-point strategy used for G-SVIEs, with a minor modification in the well-definedness step caused by the factor $1+|x|$ in Assumption 3.1(A5).

For $X \in M_G^2(0,T)$, define the Volterra operator Λ by

$$\begin{aligned}\Lambda_t(X) = \varphi(t) &+ \int_0^t b(t,s,X(s))\,ds + \int_0^t h(t,s,X(s))\,d\langle B\rangle_s \\ &+ \int_0^t \sigma(t,s,X(s))\,dB_s, \qquad 0 \leq t \leq T.\end{aligned}$$

We first show that $\Lambda(X) \in M_G^2(0,T)$. We give the details for the stochastic integral term and set

$$M(t) = \int_0^t \sigma(t,s,X(s))\,dB_s.$$

Let $\Pi_n = \{0 = t_0^n < t_1^n < \cdots < t_n^n = T\}$ be partitions of $[0,T]$ with

$$|\Pi_n| = \max_{0\leq i\leq n-1}(t_{i+1}^n - t_i^n) \longrightarrow 0.$$

Define

$$M_n(t) = \sum_{i=0}^{n-1} M(t_i^n)\mathbf{1}_{[t_i^n, t_{i+1}^n)}(t).$$

For $t \in [t_i^n, t_{i+1}^n)$, we have

$$\begin{aligned} M(t) - M(t_i^n) &= \int_0^{t_i^n} \big[\sigma(t,s,X(s)) - \sigma(t_i^n, s, X(s))\big]\, dB_s \\ &\quad + \int_{t_i^n}^t \sigma(t,s,X(s))\, dB_s. \end{aligned}$$

Using the G-Itô estimate, Assumption 3.1(A5), and the linear growth condition 3.1(A4), we obtain

$$\begin{aligned} \widehat{\mathbb{E}}\big[|M(t) - M(t_i^n)|^2\big] &\le 2\overline{\sigma}^2 \widehat{\mathbb{E}} \int_0^{t_i^n} |\sigma(t,s,X(s)) - \sigma(t_i^n, s, X(s))|^2\, ds \\ &\quad + 2\overline{\sigma}^2 \widehat{\mathbb{E}} \int_{t_i^n}^t |\sigma(t,s,X(s))|^2\, ds \\ &\le C\rho^2(|\Pi_n|)\widehat{\mathbb{E}} \int_0^T (1 + |X(s)|^2)\, ds \\ &\quad + C\widehat{\mathbb{E}} \int_{t_i^n}^t (1 + |X(s)|^2)\, ds. \end{aligned}$$

Therefore,

$$\begin{aligned} \widehat{\mathbb{E}} \int_0^T |M(t) - M_n(t)|^2\, dt &\le CT\rho^2(|\Pi_n|)\widehat{\mathbb{E}} \int_0^T (1 + |X(s)|^2)\, ds \\ &\quad + C|\Pi_n|\widehat{\mathbb{E}} \int_0^T (1 + |X(s)|^2)\, ds. \end{aligned}$$

Since $X \in M_G^2(0,T)$, the right-hand side tends to zero as $n \to \infty$. Hence

$$\widehat{\mathbb{E}} \int_0^T |M(t) - M_n(t)|^2\, dt \longrightarrow 0.$$

Thus $M \in M_G^2(0,T)$. The factor $1+|X(s)|$ is controlled by the square integrability of X.

The Lebesgue term is treated similarly. If

$$N_b(t) = \int_0^t b(t,s,X(s))\, ds,$$

then Cauchy's inequality, Assumption 3.1(A5) and the linear growth condition show that the corresponding step approximations converge to N_b in $M_G^2(0,T)$. Hence $N_b \in M_G^2(0,T)$.

For

$$N_h(t) = \int_0^t h(t,s,X(s))\, d\langle B\rangle_s,$$

the quasi-sure bound $d\langle B\rangle_s \le \overline{\sigma}^2\,ds$, combined with the preceding argument for the Lebesgue term, gives $N_h \in M_G^2(0,T)$. Hence $\Lambda(X) \in M_G^2(0,T)$.

