Gulf Journal of Mathematics
Vol 24, Issue 1 (2026) 1-15

ISSN: 2309-4966
https://doi.org/10.56947/1njkmv81

FIXED POINTS OF A NEW GERAGHTY CONTRACTION

LOITONGBAM MELEI SINGH[1], YUMNAM ROHEN[2*], THANGJAM BIMOL[3] and N. PRIYOBARTA[4]

ABSTRACT. In this paper, we introduce a new type of $(\alpha,\beta)-\psi$-Geraghty rational contraction in b-metric space by generalizing alpha-psi-contraction with (α,β)-admissible mapping. Further, we prove some new fixed point theorems for this new $(\alpha,\beta)-\psi$- Geraghty rational contraction. Some examples are illustrated for the suitability of the result. Further, we apply our results in solving integral equation.

Keywords. Fixed point, b-metric space, (α,β)-admissible, Geraghty contraction

2020 Mathematics Subject Classification. Primary 47H10 ; Secondary 54H25

1. Introduction and preliminaries

Generalized forms of metric space are one of the important tools for the development of the Banach fixed point theorem [4]. Within the framework of modern nonlinear analysis, fixed point theory serves as a crucial tool for addressing problems in mathematical modeling, optimization, integral and differential equations. The foundation of this concept lies in the renowned Banach contraction principle, ensuring the existence and uniqueness of fixed points for contractive mappings in complete metric spaces. Indeed, Banach [4] developed a simple and effective approach to locate fixed points. This technique plays a crucial role in solving a variety of problems in different research domains. This inspired several researchers to extend and enhance the scope of metric fixed point theory.

In recent decades, researchers have focused on extending these theorems to various type of generalized metric spaces. Also, researchers have numerous generalizations and extension of metric space. Chand et al. [6] investigated Paired-Kannan contraction in metric space as a generalization of the Banach contraction principle. Czerwik [10] introduced a new version of metric space by generalizing the triangle inequality and this concept was originally introduced as b-metric space by Bakhtin [3]. He established the contraction mapping principle in b-metric spaces, which extends the classical Banach contraction principle in metric spaces. There are many results on various types of contraction mappings in b-metric space.

Date: Received: Jun 27, 2026; Revised: Jul 25, 2026; Accepted: Aug 1, 2026.
* Corresponding author

As an important achievement, Samet et al. [18] introduced the concept of α-ψ-contraction. The work of Samet et al. [18] is extended to a new direction by Chandok [7] by introducing (α, β)-admissible mapping. Samet et al. [18] results are further extended in S-metric spaces by Shanu and Rohen [19] and Priyobarta et al. [16]. In depth study about α-admissible in S-metric space is discussed in Priyobarta et al. [16]. Chandok's work is also extended to S-metric space by Deepchand and Rohen [5]. In another development Samet et al. [18] result is further improved by Zoto et al. [21] and introduced the concept of α_{qs^p}-admissible mappings. Melei and Rohen [20] introduced $(\alpha_{qs^p}, \beta_{qs^p})$-admissible mapping in b-metric spaces, which extends the classical notions of α-admissible, (α, β)-admissible, and α_{qs^p}-admissible mappings. Some other forms of the generalizations of α-admissible mappings can be found in ([17],[13], [14]), and references there in.

In 1973, M.A. Geraghty [11] introduced another new development of the Banach contraction principle which is known as Geraghty contraction. The concept of Geraghty is also extended by different researchers. Some of them can be found in ([1], [2], [12],[15]) and references there in.

Motivated by the above-mentioned works, as well as contribution of Samet et al.[18] , Chandok [7] , Zoto et al.[21] , Melei and Rohen [20], Geraghty [11] etc., in this paper we introduce the concept of $(\alpha_{qs^p}, \beta_{qs^p}) - \psi$-Geraghty rational contraction which extends the classical Banach and Geraghty contractions. Further, we examine the existence of fixed point in b-metric spaces. Furthermore, we apply our results in solving integral equation.

We recall some basic definitions and results that will be used in proving our main result.

Definition 1.1. [3] Let $\mathbb{H}$ be a non empty set. $\delta\colon \mathbb{H} \times \mathbb{H} \to [0, +\infty)$ is called a b-metric if

(i) $\delta(\vartheta, \varphi,) = 0$ iff $\vartheta = \varphi$
(ii) $\delta(\vartheta, \varphi) = \delta(\varphi, \vartheta)$
(iii) $\delta(\vartheta, \varphi) \leq s[\delta(\vartheta, z) + \delta(z, \varphi)]$, $\forall\ \vartheta, \varphi, z \in \mathbb{H}$ and for some $s \geq 1$.

$(\mathbb{H}, \delta)$ is called a b-metric space with parameter s.

Note that Stephen Czerwik [10] made b-metric space popular among researchers in fixed point theory.

Definition 1.2. ([18],[7],[21], [20])
Let $\mathbb{H}$ be a non empty set and $\alpha\colon \mathbb{H} \times \mathbb{H} \to [0, +\infty)$. $\mathbb{T}\colon \mathbb{H} \to \mathbb{H}$ is called

(i) α-admissible if $\alpha(\vartheta, \varphi) \geq 1 \Rightarrow \alpha(\mathbb{T}\vartheta, \mathbb{T}\varphi) \geq 1\ \forall\ \vartheta, \varphi \in \mathbb{H}$.
(ii) (α, β)-admissible if $\alpha(\vartheta, \varphi) \geq 1$ and $\beta(\vartheta, \varphi) \geq 1 \Rightarrow \alpha(\mathbb{T}\vartheta, \mathbb{T}\varphi) \geq 1$ and $\beta(\mathbb{T}\vartheta, \mathbb{T}\varphi) \geq 1\ \forall\ \vartheta, \varphi \in \mathbb{H}$.
(iii) α_{qs^p}-admissible if $\alpha(\vartheta, \varphi) \geq qs^p \Rightarrow \alpha(\mathbb{T}\vartheta, \mathbb{T}\varphi) \geq qs^p\ \forall\ \vartheta, \varphi \in \mathbb{H}$, $q \geq 1$ and $p \geq 2$.
(iv) $(\alpha_{qs^p}, \beta_{qs^p})$-admissible mapping if $\alpha(\vartheta, \varphi) \geq qs^p$ and $\beta(\vartheta, \varphi) \geq qs^p \Rightarrow \alpha(\mathbb{T}\vartheta, \mathbb{T}\varphi) \geq qs^p$ and $\beta(\mathbb{T}\vartheta, \mathbb{T}\varphi) \geq qs^p\ \forall\ \vartheta, \varphi \in \mathbb{H}$, $q \geq 1$ and $p \geq 2$.

Definition 1.3. [20] Let $(\mathbb{H}, \delta)$ be a b-metric space with $s \geq 1$, and $\alpha, \beta\colon \mathbb{H} \times \mathbb{H} \to [0, +\infty)$. A self Mapping $\mathbb{T}\colon \mathbb{H} \to \mathbb{H}$ is $(\alpha_{qs^p}, \beta_{qs^p})$-regular if $\{\vartheta_\varrho\} \in \mathbb{H}$

s.t. $\vartheta_\varrho \to \vartheta \in \mathbb{H}, \alpha(\vartheta_\varrho, \vartheta_{\varrho+1}) \geq qs^p, \beta(\vartheta_\varrho, \vartheta_{\varrho+1}) \geq qs^p \ \forall \ \varrho, \ \exists \ \{\vartheta_{\varrho_k}\}$ of $\{\vartheta_\varrho\}$ s.t. $\alpha(\vartheta_{\varrho_k}, \vartheta_{\varrho_k+1}) \geq qs^p, \beta(\vartheta_{\varrho_k}, \vartheta_{\varrho_K+1}) \geq qs^p \ \forall \ k \in \mathbb{N}$ and $\alpha(\vartheta, \mathbb{T}\vartheta) \geq qs^p, \beta(\vartheta, \mathbb{T}\vartheta) \geq qs^p$ where $q \geq 1$ and $p \geq 2$.