We now prove that Λ is a contraction under a suitable equivalent norm. Let $X, Y \in M_G^2(0,T)$. By the Lipschitz condition in Assumption 3.1(A3), Cauchy's inequality, the bound on $d\langle B\rangle$, and the G-Itô estimate, there exists a constant $C > 0$ such that, for every $\beta > 0$,

$$\begin{aligned}
&\widehat{\mathbb{E}}\int_0^T e^{-\beta t}|\Lambda_t(X)-\Lambda_t(Y)|^2\,dt \\
&\quad\le C\widehat{\mathbb{E}}\int_0^T e^{-\beta t}\int_0^t |X(s)-Y(s)|^2\,ds\;dt \\
&\quad= C\widehat{\mathbb{E}}\int_0^T \left(\int_s^T e^{-\beta t}\,dt\right)|X(s)-Y(s)|^2\,ds \\
&\quad\le \frac{C}{\beta}\widehat{\mathbb{E}}\int_0^T e^{-\beta s}|X(s)-Y(s)|^2\,ds.
\end{aligned}$$

Define

$$\|X\|_\beta^2 = \widehat{\mathbb{E}}\int_0^T e^{-\beta t}|X(t)|^2\,dt.$$

Choosing $\beta > 2C$, we obtain

$$\|\Lambda(X)-\Lambda(Y)\|_\beta^2 \le \frac{1}{2}\|X-Y\|_\beta^2.$$

Thus Λ is a contraction on $(M_G^2(0,T), \|\cdot\|_\beta)$. Banach's fixed-point theorem yields a unique fixed point $X \in M_G^2(0,T)$, hence the unique solution of (3.1).

It remains to prove the uniform second-moment estimate. By (3.1), Cauchy's inequality, the bound on $d\langle B\rangle$, the G-Itô estimate and the linear growth condition, we get

$$\widehat{\mathbb{E}}\big[|X(t)|^2\big] \le C\widehat{\mathbb{E}}\big[|\varphi(t)|^2\big] + C\int_0^t \Big(1+\widehat{\mathbb{E}}\big[|X(s)|^2\big]\Big)\,ds.$$

Using Assumption 3.1(A1), we obtain

$$\widehat{\mathbb{E}}\big[|X(t)|^2\big] \le C + C\int_0^t \widehat{\mathbb{E}}\big[|X(s)|^2\big]\,ds.$$

Gronwall's inequality yields

$$\sup_{0\le t\le T}\widehat{\mathbb{E}}\big[|X(t)|^2\big] < \infty.$$

□

In what follows, X always denotes the unique solution of (3.1).

4. Moment and regularity estimates

This section records the estimates for the exact solution which will be used in the error analysis. The uniform second-moment estimate follows from the basic assumptions. For the mean-square regularity estimate, we impose the following additional regularity assumptions.

Assumption 4.1. The following conditions hold.

(B1) The free term φ is Hölder continuous in mean square; that is, there exist constants $K_\varphi > 0$ and $\gamma \in (0,1]$ such that

$$\widehat{\mathbb{E}}\big[|\varphi(t)-\varphi(r)|^2\big] \le K_\varphi |t-r|^{2\gamma}, \qquad t,r \in [0,T].$$

(B2) There exists a constant $K > 0$ such that, for every $\ell = b, h, \sigma$,

$$|\ell(t,s,x)-\ell(t',s',x)|^2 \le K(1+|x|^2)\left(|t-t'|^{2\gamma}+|s-s'|^{2\gamma}\right),$$

for all $(t,s),(t',s') \in \Delta_T$ and $x \in \mathbb{R}$.

Lemma 4.2. *Under Assumption 3.1, the solution X of (3.1) satisfies*

$$\sup_{0\le t\le T} \widehat{\mathbb{E}}\big[|X(t)|^2\big] \le C.$$

Proof. From (3.1), Cauchy's inequality, the quasi-sure bound

$$d\langle B\rangle_s \le \overline{\sigma}^2\, ds,$$

the G-Itô estimate and the linear growth condition in Assumption 3.1(A4), we obtain

$$\widehat{\mathbb{E}}\big[|X(t)|^2\big] \le C\widehat{\mathbb{E}}\big[|\varphi(t)|^2\big] + C\int_0^t \Big(1+\widehat{\mathbb{E}}\big[|X(s)|^2\big]\Big)\, ds.$$

Using Assumption 3.1(A1), this gives

$$\widehat{\mathbb{E}}\big[|X(t)|^2\big] \le C + C\int_0^t \widehat{\mathbb{E}}\big[|X(s)|^2\big]\, ds.$$

Gronwall's inequality yields the desired estimate. □

Lemma 4.3. *Let Assumptions 3.1 and 4.1 hold, and set*

$$\kappa := \min\{2\gamma, 1\}.$$

Then the solution X satisfies

$$\widehat{\mathbb{E}}\big[|X(t)-X(r)|^2\big] \le C|t-r|^{\kappa}, \qquad t,r\in[0,T].$$

Proof. Assume without loss of generality that $0 \le r \le t \le T$. We write

$$X(t)-X(r) = \varphi(t)-\varphi(r) + I_b + I_h + I_\sigma,$$

where

$$I_b = \int_0^t b(t,s,X(s))\, ds - \int_0^r b(r,s,X(s))\, ds,$$

$$I_h = \int_0^t h(t,s,X(s))\, d\langle B\rangle_s - \int_0^r h(r,s,X(s))\, d\langle B\rangle_s,$$

and

$$I_\sigma = \int_0^t \sigma(t,s,X(s))\, dB_s - \int_0^r \sigma(r,s,X(s))\, dB_s.$$

For the Lebesgue term, we have

$$I_b = \int_0^r \big(b(t,s,X(s)) - b(r,s,X(s))\big)\, ds + \int_r^t b(t,s,X(s))\, ds.$$

By Cauchy's inequality, Assumption 4.1(B2), the linear growth condition 3.1(A4), and Lemma 4.2, we obtain

$$\widehat{\mathbb{E}}\big[|I_b|^2\big] \le C|t-r|^{2\gamma} + C|t-r|.$$

For the quadratic variation term,

$$I_h = \int_0^r \big(h(t,s,X(s)) - h(r,s,X(s))\big)\, d\langle B\rangle_s + \int_r^t h(t,s,X(s))\, d\langle B\rangle_s.$$

Using

$$d\langle B\rangle_s \le \overline{\sigma}^2\, ds$$

quasi-surely, together with Cauchy's inequality, Assumption 4.1(B2), the linear growth condition 3.1(A4), and Lemma 4.2, we get

$$\widehat{\mathbb{E}}\big[|I_h|^2\big] \le C|t-r|^{2\gamma} + C|t-r|.$$

Finally, for the stochastic integral term,

$$I_\sigma = \int_0^r \big(\sigma(t,s,X(s)) - \sigma(r,s,X(s))\big)\, dB_s + \int_r^t \sigma(t,s,X(s))\, dB_s.$$

By the G-Itô estimate, Assumption 4.1(B2), the linear growth condition 3.1(A4), and Lemma 4.2, we obtain

$$\begin{aligned}\widehat{\mathbb{E}}\big[|I_\sigma|^2\big] &\le 2\overline{\sigma}^2 \widehat{\mathbb{E}} \int_0^r |\sigma(t,s,X(s)) - \sigma(r,s,X(s))|^2\, ds \\ &\quad + 2\overline{\sigma}^2 \widehat{\mathbb{E}} \int_r^t |\sigma(t,s,X(s))|^2\, ds \\ &\le C|t-r|^{2\gamma} + C|t-r|.\end{aligned}$$