Theorem 1.4. [11] *Let $(\mathbb{H}, \delta)$ be a complete metric space and $\mathbb{T}\colon \mathbb{H} \to \mathbb{H}$ be a mapping. Suppose that there exists $g \in \mathcal{G}$ s. t.*

$$\delta(\mathbb{T}\vartheta, \mathbb{T}\varphi) \leq g(\delta(\vartheta, \varphi)) \ \forall \ \vartheta, \varphi \in \mathbb{H}.$$

Then $\mathbb{T}$ has a unique fixed point in $\mathbb{H}$.

Remark 1.5. In this paper, we use the following terms which are defined below.

(i) $F(\mathbb{T})$: family of fixed points of $\mathbb{T}$.

(ii) Ψ: family of functions and non-decreasing function $\psi\colon [0, +\infty) \to [0, +\infty)$, $\lim\limits_{\varrho\to\infty} \psi^\varrho = 0$ and $\psi(t) < t \ \forall \ t \in (0, +\infty)$, $\sum\limits_{i=1}^{\infty}$ is convergent for $t \geq 0$.

(iii) $\mathcal{G}$: family of functions $g\colon [0, +\infty) \to [0, 1)$, $g(t_\varrho) \to 1 \Rightarrow t_\varrho \to 0$, $\{t_\varrho\}$ bounded sequence of positive reals.

2. Main results

Here, we define $(\alpha_{qs^p}, \beta_{qs^p})$-Geraghty rational contraction in b-metric space. Using this new contraction, we prove some fixed point results in b-metric space.

Definition 2.1. Let $(\mathbb{H}, \delta)$ be a b-metric space with parameter $s \geq 1$ and $\alpha, \beta\colon \mathbb{H} \times \mathbb{H} \to [0, +\infty)$. We say that $\mathbb{T}\colon \mathbb{H} \to \mathbb{H}$ is a $(\alpha_{qs^p}, \beta_{qs^p})$-Geraghty rational contractive type-I mapping if $\mathbb{T}$ is an $(\alpha_{qs^p}, \beta_{qs^p})$ -admissible mapping s.t.

$$\alpha(\vartheta, \mathbb{T}\vartheta)\beta(\varphi, \mathbb{T}\varphi)\psi(\delta(\mathbb{T}\vartheta, \mathbb{T}\varphi)) \leq g(\psi(\mathbf{M}(\vartheta, \varphi)))\psi(\mathbf{M}(\vartheta, \varphi)), \ \forall \ \vartheta, \varphi \in \mathbb{H} \quad (2.1)$$

where

$$\mathbf{M}(\vartheta, \varphi) = \frac{1}{s} \max \left\{ \delta(\vartheta, \varphi), \delta(\vartheta, \mathbb{T}\vartheta), \delta(\varphi, \mathbb{T}\varphi), \frac{\delta(\vartheta, \mathbb{T}\vartheta)\delta(\varphi, \mathbb{T}y)}{1 + \delta(\vartheta, \varphi)}, \frac{\delta(\vartheta, \mathbb{T}\vartheta)\delta(\varphi, \mathbb{T}\varphi)}{1 + \delta(\mathbb{T}\vartheta, \mathbb{T}\varphi)} \right\}$$

and $\psi \in \Psi, g \in \mathcal{G}$.

Theorem 2.2. *Let $(\mathbb{H}, \delta)$ be a complete b-metric space, $\alpha, \beta\colon \mathbb{H} \times \mathbb{H} \to [0, +\infty)$ and $\mathbb{T}\colon \mathbb{H} \to \mathbb{H}$ be continuous or $\mathbb{H}$ is $(\alpha_{qs^p}, \beta_{qs^p})$-regular and $(\alpha_{qs^p}, \beta_{qs^p})$-admissible mapping. Let $\mathbb{T}$ holds (2.1) and $\exists \ \vartheta_0 \in \mathbb{H}$ s.t. $\alpha(\vartheta_0, \mathbb{T}\vartheta_0) \geq qs^p$ and $\beta(\vartheta_0, \mathbb{T}\vartheta_0) \geq qs^p$, then $\mathbb{T}\vartheta = \vartheta, \ \forall \ \vartheta \in \mathbb{H}$. If $\forall \ \vartheta, \varphi \in F(\mathbb{T})$ s.t. $\alpha(\vartheta, \mathbb{T}\vartheta) \geq qs^p, \ \alpha(\varphi, \mathbb{T}\varphi) \geq qs^p$ and $\beta(\vartheta, \mathbb{T}\vartheta) \geq qs^p, \ \beta(\varphi, \mathbb{T}\varphi) \geq qs^p$ then fixed point of $\mathbb{T}$ is unique.*

Proof. Let $\vartheta_0 \in \mathbb{H}$ s.t. $\alpha(\vartheta_0, \mathbb{T}\vartheta_0) \geq qs^p, \ \beta(\vartheta_0, \mathbb{T}\vartheta_0) \geq qs^p$. Construct $\{\vartheta_\varrho\} \in \mathbb{H}$ by $\vartheta_\varrho = \mathbb{T}^\varrho\vartheta_0 = \mathbb{T}\vartheta_{\varrho-1}$, for $\varrho \in \mathbb{N}$.

If $\vartheta_{\varrho_0} = \vartheta_{\varrho_0+1}$, for some $\varrho_0 \in \mathbb{N}$, then ϑ_{ϱ_0} is a fixed point of $\mathbb{T}$. Consequently, suppose $\vartheta_\varrho \neq \vartheta_{\varrho+1} \ \forall \ \varrho \in \mathbb{N}$.

Since $\mathbb{T}$ is $(\alpha_{qs^p}, \beta_{qs^p})-$admissible mapping, $\alpha(\vartheta_0, \mathbb{T}\vartheta_0) = \alpha(\vartheta_0, \vartheta_1) \geq qs^p, \alpha(\mathbb{T}\vartheta_0, \mathbb{T}\vartheta_1) = \alpha(\vartheta_1, \vartheta_2) \geq qs^p, \alpha(\mathbb{T}\vartheta_1, \mathbb{T}\vartheta_2) = \alpha(\vartheta_2, \vartheta_3) \geq qs^p$. Hence, by induction, $\alpha(\vartheta_\varrho, \vartheta_{\varrho+1}) \geq$

$qs^p \; \forall \; \varrho \geq 0$.

Similarly, $\beta(\vartheta_\varrho, \vartheta_{\varrho+1}) \geq qs^p \; \forall \; \varrho \geq 0$.

Considering (2.1), we have

$$\begin{aligned}
\psi(\delta(\vartheta_{\varrho+1}, \vartheta_{\varrho+2})) &\leq (qs^p)^2 \psi(\delta(\mathbb{T}\vartheta_\varrho, \mathbb{T}\vartheta_{\varrho+1})) \\
&\leq \alpha(\vartheta_\varrho, \mathbb{T}\vartheta_\varrho)\beta(\vartheta_{\varrho+1}, \mathbb{T}\vartheta_{\varrho+1})\psi(\delta(\mathbb{T}\vartheta_\varrho, \mathbb{T}\vartheta_{\varrho+1})) \\
&\leq g(\psi(\mathbf{M}(\vartheta_\varrho, \vartheta_{\varrho+1})))\psi(\mathbf{M}(\vartheta_\varrho, \vartheta_{\varrho+1}))
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{M}(\vartheta_\varrho, \vartheta_{\varrho+1}) &= \frac{1}{s} \max \left\{ \delta(\vartheta_\varrho, \vartheta_{\varrho+1}), \delta(\vartheta_\varrho, \mathbb{T}\vartheta_\varrho), \delta(\vartheta_{\varrho+1}, \mathbb{T}\vartheta_{\varrho+1}), \frac{\delta(\vartheta_\varrho, \mathbb{T}\vartheta_\varrho)\delta(\vartheta_{\varrho+1}, \mathbb{T}\vartheta_{\varrho+1})}{1+\delta(\vartheta_\varrho, \vartheta_{\varrho+1})}, \right. \\
&\quad \left. \frac{\delta(\vartheta_\varrho, \mathbb{T}\vartheta_\varrho)\delta(\vartheta_{\varrho+1}, \mathbb{T}\vartheta_{\varrho+1})}{1+\delta(\mathbb{T}\vartheta_\varrho, \mathbb{T}\vartheta_{\varrho+1})} \right\} \\
&= \frac{1}{s} \max \left\{ \delta(\vartheta_\varrho, \vartheta_{\varrho+1}), \delta(\vartheta_\varrho, \vartheta_{\varrho+1}), \delta(\vartheta_{\varrho+1}, \vartheta_{\varrho+2}), \frac{\delta(\vartheta_\varrho, \vartheta_{\varrho+1})\delta(\vartheta_{\varrho+1}, \vartheta_{\varrho+2})}{1+\delta(\vartheta_\varrho, \vartheta_{\varrho+1})}, \right. \\
&\quad \left. \frac{\delta(\vartheta_\varrho, \vartheta_{\varrho+1})\delta(\vartheta_{\varrho+1}, \vartheta_{\varrho+2})}{1+\delta(\vartheta_{\varrho+1}, \vartheta_{\varrho+2})} \right\} \\
&= \frac{1}{s} \max\{\delta(\vartheta_\varrho, \vartheta_{\varrho+1}), \delta(\vartheta_{\varrho+1}, \vartheta_{\varrho+2})\}. \qquad (2.2)
\end{aligned}$$

If $\mathbf{M}(\vartheta_\varrho, \vartheta_{\varrho+1}) = \frac{1}{s}\delta(\vartheta_{\varrho+1}, \vartheta_{\varrho+2})$, then

$$\begin{aligned}
\psi(\delta(\vartheta_{\varrho+1}, \vartheta_{\varrho+2})) &\leq g(\psi(\mathbf{M}(\vartheta_\varrho, \vartheta_{\varrho+1})))\psi(\mathbf{M}(\vartheta_\varrho, \vartheta_{\varrho+1})) \\
&= g(\psi(\mathbf{M}(\vartheta_\varrho, \vartheta_{\varrho+1})))\psi\left(\frac{1}{s}\delta(\vartheta_{\varrho+1}, \vartheta_{\varrho+2})\right) \\
&< \psi(\delta(\vartheta_{\varrho+1}, \vartheta_{\varrho+2}))
\end{aligned}$$

a contradiction.
So, $\mathbf{M}(\vartheta_\varrho, \vartheta_{\varrho+1}) = \frac{1}{s}\delta(\vartheta_\varrho, \vartheta_{\varrho+1})$ and

$$\begin{aligned}
\psi(\delta(\vartheta_{\varrho+1}, \vartheta_{\varrho+2})) &\leq g(\psi(\mathbf{M}(\vartheta_\varrho, \vartheta_{\varrho+1})))\psi(\mathbf{M}(\vartheta_\varrho, \vartheta_{\varrho+1})) \\
&\leq g(\psi(\mathbf{M}(\vartheta_\varrho, \vartheta_{\varrho+1}))\psi\left(\frac{1}{s}\delta(\vartheta_\varrho, \vartheta_{\varrho+1}))\right) \\
&< \psi(\delta(\vartheta_\varrho, \vartheta_{\varrho+1})). \qquad (2.3)
\end{aligned}$$

By property of ψ,

$$\delta(\vartheta_{\varrho+1}, \vartheta_{\varrho+2}) < \delta(\vartheta_\varrho, \vartheta_{\varrho+1}) \qquad (2.4)$$

$\forall \; \varrho \in \mathbb{N}$. $\{\delta(\vartheta_\varrho, \vartheta_{\varrho+1})\}$ is non-increasing and $\exists$ some $r \geq 0$, s.t.

$$\lim_{\varrho \to \infty} \delta(\vartheta_\varrho, \vartheta_{\varrho+1}) = r. \qquad (2.5)$$

From (2.3),

$$\frac{\psi(\delta(\vartheta_{\varrho+1}, \vartheta_{\varrho+2}))}{\psi(\mathbf{M}(\vartheta_\varrho, \vartheta_{\varrho+1}))} \leq g(\psi(\mathbf{M}(\vartheta_\varrho, \vartheta_{\varrho+1}))) < 1. \qquad (2.6)$$

By $\varrho \to \infty$ in (2.6),

$$\lim_{\varrho\to\infty} g(\psi(\mathbf{M}(\vartheta_\varrho, \vartheta_{\varrho+1}))) = 1$$

and

$$g \in \mathcal{G}, \lim_{\varrho\to\infty} \psi(\mathbf{M}(\vartheta_\varrho, \vartheta_{\varrho+1})) = 0$$

which yields that

$$r = \lim_{\varrho\to\infty} \delta(\vartheta_\varrho, \vartheta_{\varrho+1}) = 0. \tag{2.7}$$

By contradiction method, suppose $\{\vartheta_\varrho\}$ is not a Cauchy. Then, $\exists\, \varepsilon > 0$ for which we can find $\{\vartheta_{\varrho_k}\}$,$\{\vartheta_{\mu_k}\} \in \{\vartheta_\varrho\}$ with $\varrho_k > \mu_k > k$ s.t.

$$\delta(\vartheta_{\mu_k}, \vartheta_{\varrho_k}) \geq \varepsilon. \tag{2.8}$$

We can choose ϱ_k corresponding to μ_k in with smallest integer $\varrho_k > \mu_k$ and satisfying (2.8),

$$\delta(\vartheta_{\mu_k}, \vartheta_{\varrho_k-1}) < \varepsilon. \tag{2.9}$$

Using triangle inequality and (2.9) we get

$$\begin{aligned} 0 &< \varepsilon \leq \delta(\vartheta_{\mu_k}, \vartheta_{\varrho_k}) \\ &\leq s\delta(\vartheta_{\mu_k}, \vartheta_{\mu_k-1}) + s\delta(\vartheta_{\mu_k-1}, \vartheta_{\varrho_k}) \\ &\leq s\delta(\vartheta_{\mu_k}, \vartheta_{\mu_k-1}) + s^2\delta(\vartheta_{\mu_k-1}, \vartheta_{\varrho_k-1}) + s^2\delta(\vartheta_{\varrho_k-1}, \vartheta_{\varrho_k}). \end{aligned} \tag{2.10}$$

By $k \to \infty$ in(2.10) and using (2.7),

$$\frac{\varepsilon}{s^2} \leq \lim_{k\to\infty} sup\ \delta(\vartheta_{\mu_k-1}, \vartheta_{\varrho_k-1}). \tag{2.11}$$

By the triangle inequality,

$$\delta(\vartheta_{\mu_k-1}, \vartheta_{\varrho_k-1}) \leq s\delta(\vartheta_{\mu_k-1}, \vartheta_{\mu_k}) + s\delta(\vartheta_{\mu_k}, \vartheta_{\varrho_k-1}). \tag{2.12}$$

By $k \to \infty$ in (2.12) and using (2.7) and (2.9)

$$\frac{\varepsilon}{s^2} \leq \lim_{k\to\infty} sup\ \delta(\vartheta_{\mu_k-1}, \vartheta_{\varrho_k-1}) \leq \varepsilon s. \tag{2.13}$$

Also,

$$\begin{aligned} 0 < \varepsilon \leq \delta(\vartheta_{\mu_k}, \vartheta_{\varrho_k}) &\leq s\delta(\vartheta_{\mu_k}, \vartheta_{\varrho_k-1}) + s\delta(\vartheta_{\varrho_k-1}, \vartheta_{\varrho_k}) \\ &\leq \varepsilon s + s\delta(\vartheta_{\varrho_k-1}, \vartheta_{\varrho_k}). \end{aligned}$$

By $k \to \infty$ and using (2.7),

$$\varepsilon \leq \lim_{k\to\infty} sup\ \delta(\vartheta_{\mu_k}, \vartheta_{\varrho_k}) \leq \varepsilon s. \tag{2.14}$$