Combining these estimates with Assumption 4.1(B1), we conclude that

$$\widehat{\mathbb{E}}\big[|X(t) - X(r)|^2\big] \le C\left(|t-r|^{2\gamma} + |t-r|\right).$$

Since $\kappa = \min\{2\gamma, 1\}$, after changing the constant C, this gives

$$\widehat{\mathbb{E}}\big[|X(t) - X(r)|^2\big] \le C|t-r|^{\kappa}.$$

$\square$

5. Euler–Maruyama scheme

Let

$$\pi = \{0 = t_0 < t_1 < \cdots < t_N = T\}$$

be a partition of $[0,T]$. Set

$$\Delta_j = t_{j+1} - t_j, \qquad |\pi| = \max_{0 \le j \le N-1} \Delta_j.$$

For $s \in [t_j, t_{j+1})$, define the left grid projection

$$\tau_\pi(s) = t_j.$$

At the terminal point, we set $\tau_\pi(T) = t_{N-1}$. We also write

$$\Delta B_j = B_{t_{j+1}} - B_{t_j}, \qquad \Delta\langle B\rangle_j = \langle B\rangle_{t_{j+1}} - \langle B\rangle_{t_j}.$$

Definition 5.1. The Euler–Maruyama approximation of (3.1) is the grid process $(X_i^\pi)_{i=0}^N$ defined by

$$X_0^\pi = \varphi(0),$$

and, for $i = 1, \dots, N$,

$$\begin{aligned} X_i^\pi = \varphi(t_i) &+ \sum_{j=0}^{i-1} b(t_i, t_j, X_j^\pi)\Delta_j + \sum_{j=0}^{i-1} h(t_i, t_j, X_j^\pi)\Delta\langle B\rangle_j \\ &+ \sum_{j=0}^{i-1} \sigma(t_i, t_j, X_j^\pi)\Delta B_j. \end{aligned} \tag{5.1}$$

We introduce the piecewise constant interpolation

$$\overline{X}^\pi(s) = X_j^\pi, \qquad s \in [t_j, t_{j+1}),$$

with $\overline{X}^\pi(T) = X_{N-1}^\pi$. The value assigned at T is immaterial for the integrals. We also introduce the continuous-time Volterra interpolation

$$\begin{aligned} X^\pi(t) = \varphi(t) &+ \int_0^t b(t, \tau_\pi(s), \overline{X}^\pi(s))\, ds \\ &+ \int_0^t h(t, \tau_\pi(s), \overline{X}^\pi(s))\, d\langle B\rangle_s + \int_0^t \sigma(t, \tau_\pi(s), \overline{X}^\pi(s))\, dB_s. \end{aligned} \tag{5.2}$$

Then $X^\pi(t_i) = X_i^\pi$ for every grid point t_i.

Lemma 5.2. *Under Assumption 3.1, the grid values are well defined and satisfy*

$$X_i^\pi \in L_G^2(\Omega_{t_i}), \qquad i = 0, 1, \dots, N.$$

Moreover, for each fixed $t \in [0, T]$, the processes

$$s \longmapsto b(t, \tau_\pi(s), \overline{X}^\pi(s)), \qquad s \longmapsto h(t, \tau_\pi(s), \overline{X}^\pi(s)),$$

and

$$s \longmapsto \sigma(t, \tau_\pi(s), \overline{X}^\pi(s))$$

belong to the appropriate G-integrability spaces on $[0, t]$. Hence the interpolation $X^\pi(t)$ in (5.2) is well defined for every $t \in [0, T]$.