We have

$$\begin{aligned}\mathbf{M}(\vartheta_{\mu_k-1},\vartheta_{\varrho_k-1}) &= \frac{1}{s}\max\{\delta(\vartheta_{\mu_k-1},\vartheta_{\varrho_k-1}),\delta(\vartheta_{\mu_k-1},\mathbb{T}\vartheta_{\mu_k-1}),\delta(\vartheta_{\varrho_k-1},\mathbb{T}\vartheta_{\varrho_k-1}),\\ &\quad \frac{\delta(\vartheta_{\mu_k-1},\mathbb{T}\vartheta_{\mu_k-1})\delta(\vartheta_{\varrho_k-1},\mathbb{T}\vartheta_{\varrho_k-1})}{1+\delta(\vartheta_{\mu_k-1},\vartheta_{\varrho_k-1})},\frac{\delta(\vartheta_{\mu_k-1},\mathbb{T}\vartheta_{\mu_k-1})\delta(\vartheta_{\varrho_k-1},\mathbb{T}\vartheta_{\varrho_k-1})}{1+\delta(\mathbb{T}\vartheta_{\mu_k-1},\mathbb{T}\vartheta_{\varrho_k-1})}\Big\}\\ &= \frac{1}{s}\max\{\delta(\vartheta_{\mu_k-1},\vartheta_{\varrho_k-1}),\delta(\vartheta_{\mu_k-1},\vartheta_{\mu_k}),\delta(\vartheta_{\varrho_k-1},\vartheta_{\varrho_k}),\\ &\quad \frac{\delta(\vartheta_{\mu_k-1},\vartheta_{\mu_k})\delta(\vartheta_{\varrho_k-1},\vartheta_{\varrho_k})}{1+\delta(\vartheta_{\mu_k-1},\vartheta_{\varrho_k-1})},\frac{\delta(\vartheta_{\mu_k-1},\vartheta_{\mu_k})\delta(\vartheta_{\varrho_k-1},\vartheta_{\varrho_k})}{1+\delta(\vartheta_{\mu_k},\vartheta_{\varrho_k})}\Big\}. \qquad (2.15)\end{aligned}$$

By $k\to\infty$ in (2.15), using (2.7) and (2.13),

$$\begin{aligned}\lim_{k\to\infty} sup\ \mathbf{M}(\vartheta_{\mu_k-1},\vartheta_{\varrho_k-1}) &= \lim_{k\to\infty} sup\ \frac{1}{s}\max\{\delta(\vartheta_{\mu_k-1},\vartheta_{\varrho_k-1}),\delta(\vartheta_{\mu_k-1},\vartheta_{\mu_k}),\delta(\vartheta_{\varrho_k-1},\vartheta_{\varrho_k}),\\ &\quad \frac{\delta(\vartheta_{\mu_k-1},\vartheta_{\mu_k})\delta(\vartheta_{\varrho_k-1},\vartheta_{\varrho_k})}{1+\delta(\vartheta_{\mu_k-1},\vartheta_{\varrho_k-1})},\frac{\delta(\vartheta_{\mu_k-1},\vartheta_{\mu_k})\delta(\vartheta_{\varrho_k-1},\vartheta_{\varrho_k})}{1+\delta(\vartheta_{\mu_k},\vartheta_{\varrho_k})}\Big\}\\ &\le \frac{1}{s}\max\{\varepsilon s,0,0,0,0\}\\ &\le \varepsilon. \qquad (2.16)\end{aligned}$$

Using the $(\alpha_{qs^p},\beta_{qs^p})$- Geraghty rational contractive type-I condition,

$$\begin{aligned}\psi(\delta(\vartheta_{\mu_k},\vartheta_{\varrho_k})) &\le (qs^p)^2\psi\left(\delta(\vartheta_{\mu_k},\vartheta_{\varrho_k})\right)\\ &= (qs^p)^2\psi\left(\delta(\mathbb{T}\vartheta_{\mu_k-1},\mathbb{T}\vartheta_{\varrho_k-1})\right)\\ &\le \alpha(\vartheta_{\mu_k-1},\mathbb{T}\vartheta_{\mu_k-1})\beta(\vartheta_{\varrho_k-1},\mathbb{T}\vartheta_{\varrho_k-1})\psi\left(\delta(\mathbb{T}\vartheta_{\mu_k-1},\mathbb{T}\vartheta_{\varrho_k-1})\right)\\ &\le g(\psi(\mathbf{M}(\vartheta_{\mu_k-1},\vartheta_{\varrho_k-1})))\psi\left(\mathbf{M}(\vartheta_{\mu_k-1},\vartheta_{\varrho_k-1})\right). \qquad (2.17)\end{aligned}$$

Taking the upper limit in (2.17), using (2.14) and (2.16), we get

$$\begin{aligned}\psi(\varepsilon) &\le (qs^p)^2\psi(\varepsilon)\\ &\le (qs^p)^2\psi\left(\lim_{k\to\infty} sup\ \delta(\vartheta_{\mu_k},\vartheta_{\varrho_k})\right)\\ &\le \lim_{k\to\infty} sup\ g(\psi(\mathbf{M}(\vartheta_{\mu_k-1},\vartheta_{\varrho_k-1})))\psi\left(\lim_{k\to\infty} sup\ \mathbf{M}(\vartheta_{\mu_k-1},\mathbb{T}\vartheta_{\varrho_k-1})\right)\\ &\le \lim_{k\to\infty} sup\ g(\psi(\mathbf{M}(\vartheta_{\mu_k-1},\vartheta_{\varrho_k-1})))\psi(\varepsilon)\\ &< \psi(\varepsilon)\end{aligned}$$

a contradiction, since $\varepsilon>0$. Therefore, $\{\vartheta_\varrho\}$ is a Cauchy. $\exists\,\eta\in\mathbb{H}$ s.t. $\vartheta_\varrho\to\eta$ ($\mathbb{H}$ complete). Therefore, $\eta=\lim_{\varrho\to\infty}\vartheta_{\varrho+1}=\lim_{\varrho\to\infty}\mathbb{T}\vartheta_\varrho=\mathbb{T}\lim_{\varrho\to\infty}\vartheta_\varrho=\mathbb{T}\eta$ ($\mathbb{T}$ continuous).
Suppose $\mathbb{H}$ is $(\alpha_{qs^p},\beta_{qs^p})-$regular. $\exists\ \{\vartheta_{\varrho_k}\}\in\{\vartheta_\varrho\}$ s.t. $\alpha(\vartheta_{\varrho_k-1},\vartheta_{\varrho_k})\ge qs^p$, $\beta(\vartheta_{\varrho_k-1},\vartheta_{\varrho_k})\ge qs^p$ $\forall\ k\in\mathbb{N}$ and $\alpha(\eta,\mathbb{T}\eta)\ge qs^p$, $\beta(\eta,\mathbb{T}\eta)\ge qs^p$.
From (2.1) with $(\vartheta,\varphi)=(\vartheta_{\varrho_k},\eta)$,

$$\begin{aligned}\psi(\delta(\vartheta_{\varrho_k+1},\mathbb{T}\eta)) &= \psi(\delta(\mathbb{T}\vartheta_{\varrho_k},\mathbb{T}\eta))\\ &\le \alpha(\vartheta_{\varrho_k},\mathbb{T}\vartheta_{\varrho_k})\beta(\eta,\mathbb{T}\eta)\psi(\delta(\mathbb{T}\vartheta_{\varrho_k},\mathbb{T}\eta))\\ &\le g(\psi(\mathbf{M}(\vartheta_{\varrho_k},\eta)))\psi(\mathbf{M}(\vartheta_{\varrho_k},\eta)) \qquad (2.18)\end{aligned}$$

where

$$\begin{aligned}\mathbf{M}(\vartheta_{\varrho_k},\eta) &= \frac{1}{s}\max\left\{\delta(\vartheta_{\varrho_k},\eta),\delta(\vartheta_{\varrho_k},\mathbb{T}\vartheta_{\varrho_k}),\delta(\eta,\mathbb{T}\eta),\frac{\delta(\vartheta_{\varrho_k},\mathbb{T}\vartheta_{\varrho_k})\delta(\eta,\mathbb{T}\eta)}{1+\delta(\vartheta_{\varrho_k},\eta)},\right.\\ &\qquad\left.\frac{\delta(\vartheta_{\varrho_k},\mathbb{T}\vartheta_{\varrho_k})\delta(\eta,\mathbb{T}\eta)}{1+\delta(\mathbb{T}\vartheta_{\varrho_k},\mathbb{T}\eta)}\right\}\\ &= \frac{1}{s}\max\left\{\delta(\vartheta_{\varrho_k},\eta),\delta(\vartheta_{\varrho_k},\vartheta_{\varrho_k+1}),\delta(\eta,\mathbb{T}\eta),\frac{\delta(\vartheta_{\varrho_k},\vartheta_{\varrho_k+1})\delta(\eta,\mathbb{T}\eta)}{1+\delta(\vartheta_{\varrho_k},\eta)},\right.\\ &\qquad\left.\frac{\delta(\vartheta_{\varrho_k},\vartheta_{\varrho_k+1})\delta(\eta,\mathbb{T}\eta)}{1+\delta(\vartheta_{\varrho_k+1},\mathbb{T}\eta)}\right\}. \end{aligned} \tag{2.19}$$