Proof. We argue by induction. By Assumption 3.1(A1), we have

$$X_0^\pi = \varphi(0) \in L_G^2(\Omega_0).$$

Assume that $X_j^\pi \in L_G^2(\Omega_{t_j})$ for $j = 0, \dots, i-1$. Since the coefficients are deterministic and satisfy the linear growth condition, the random variables

$$b(t_i, t_j, X_j^\pi), \qquad h(t_i, t_j, X_j^\pi), \qquad \sigma(t_i, t_j, X_j^\pi)$$

belong to $L_G^2(\Omega_{t_j})$. Therefore the sums in (5.1) are well defined, and the stochastic term is the G-Itô integral of an adapted step process. Hence

$$X_i^\pi \in L_G^2(\Omega_{t_i}).$$

The same argument shows that, for fixed $t \in [0, T]$, the integrands in (5.2) are adapted step processes with the required integrability. Thus the interpolation $X^\pi(t)$ is well defined. □

Algorithm 1 Euler–Maruyama scheme for a G-SVIE

1: Choose a partition $\pi = \{0 = t_0 < \cdots < t_N = T\}$.
2: Set $X_0^\pi = \varphi(0)$.
3: **for** $i = 1, \dots, N$ **do**
4: Compute X_i^π from the recursion (5.1).
5: **end for**
6: Return $(X_i^\pi)_{i=0}^N$.

Proposition 5.3. *Under Assumption 3.1, the Euler–Maruyama approximation satisfies*

$$\max_{0\le i\le N} \widehat{\mathbb{E}}\big[|X_i^\pi|^2\big] \le C.$$

Consequently,

$$\sup_{0\le t\le T} \widehat{\mathbb{E}}\big[|X^\pi(t)|^2\big] \le C.$$

Proof. For $i = 0$, the assertion follows from $X_0^\pi = \varphi(0)$ and Assumption 3.1(A1). Let $i \ge 1$. The scheme (5.1) can be written as

$$\begin{aligned} X_i^\pi = \varphi(t_i) &+ \int_0^{t_i} b(t_i, \tau_\pi(s), \overline{X}^\pi(s))\, ds \\ &+ \int_0^{t_i} h(t_i, \tau_\pi(s), \overline{X}^\pi(s))\, d\langle B\rangle_s + \int_0^{t_i} \sigma(t_i, \tau_\pi(s), \overline{X}^\pi(s))\, dB_s. \end{aligned}$$

Using Cauchy's inequality, the quasi-sure estimate

$$d\langle B\rangle_s \le \overline{\sigma}^2\, ds,$$

the G-Itô estimate, Assumption 3.1(A1) and the linear growth condition in Assumption 3.1(A4), we obtain

$$\widehat{\mathbb{E}}\big[|X_i^\pi|^2\big] \le C + C\int_0^{t_i} \Big(1 + \widehat{\mathbb{E}}\big[|\overline{X}^\pi(s)|^2\big]\Big)\, ds.$$

Since $\overline{X}^\pi(s) = X_j^\pi$ for $s \in [t_j, t_{j+1})$, this gives

$$\widehat{\mathbb{E}}\big[|X_i^\pi|^2\big] \le C + C\sum_{j=0}^{i-1} \Delta_j \widehat{\mathbb{E}}\big[|X_j^\pi|^2\big].$$

By a discrete Gronwall inequality,

$$\max_{0\le i\le N} \widehat{\mathbb{E}}\big[|X_i^\pi|^2\big] \le C.$$

It remains to estimate the continuous-time interpolation. Repeating the same argument in (5.2), with t in place of t_i, yields

$$\widehat{\mathbb{E}}\big[|X^\pi(t)|^2\big] \le C + C\int_0^{t} \Big(1 + \widehat{\mathbb{E}}\big[|\overline{X}^\pi(s)|^2\big]\Big)\, ds.$$

Using the already proved bound for the grid values, we obtain

$$\widehat{\mathbb{E}}\big[|X^\pi(t)|^2\big] \le C, \qquad 0 \le t \le T.$$

Taking the supremum over $t \in [0, T]$ gives the result. $\square$

6. Strong convergence analysis

We now prove the main convergence theorem. The proof is based on the mean-square regularity estimate in Lemma 4.3, a local discretization estimate and a discrete Gronwall argument.

Lemma 6.1. *Under Assumptions 3.1 and 4.1, let*

$$\kappa = \min\{2\gamma, 1\}.$$

Then, for every $\ell = b, h, \sigma$, every $u \in [0,T]$ and every $s \in [0,u]$,

$$\widehat{\mathbb{E}}\left[|\ell(u,s,X(s)) - \ell(u,\tau_\pi(s),\overline{X}^\pi(s))|^2\right] \le C\left(|\pi|^\kappa + \widehat{\mathbb{E}}\left[|X(\tau_\pi(s)) - \overline{X}^\pi(s)|^2\right]\right).$$

Proof. Let $\ell = b, h, \sigma$. By Assumption 4.1(B2) and the Lipschitz condition in Assumption 3.1(A3), we have

$$\begin{aligned}&|\ell(u,s,X(s)) - \ell(u,\tau_\pi(s),\overline{X}^\pi(s))|^2\\ &\quad\le C(1+|X(s)|^2)|s-\tau_\pi(s)|^{2\gamma} + C|X(s) - \overline{X}^\pi(s)|^2.\end{aligned}$$

Moreover,

$$|X(s) - \overline{X}^\pi(s)|^2 \le 2|X(s) - X(\tau_\pi(s))|^2 + 2|X(\tau_\pi(s)) - \overline{X}^\pi(s)|^2.$$

Taking $\widehat{\mathbb{E}}$ on both sides and using Lemmas 4.2 and 4.3 with $|s-\tau_\pi(s)| \le |\pi|$ yields

$$\begin{aligned}&\widehat{\mathbb{E}}\left[|\ell(u,s,X(s)) - \ell(u,\tau_\pi(s),\overline{X}^\pi(s))|^2\right]\\ &\quad\le C|\pi|^{2\gamma} + C|\pi|^\kappa + C\widehat{\mathbb{E}}\left[|X(\tau_\pi(s)) - \overline{X}^\pi(s)|^2\right].\end{aligned}$$

Since $\kappa = \min\{2\gamma, 1\}$, the estimate follows by enlarging C. □

Theorem 6.2. *Let Assumptions 3.1 and 4.1 hold. Let X be the solution of (3.1), and let $(X_i^\pi)_{i=0}^N$ be defined by (5.1). Then*

$$\max_{0\le i\le N} \widehat{\mathbb{E}}\left[|X(t_i) - X_i^\pi|^2\right] \le C|\pi|^\kappa, \qquad \kappa = \min\{2\gamma, 1\}.$$

Consequently,

$$\max_{0\le i\le N}\left(\widehat{\mathbb{E}}\left[|X(t_i) - X_i^\pi|^2\right]\right)^{1/2} \le C|\pi|^{\kappa/2}.$$

Proof. For $i = 0$, we have

$$X(0) = \varphi(0) = X_0^\pi.$$

Let $i \ge 1$ and set

$$e_i = X(t_i) - X_i^\pi.$$

Using (5.1), we write

$$\begin{aligned}X_i^\pi &= \varphi(t_i) + \int_0^{t_i} b(t_i,\tau_\pi(s),\overline{X}^\pi(s))\,ds\\ &\quad + \int_0^{t_i} h(t_i,\tau_\pi(s),\overline{X}^\pi(s))\,d\langle B\rangle_s + \int_0^{t_i} \sigma(t_i,\tau_\pi(s),\overline{X}^\pi(s))\,dB_s.\end{aligned}$$

Subtracting this identity from (3.1) at $t = t_i$ gives

$$
\begin{aligned}
e_i = &\int_0^{t_i} \big[b(t_i, s, X(s)) - b(t_i, \tau_\pi(s), \overline{X}^\pi(s))\big]\, ds \\
&+ \int_0^{t_i} \big[h(t_i, s, X(s)) - h(t_i, \tau_\pi(s), \overline{X}^\pi(s))\big]\, d\langle B\rangle_s \\
&+ \int_0^{t_i} \big[\sigma(t_i, s, X(s)) - \sigma(t_i, \tau_\pi(s), \overline{X}^\pi(s))\big]\, dB_s.
\end{aligned}
$$