Therefore,

$$\psi(\delta(\vartheta_{\varrho_k+1},\mathbb{T}\eta)) \leq g(\psi(\mathbf{M}(\vartheta_{\varrho_k},\eta)))\psi(\mathbf{M}(\vartheta_{\varrho_k},\eta)). \tag{2.20}$$

By $k\to\infty$ in (2.20),

$$\begin{aligned}\psi(\delta(\eta,\mathbb{T}\eta)) &\leq \lim_{k\to\infty} g(\psi(\mathbf{M}(\vartheta_{\varrho_k},\eta)))\psi\left(\frac{1}{s}\delta(\eta,\mathbb{T}\eta)\right)\\ &\leq \lim_{k\to\infty} g(\psi(\mathbf{M}(\vartheta_{\varrho_k},\eta)))\psi(\delta(\eta,\mathbb{T}\eta))\\ \Rightarrow 1 &\leq \lim_{k\to\infty} g(\psi(\mathbf{M}(\vartheta_{\varrho_k},\eta)))\\ \Rightarrow \lim_{k\to\infty} g(\psi(\mathbf{M}(\vartheta_{\varrho_k},\eta))) &= 1.\end{aligned}$$

Consequently, we get $\lim_{k\to\infty}\mathbf{M}(\vartheta_{\varrho_k},\eta)=0$. So, $\eta=\mathbb{T}\eta$.
Let $\eta,\zeta\in Fix(\mathbb{T})$ s.t. $\eta\neq\zeta$ and $\alpha(\eta,\mathbb{T}\eta)\geq qs^p,\alpha(\zeta,\mathbb{T}\zeta)\geq qs^p$ and $\beta(\eta,\mathbb{T}\eta)\geq qs^p,\beta(\zeta,T\zeta)\geq qs^p$.
From (2.1),

$$\begin{aligned}\psi(\delta(\eta,\zeta)) &= \psi(\delta(\mathbb{T}\eta,\mathbb{T}\zeta))\\ &\leq \alpha(\eta,\mathbb{T}\eta)\beta(\zeta,\mathbb{T}\zeta)\psi(\delta(\mathbb{T}\eta,\mathbb{T}\zeta))\\ &\leq g(\psi(\mathbf{M}(\eta,\zeta)))\psi(\mathbf{M}(\eta,\zeta))\end{aligned}$$

where,

$$\mathbf{M}(\eta,\zeta)=\frac{1}{s}\max\left\{\delta(\eta,\zeta),\delta(\eta,\mathbb{T}\eta),\delta(\zeta,\mathbb{T}\zeta),\frac{\delta(\eta,\mathbb{T}\eta)\delta(\zeta,\mathbb{T}\zeta)}{1+\delta(\eta,\zeta)},\frac{\delta(\eta,\mathbb{T}\eta)\delta(\zeta,\mathbb{T}\zeta)}{1+\delta(\mathbb{T}\eta,\mathbb{T}\zeta)}\right\}.$$

Hence,

$$\psi(\delta(\eta,\zeta))\leq g(\psi(\mathbf{M}(\eta,\zeta)))\psi\left(\frac{1}{s}\delta(\eta,\zeta)\right)<\psi(\delta(\eta,\zeta)),$$

a contradiction. Hence, $\mathbb{T}$ has a unique fixed point. □

We now present a supporting example of Theorem 2.2.

Example 2.3. Let $\mathbb{H}=[0,+\infty)$, $\delta(x,y)=(x-y)^2$ $\forall$ $x,y\in\mathbb{H}$ and $\mathbb{T}\colon\mathbb{H}\to\mathbb{H}$ define as

$$\mathbb{T}\vartheta=\begin{cases}\frac{1-\vartheta^2}{32}, & \vartheta\in[0,1]\\ 9\vartheta, & otherwise.\end{cases}$$

Also define $\alpha, \beta \colon \mathbb{H} \times \mathbb{H} \to [0, +\infty)$, $\psi \colon [0, +\infty) \to [0, +\infty)$ and $g \colon [0, +\infty) \to [0, 1)$ as

$$\alpha(\vartheta, \varphi) = \begin{cases} 4, & (\vartheta, \varphi) \in [0, 1] \\ 0, & otherwise \end{cases}$$

$$\beta(\vartheta, \varphi) = \begin{cases} 4, & (\vartheta, \varphi) \in [0, 1] \\ 0, & otherwise \end{cases}$$

$g(t) = \frac{1}{4}$ and $\psi(t) = t$. Here $q = 1, s = 2, p = 2$.

Let $\vartheta, \varphi \in \mathbb{H}$, if $\alpha(\vartheta, \varphi) \geq 4$ and $\beta(\vartheta, \varphi) \geq 4$, then $\vartheta, \varphi \in [0, 1]$. $\mathbb{T}\vartheta \leq 1$, $\forall\ \vartheta \in [0, 1]$. Therefore, $\alpha(\mathbb{T}\vartheta, \mathbb{T}\varphi) \geq 4$ and $\beta(\mathbb{T}\vartheta, \mathbb{T}\varphi) \geq 4$ and $\alpha(0, \mathbb{T}0) \geq 4$ and $\beta(0, \mathbb{T}0) \geq 4$. Hence, $\mathbb{T}$ is $(\alpha_{qs^p}, \beta_{qs^p})$-admissible.

If $\{\vartheta_\varrho\} \in \mathbb{H}$ s.t. $\alpha(\vartheta_\varrho, \vartheta_{\varrho+1}) \geq 4$, $\beta(\vartheta_\varrho, \vartheta_{\varrho+1}) \geq 4$ and $\vartheta_\varrho \to \vartheta \in \mathbb{H}$, $\forall\ \varrho \in \mathbb{N} \cup \{0\}$, then $\vartheta_\varrho \subseteq [0, 1]$ and hence $\vartheta \in [0, 1] \Rightarrow \alpha(\vartheta, \mathbb{T}\vartheta) \geq 4$, $\beta(\vartheta, \mathbb{T}\vartheta) \geq 4$.
Let $\vartheta, \varphi \in [0, 1]$.

$$\begin{aligned} \alpha(\vartheta, \mathbb{T}\vartheta)\beta(\vartheta, \mathbb{T}\vartheta)\psi(\delta(\mathbb{T}\vartheta, \mathbb{T}\varphi)) &= 4 \times 4\delta(\mathbb{T}\vartheta, \mathbb{T}\varphi) \\ &= 16(\mathbb{T}\vartheta - \mathbb{T}\varphi)^2 \\ &= 16 \times \frac{1}{32^2}(\vartheta - \varphi)^2(\vartheta + \varphi)^2 \\ &\leq \frac{1}{4} \times \frac{1}{2}(\vartheta - \varphi)^2 \\ &\leq g(\psi(\mathbf{M}(\vartheta, \varphi)))\psi(\mathbf{M}(\vartheta, \varphi)). \end{aligned}$$

Hence,the given inequality is satisfied. Otherwise, if $\alpha(\vartheta, \mathbb{T}\vartheta)\beta(\varphi, \mathbb{T}\varphi) = 0$. Then, $\alpha(\vartheta, \mathbb{T}\vartheta)\beta(\varphi, \mathbb{T}\varphi)\psi(\delta(\mathbb{T}\vartheta, \mathbb{T}\varphi)) = 0 \leq g(\psi(\mathbf{M}(\vartheta, \varphi)))\psi(\mathbf{M}(\vartheta, \varphi))$. Therefore, all the conditions of Theorem 2.2 are satisfied and the fixed point of $\mathbb{T}$ is $\sqrt{257} - 16$.