Using Cauchy's inequality, the bound

$$
d\langle B\rangle_s \le \overline{\sigma}^2\, ds,
$$

the G-Itô estimate, and Lemma 6.1 with $u = t_i$, we obtain

$$
\widehat{\mathbb{E}}\big[|e_i|^2\big] \le C|\pi|^\kappa + C \int_0^{t_i} \widehat{\mathbb{E}}\left[|X(\tau_\pi(s)) - \overline{X}^\pi(s)|^2\right]\, ds.
$$

For $s \in [t_j, t_{j+1})$, we have

$$
X(\tau_\pi(s)) - \overline{X}^\pi(s) = X(t_j) - X_j^\pi = e_j.
$$

Therefore,

$$
\widehat{\mathbb{E}}\big[|e_i|^2\big] \le C|\pi|^\kappa + C \sum_{j=0}^{i-1} \Delta_j \widehat{\mathbb{E}}\big[|e_j|^2\big].
$$

By the discrete Gronwall lemma,

$$
\max_{0\le i\le N} \widehat{\mathbb{E}}\big[|e_i|^2\big] \le C|\pi|^\kappa.
$$

The root-mean-square estimate follows immediately. $\square$

Corollary 6.3. *Under the assumptions of Theorem 6.2, the interpolation X^π defined by (5.2) satisfies*

$$
\sup_{0\le t\le T} \widehat{\mathbb{E}}\left[|X(t) - X^\pi(t)|^2\right] \le C|\pi|^\kappa.
$$

Consequently,

$$
\widehat{\mathbb{E}}\left[\int_0^T |X(t) - X^\pi(t)|^2\, dt\right] \le C|\pi|^\kappa.
$$

Proof. Let $t \in [0, T]$. Subtracting (5.2) from (3.1), we get

$$
\begin{aligned}
X(t) - X^\pi(t) = &\int_0^t \big[b(t, s, X(s)) - b(t, \tau_\pi(s), \overline{X}^\pi(s))\big]\, ds \\
&+ \int_0^t \big[h(t, s, X(s)) - h(t, \tau_\pi(s), \overline{X}^\pi(s))\big]\, d\langle B\rangle_s \\
&+ \int_0^t \big[\sigma(t, s, X(s)) - \sigma(t, \tau_\pi(s), \overline{X}^\pi(s))\big]\, dB_s.
\end{aligned}
$$

Repeating the estimates used in the proof of Theorem 6.2, now using Lemma 6.1 with $u = t$, yields

$$\widehat{\mathbb{E}}\big[|X(t) - X^\pi(t)|^2\big] \le C|\pi|^\kappa + C\int_0^t \widehat{\mathbb{E}}\left[|X(\tau_\pi(s)) - \overline{X}^\pi(s)|^2\right] ds.$$

For $s \in [t_j, t_{j+1})$, we have

$$X(\tau_\pi(s)) - \overline{X}^\pi(s) = X(t_j) - X_j^\pi.$$

Therefore, by Theorem 6.2,

$$\begin{aligned}\int_0^t \widehat{\mathbb{E}}\left[|X(\tau_\pi(s)) - \overline{X}^\pi(s)|^2\right] ds &\le T \max_{0\le j\le N} \widehat{\mathbb{E}}\big[|X(t_j) - X_j^\pi|^2\big] \\ &\le C|\pi|^\kappa.\end{aligned}$$

Hence

$$\widehat{\mathbb{E}}\big[|X(t) - X^\pi(t)|^2\big] \le C|\pi|^\kappa, \qquad 0 \le t \le T.$$

Taking the supremum over $t \in [0, T]$ gives the first assertion. Finally,

$$\begin{aligned}\widehat{\mathbb{E}}\left[\int_0^T |X(t) - X^\pi(t)|^2\, dt\right] &\le \int_0^T \widehat{\mathbb{E}}\left[|X(t) - X^\pi(t)|^2\right] dt \\ &\le C|\pi|^\kappa.\end{aligned}$$

□

7. Numerical implementation under volatility uncertainty

This section gives a numerical illustration of Theorem 6.2 under volatility uncertainty. In the G-framework, the quadratic variation is uncertain; for simulation, we replace the volatility interval by a finite grid

$$\underline{\sigma} = \sigma_1 < \sigma_2 < \cdots < \sigma_I = \overline{\sigma}.$$

For each fixed volatility level σ_r, we simulate

$$\Delta B_j^{(r)} = \sigma_r\sqrt{\Delta_j}\, Z_j, \qquad \Delta\langle B\rangle_j^{(r)} = \sigma_r^2\Delta_j, \qquad Z_j \sim N(0,1),$$

and approximate the upper expectation by taking the maximum over the simulated volatility levels.

We consider the linear Volterra-memory equation

$$\begin{aligned}X(t) = 1 &+ \int_0^t 0.2(t-s)X(s)\, ds \\ &+ \int_0^t 0.1(1+t-s)X(s)\, d\langle B\rangle_s + \int_0^t 0.3e^{-(t-s)}X(s)\, dB_s, \qquad 0 \le t \le T.\end{aligned} \tag{7.1}$$

The coefficients are Lipschitz continuous in the state variable, have linear growth, and are Lipschitz continuous in the Volterra variables. Hence Assumptions 3.1 and 4.1 hold with $\gamma = 1$. Therefore $\kappa = 1$, and Theorem 6.2 predicts the root-mean-square order $1/2$.

Since no closed-form solution is available, the error is computed against a reference approximation with $N_{\text{ref}} = 6400$. We take $T = 1$, $\underline{\sigma} = 0.5$, $\overline{\sigma} = 1$,

$M = 10000$ Monte Carlo paths, $I = 5$ equally spaced volatility levels, and coarse grids $N = 50, 100, 200, 400, 800$. Consistently with Theorem 6.2, define

$$\mathrm{Err}(N) = \max_{0\le k\le N} \left(\max_{1\le r\le I} \frac{1}{M} \sum_{m=1}^{M} |X_k^{\pi,\sigma_r,m} - X^{\pi_{\mathrm{ref}},\sigma_r,m}(t_k)|^2 \right)^{1/2}.$$

TABLE 1. Strong error for the Volterra-memory G-SVIE test equation.

N	$\lvert\pi\rvert$	$\mathrm{Err}(N)$	Observed rate
50	1/50	8.9139×10^{-3}	–
100	1/100	5.8923×10^{-3}	0.597
200	1/200	3.9423×10^{-3}	0.580
400	1/400	2.6623×10^{-3}	0.566
800	1/800	1.8174×10^{-3}	0.551

The error decreases as the mesh is refined, and the observed rates are close to the theoretical root-mean-square order 1/2.

Remark 7.1. The finite maximum over volatility levels is only a practical approximation of the upper expectation. Thus the table illustrates the convergence behavior of the scheme, not an exact computation of the sublinear expectation. The rigorous convergence statement is Theorem 6.2.

8. Conclusion

In this paper, we studied Euler–Maruyama approximations for stochastic Volterra integral equations driven by a G-Brownian motion. The proposed explicit Volterra scheme discretizes the Lebesgue integral, the quadratic variation integral and the G-Itô integral. Under Lipschitz continuity, linear growth and Hölder-type regularity assumptions, we established uniform moment estimates and strong mean-square convergence under sublinear expectation.

More precisely, we proved convergence at the grid points with rate $|\pi|^\kappa$, where $\kappa = \min\{2\gamma, 1\}$, and obtained the corresponding uniform-in-time estimate for the continuous Volterra interpolation. In the time-Lipschitz case, this gives the classical root-mean-square order 1/2.

A numerical experiment under volatility uncertainty illustrated the predicted convergence behavior. The finite volatility approximation used in the simulation is only a practical approximation of the upper expectation, while the rigorous convergence result remains the one proved under the G-expectation framework.

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