Definition 2.4. Let $(\mathbb{H}, \delta)$ be a b-metric space with $s \geq 1$ and $\alpha, \beta \colon \mathbb{H} \times \mathbb{H} \to [0, +\infty)$. $\mathbb{T} \colon \mathbb{H} \to \mathbb{H}$ is called $(\alpha_{qs^p}, \beta_{qs^p})$-Geraghty rational contractive type-II mapping if $\mathbb{T}$ is an $(\alpha_{qs^p}, \beta_{qs^p})$-admissible mapping s.t.

$$[\psi(\delta(\mathbb{T}\vartheta, \mathbb{T}\varphi)) + l]^{\alpha(\vartheta, \mathbb{T}\vartheta)\beta(\varphi, \mathbb{T}\varphi)} \leq g\left(\psi(\mathbf{M}(\vartheta, \varphi))\right)\psi(\mathbf{M}(\vartheta, \varphi)) + l\ \forall\ \vartheta, \varphi \in \mathbb{H} \tag{2.21}$$

where

$$\mathbf{M}(\vartheta, \varphi) = \frac{1}{s} \max\left\{\delta(\vartheta, \varphi), \delta(\vartheta, \mathbb{T}\vartheta), \delta(\varphi, \mathbb{T}\varphi), \frac{\delta(\vartheta, \mathbb{T}\vartheta)\delta(\varphi, \mathbb{T}\varphi)}{1 + \delta(\vartheta, \varphi)}, \frac{\delta(\vartheta, \mathbb{T}\vartheta)\delta(\varphi, \mathbb{T}\varphi)}{1 + \delta(\mathbb{T}\vartheta, T\varphi)}\right\}$$

and $\psi \in \Psi, g \in \mathcal{G}\ l \geq 1$.

Theorem 2.5. *Let $(\mathbb{H}, \delta)$ be a complete b-metric space with $s \geq 1$ and $\alpha, \beta \colon \mathbb{H} \times \mathbb{H} \to [0, +\infty)$. Suppose $\mathbb{T} \colon \mathbb{H} \to \mathbb{H}$ is continuous or $\mathbb{H}$ is $(\alpha_{qs^p}, \beta_{qs^p})-$regular and $(\alpha_{qs^p}, \beta_{qs^p})-$admissible mapping. Let $\mathbb{T}$ holds (2.21) and $\exists\ \vartheta_0 \in \mathbb{H}$ s.t. $\alpha(\vartheta_0, \mathbb{T}\vartheta_0) \geq qs^p$ and $\beta(\vartheta_0, \mathbb{T}\vartheta_0) \geq qs^p$, then $\mathbb{T}\vartheta = \vartheta$, $\forall\ \vartheta \in \mathbb{H}$. If $\forall\ \vartheta, \varphi \in F(\mathbb{T})$ s.t. $\alpha(\vartheta, \mathbb{T}\vartheta) \geq qs^p$, $\alpha(\varphi, \mathbb{T}\varphi) \geq qs^p$ and $\beta(\vartheta, \mathbb{T}\vartheta) \geq qs^p$, $\beta(\varphi, \mathbb{T}\varphi) \geq qs^p$, then fixed point of $\mathbb{T}$ is unique.*

Proof. Let $\vartheta_0 \in \mathbb{H}$ s.t. $\alpha(\vartheta_0, \mathbb{T}\vartheta_0) \geq qs^p$ and $\beta(\vartheta_0, \mathbb{T}\vartheta_0) \geq qs^p$. Construct $\{\vartheta_\varrho\} \in \mathbb{H}$ by $\vartheta_\varrho = \mathbb{T}^\varrho\vartheta_0 = \mathbb{T}\vartheta_{\varrho-1}$, $\forall\ \varrho \in \mathbb{N}$.

If $\vartheta_{\varrho_0} = \vartheta_{\varrho_0+1}$, for some $\varrho_0 \in \mathbb{N}$, then ϑ_{ϱ_0} is a fixed point of $\mathbb{T}$. Let $\vartheta_\varrho \neq \vartheta_{\varrho+1}$ $\forall\ \varrho \in \mathbb{N}$.

Since $\mathbb{T}$ is $(\alpha_{qs^p}, \beta_{qs^p})-$admissible mapping, $\alpha(\vartheta_0, \mathbb{T}\vartheta_0) = \alpha(\vartheta_0, \vartheta_1) \geq qs^p, \alpha(\mathbb{T}\vartheta_0, \mathbb{T}\vartheta_1) = \alpha(\vartheta_1, \vartheta_2) \geq qs^p, \alpha(\mathbb{T}\vartheta_1, \mathbb{T}\vartheta_2) = \alpha(\vartheta_2, \vartheta_3) \geq qs^p$. Hence, by induction, $\alpha(\vartheta_\varrho, \vartheta_{\varrho+1}) \geq qs^p\ \forall\ n \geq 0$.
Similarly, $\beta(\vartheta_\varrho, \vartheta_{\varrho+1}) \geq qs^p\ \forall\ \varrho \geq 0$.
Considering (2.21), we have

$$\begin{aligned}
\psi(\delta(\vartheta_{\varrho+1}, \vartheta_{\varrho+2})) + l &= \psi(\delta(\mathbb{T}\vartheta_\varrho, \mathbb{T}\vartheta_{\varrho+1})) + l \\
&\leq [\psi(\delta(\mathbb{T}\vartheta_\varrho, \mathbb{T}\vartheta_{\varrho+1})) + l]^{(qs^p)^2} \\
&\leq [\psi(\delta(\mathbb{T}\vartheta_\varrho, \mathbb{T}\vartheta_{\varrho+1}))]^{\alpha(\vartheta_\varrho, \mathbb{T}\vartheta_\varrho)\beta(\vartheta_{\varrho+1}, \mathbb{T}\vartheta_{\varrho+1})} \\
&\leq g(\psi(\mathbf{M}(\vartheta_\varrho, \vartheta_{\varrho+1})))\psi(\mathbf{M}(\vartheta_\varrho, \vartheta_{\varrho+1})) + l \\
\Rightarrow \psi(\delta(\vartheta_{\varrho+1}, \vartheta_{\varrho+2})) &\leq g(\psi(\mathbf{M}(\vartheta_\varrho, \vartheta_{\varrho+1})))\psi(\mathbf{M}(\vartheta_\varrho, \vartheta_{\varrho+1})).
\end{aligned}$$

Following the same lines as in Theorem 2.2, we obtain

$$\lim_{k\to\infty} sup\ \mathbf{M}(\vartheta_{\mu_k-1}, \vartheta_{\varrho_k-1}) \leq \varepsilon \tag{2.22}$$

Using the $(\alpha_{qs^p}, \beta_{qs^p})$-Geraghty rational contractive type-II condition, we have

$$\begin{aligned}
\psi(\delta(\vartheta_{\mu_k}, \vartheta_{\varrho_k})) + l &= \psi(\delta(\mathbb{T}\vartheta_{\mu_k-1}, \mathbb{T}\vartheta_{\varrho_k-1})) + l \\
&\leq [\psi(\delta(\mathbb{T}\vartheta_{\mu_k-1}, \mathbb{T}\vartheta_{\varrho_k-1})) + l]^{(qs^p)^2} \\
&\leq [\psi\left(\delta(\mathbb{T}\vartheta_{\mu_k-1}, \mathbb{T}\vartheta_{\varrho_k-1})\right) + l]^{\alpha(\vartheta_{\mu_k-1}, \mathbb{T}\vartheta_{\mu_k-1})\beta(\vartheta_{\varrho_k-1}, \mathbb{T}\vartheta_{\varrho_k-1})} \\
&\leq g(\psi(\mathbf{M}(\vartheta_{\mu_k-1}, \vartheta_{\varrho_k-1})))\psi\left(\mathbf{M}(\vartheta_{\mu_k-1}, \vartheta_{\varrho_k-1})\right) + l. \qquad (2.23)
\end{aligned}$$

Taking the upper limit in (2.23), using (2.14) and (2.22), we get

$$\begin{aligned}
\psi(\varepsilon) + l &\leq (\psi(\varepsilon) + l)^{(qs^p)^2} \\
&\leq \left[\psi\left(\lim_{k\to\infty} sup\ \delta(\vartheta_{\mu_k}, \vartheta_{\varrho_k})\right) + l\right]^{(qs^p)^2} \\
&\leq \lim_{k\to\infty} sup\ g(\psi(\mathbf{M}(\vartheta_{\mu_k-1}, \vartheta_{\varrho_k-1})))\psi\left(\lim_{k\to\infty} sup\ \mathbf{M}(\vartheta_{\mu_k-1}, \vartheta_{\varrho_k-1})\right) + l \\
&\leq \lim_{k\to\infty} sup\ g(\psi(\mathbf{M}(\vartheta_{\mu_k-1}, \vartheta_{\varrho_k-1})))\psi(\varepsilon s) + l \\
&< \psi(\varepsilon) + l.
\end{aligned}$$

a contradiction. Hence, $\{x_\varrho\}$ is a Cauchy . Since $\mathbb{H}$ is complete, $\exists\ \eta \in \mathbb{H}$ s.t. $\vartheta_\varrho \to \eta$. Since $\mathbb{T}$ is continuous, $\eta = \lim_{\varrho\to\infty} \vartheta_{\varrho+1} = \lim_{\varrho\to\infty} \mathbb{T}\vartheta_\varrho = \mathbb{T} \lim_{\varrho\to\infty} \vartheta_\varrho = \mathbb{T}\eta$.
As $\mathbb{H}$ is $(\alpha_{qs^p}, \beta_{qs^p})-$regular, $\exists\ \{\vartheta_{\varrho_k}\}$ of $\{\vartheta_\varrho\}$ s.t. $\alpha(\vartheta_{\varrho_k-1}, \vartheta_{\varrho_k}) \geq qs^p$ and $\beta(\vartheta_{\varrho_k-1}, \vartheta_{\varrho_k}) \geq qs^p\ \forall\ k \in \mathbb{N}$ and $\alpha(\eta, \mathbb{T}\eta) \geq qs^p$ and $\beta(\eta, \mathbb{T}\eta) \geq qs^p$. From

(2.21) with $\vartheta = \vartheta_{\varrho_k}$ and $y = \eta$, we get

$$\begin{aligned}
\psi(\delta(\vartheta_{\varrho_k+1}, \mathbb{T}\eta)) + l &= \psi(\delta(\mathbb{T}\vartheta_{\varrho_k}, \mathbb{T}\eta)) + l \\
&\leq [\psi(\delta(\mathbb{T}\vartheta_{\varrho_k}, \mathbb{T}\eta)) + l]^{\alpha(\vartheta_{\varrho_k}, \mathbb{T}\vartheta_{\varrho_k})\beta(\eta, \mathbb{T}\eta)} \\
&\leq g(\psi(\mathbf{M}(\vartheta_{\varrho_k}, \eta)))\psi(\mathbf{M}(\vartheta_{\varrho_k}, \eta)) + l, \qquad (2.24)
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{M}(\vartheta_{\varrho_k}, \eta) &= \frac{1}{s}\max\Bigg\{\delta(\vartheta_{\varrho_k}, \eta), \delta(\vartheta_{\varrho_k}, \mathbb{T}\vartheta_{\varrho_k}), \delta(\eta, \mathbb{T}\eta), \frac{\delta(\vartheta_{\varrho_k}, \mathbb{T}\vartheta_{\varrho_k})\delta(\eta, \mathbb{T}\eta)}{1 + \delta(\vartheta_{\varrho_k}, \eta)}, \\
&\quad \frac{\delta(\vartheta_{\varrho_k}, \mathbb{T}\vartheta_{\varrho_k})\delta(\eta, \mathbb{T}\eta)}{1 + \delta(\mathbb{T}\vartheta_{\varrho_k}, \mathbb{T}\eta)}\Bigg\} \\
&= \frac{1}{s}\max\Bigg\{\delta(\vartheta_{\varrho_k}, \eta), \delta(\vartheta_{\varrho_k}, \vartheta_{\varrho_k+1}), \delta(\eta, \mathbb{T}\eta), \frac{\delta(\vartheta_{\varrho_k}, \vartheta_{\varrho_k+1})\delta(\eta, \mathbb{T}\eta)}{1 + \delta(\vartheta_{\varrho_k}, \eta)}, \\
&\quad \frac{\delta(\vartheta_{\varrho_k}, \vartheta_{\varrho_k+1})\delta(\eta, \mathbb{T}\eta)}{1 + \delta(\vartheta_{\varrho_k+1}, \mathbb{T}\eta)}\Bigg\}. \qquad (2.25)
\end{aligned}$$

Therefore,

$$\psi(\delta(\vartheta_{\varrho_k+1}, \mathbb{T}\eta)) \leq g(\psi(\mathbf{M}(\vartheta_{\varrho_k}, \eta)))\psi(\mathbf{M}(\vartheta_{\varrho_k}, \eta)). \qquad (2.26)$$

On taking limit $k \to \infty$ in (2.26), we have

$$\begin{aligned}
&\psi(\delta(\eta, \mathbb{T}\eta)) \leq \lim_{k\to\infty} g(\psi(\mathbf{M}(\vartheta_{\varrho_k}, \eta)))\psi(\frac{1}{s}\delta(\eta, \mathbb{T}\eta)) \\
\Rightarrow\ &\psi(\delta(\eta, \mathbb{T}\eta)) \leq \lim_{k\to\infty} g(\psi(\mathbf{M}(\vartheta_{\varrho_k}, \eta)))\psi(\delta(\eta, \mathbb{T}\eta)) \\
\Rightarrow\ &1 \leq \lim_{k\to\infty} g(\psi(\mathbf{M}(\vartheta_{\varrho_k}, \eta))) \\
\Rightarrow\ &\lim_{k\to\infty} g(\psi(\mathbf{M}(\vartheta_{\varrho_k}, \eta))) = 1.
\end{aligned}$$

Consequently, we obtain $\lim_{k\to\infty} \mathbf{M}(\vartheta_{\varrho_k}, \eta) = 0$ and hence $\delta(\eta, \mathbb{T}\eta) = 0$ i.e. $\eta = \mathbb{T}\eta$.

Suppose η, $\zeta \in Fix(\mathbb{T})$ s.t. $\eta \neq \zeta$ and $\alpha(\eta, \mathbb{T}\eta) \geq qs^p, \alpha(\zeta, \mathbb{T}\zeta) \geq qs^p$ and $\beta(\eta, \mathbb{T}\eta) \geq qs^p, \beta(\zeta, \mathbb{T}\zeta) \geq qs^p$. Now, applying (2.24), we have

$$\begin{aligned}
\psi(\delta(\eta, \zeta)) + l &= \psi(\delta(\mathbb{T}\eta, \mathbb{T}\zeta)) + l \\
&\leq [\psi(\delta(\mathbb{T}\eta, \mathbb{T}\zeta)) + l]^{\alpha(\eta, \mathbb{T}\eta)\beta(\zeta, \mathbb{T}\zeta)} \\
&\leq g(\psi(\mathbf{M}(\eta, \zeta)))\psi(\mathbf{M}(\eta, \zeta)) + l
\end{aligned}$$

where,

$$\mathbf{M}(\eta, \zeta) = \frac{1}{s}\max\left\{\delta(\eta, \zeta), \delta(\eta, \mathbb{T}\eta), \delta(\zeta, \mathbb{T}\zeta), \frac{\delta(\eta, \mathbb{T}\eta)\delta(\zeta, \mathbb{T}\zeta)}{1 + \delta(\eta, \zeta)}, \frac{\delta(\eta, \mathbb{T}\eta)\delta(\zeta, \mathbb{T}\zeta)}{1 + \delta(\mathbb{T}\eta, \mathbb{T}\zeta)}\right\}.$$

Hence,

$$\begin{aligned}
\psi(\delta(\eta, \zeta)) &\leq g(\psi(\mathbf{M}(\eta, \zeta)))\psi\left(\frac{1}{s}\delta(\eta, \zeta)\right) \\
&< \psi(\delta(\eta, \zeta)),
\end{aligned}$$

a contradiction unless $\delta(\eta, \zeta) = 0$ i.e. $\eta = \zeta$. Hence, $\mathbb{T}$ has a unique fixed point. $\square$

Corollary 2.6. *Let $(\mathbb{H}, \delta)$ be a complete b-metric space with $s \geq 1$ and $\mathbb{T}\colon \mathbb{H} \to \mathbb{H}$ s.t.*

$$\delta(\mathbb{T}\vartheta, \mathbb{T}\varphi) \leq g(\delta(\vartheta, \varphi))\delta(\vartheta, \varphi), \ \forall \ \vartheta, \varphi \in \mathbb{H} \text{ and } g \in \mathcal{G}$$

Then $\mathbb{T}$ has unique fixed point.

Remark 2.7. Taking $s = 1$ and $q = 1$ in our Theorems 2.1 *and* 2.2, we find Chandok's [7] Theorems 2.1 *and* 2.2. Also if we take $\alpha(\vartheta, \varphi) = 1$, $\beta(\vartheta, \varphi) = 1$ and $\psi(t) = t$ in our result, we find Geraghty's [11] result. So, our results generalize various results in the literature.

3. Application

Finding the solution of differential and integral equations can be difficult. However, by utilizing fixed point results, it becomes easier to find their solutions. Within this frame work, we utilize Corollary 2.6 to determine whether the following integral equation has a solution.

$$\vartheta(\ell) = f(\ell) + \int_0^1 \mathbb{K}(\ell, \hbar)\gamma(\ell, \vartheta(\hbar))d\hbar, \ \ell \in [0, 1] = J \tag{3.1}$$

where $\gamma\colon J \times \mathbb{R} \to \mathbb{R}$ and $f\colon J \to \mathbb{R}$ are continuous, $\mathbb{K}\colon J \times J \to [0, +\infty)$ is a function, and $\mathbb{K}(s, .) \in L^1(J)\ \forall\ \ell \in J$ (see, [8], [9]).
Let $\mathbb{H} = C(J, \mathbb{R}) = \{\vartheta\colon J \to \mathbb{R}, \textit{ be continuous}\}$ and $\delta\colon \mathbb{H} \times \mathbb{H} \to [0, +\infty)$ be b-metric given by

$$\delta(\vartheta, \varphi) = \left(\sup_{\ell \in J} |\vartheta(\ell) - \varphi(\ell)|\right)^r, \ \forall \ \vartheta, \varphi \in \mathbb{H}. \tag{3.2}$$

Then, $(\mathbb{H}, \delta)$ is a b-metric space with parameter $s = r \geq 1$.
Let $\mathbb{T}\colon \mathbb{H} \to \mathbb{H}$ given by

$$\mathbb{T}(\vartheta(\ell)) = f(\ell) + \int_0^1 K(\ell, \hbar)\gamma(\ell, \vartheta(\hbar))d\hbar, \ \ell \in J. \tag{3.3}$$

Now, we present a result for existence of a solution of (3.1).

Theorem 3.1. *Suppose that*
(i) $\exists\ \mu\colon \mathbb{H} \times \mathbb{H} \to [0, +\infty)\ \forall\ \ell \in J$

$$\begin{aligned} 0 &\leq |\gamma(\ell, \vartheta(\hbar)) - \gamma(\ell, \varphi(\hbar))| \\ &\leq \mu(\vartheta, \varphi)\, |\vartheta(\hbar) - \varphi(\hbar)|\,, \ \forall \ \vartheta, \varphi \in \mathbb{H} \end{aligned}$$

(ii) $\exists\ g\colon [0, +\infty) \to [0, 1)$ *with*

$$\sup_{\ell \in J} \int_0^1 \mathbb{K}(\ell, \hbar)\mu(\vartheta, \varphi)d\hbar \leq \sqrt[r]{g\left(\left(\sup_{\ell \in J} |\vartheta(\ell) - \varphi(\ell)|\right)^r\right)} \leq 1$$

Then, the integral equation given in (3.1) has a unique solution.

Proof. By (i) and (ii), we have

$$\begin{aligned}
\delta(\mathbb{T}(\vartheta(\ell)),\mathbb{T}(\varphi(\ell))) &= \left(\sup_{\ell\in J}|\mathbb{T}(\vartheta(\ell))-\mathbb{T}(\varphi(\ell))|\right)^r\\
&= \left(\sup_{\ell\in J}\left|\int_0^1\mathbb{K}(\ell,\hbar)\gamma(\ell,\vartheta(\hbar))d\hbar-\int_0^1\mathbb{K}(\ell,\hbar)\gamma(\ell,\varphi(\hbar))d\hbar\right|\right)^r\\
&= \left(\sup_{\ell\in J}\left|\int_0^1\mathbb{K}(\ell,\hbar)\left[\gamma(\ell,\vartheta(\hbar))-\gamma(\ell,\varphi(\hbar))\right]d\hbar\right|\right)^r\\
&\leq \left(\sup_{\ell\in J}\int_0^1\mathbb{K}(\ell,\hbar)\left|\gamma(\ell,\vartheta(\hbar))-\gamma(\ell,\varphi(\hbar))\right|d\hbar\right)^r\\
&\leq \left(\sup_{\ell\in J}\int_0^1\mathbb{K}(\ell,\hbar)\mu(\vartheta,\varphi)\left|\vartheta(\hbar)-\varphi(\hbar)\right|d\hbar\right)^r\\
&\leq \delta(\vartheta,\varphi)g(\delta(\vartheta,\varphi))
\end{aligned}$$

Thus, $\delta(\mathbb{T}\vartheta,\mathbb{T}\varphi)\leq g(\delta(\vartheta,\varphi))\delta(\vartheta,\varphi)$, $\forall\ \vartheta,\varphi\in\mathbb{H}$. This shows that $\mathbb{T}$ satisfied Corollary 2.6. Therefore, $\mathbb{T}$ has unique fixed point i.e. the integral (3.1) has a unique solution. □

We now present a supporting example of Theorem 3.1.

Example 3.2. Consider the following integral equation

$$\vartheta(\ell)=\frac{\ell^2}{3+\ell^5}+\frac{1}{8}\int_0^1\frac{\hbar cos3\hbar}{5(3+\ell)}\frac{|\ell|}{1+|\ell|}d\hbar \tag{3.4}$$

This equation is obtained from (3.1) by choosing

$$f(\ell)=\frac{\ell^2}{3+\ell^5},\ \mathbb{K}(\ell,\hbar)=\frac{\hbar}{(3+\ell)},\ and\ \gamma(\ell,\vartheta(\hbar))=\frac{|\ell|\,cos3\ell}{5(1+|\ell|)}.$$

Let $\mathbb{T}\colon\mathbb{H}\to\mathbb{H}$ given by

$$\mathbb{T}(\vartheta(\ell))=f(\ell)+\int_0^1\mathbb{K}(\ell,\hbar)\gamma(\ell,\vartheta(\hbar))d\hbar,\ \ell\in J. \tag{3.5}$$

Suppose $\mu(\vartheta,\varphi)=\frac{1}{5}$, we get

$$\begin{aligned}
|\gamma(\ell,\vartheta(\hbar)-\gamma(\ell,\varphi(\hbar)| &= \left|\frac{|\vartheta|\,cos3\hbar}{5(1+|\vartheta|)}-\frac{|\varphi|\,cos3\hbar}{5(1+|\varphi|)}\right|\\
&\leq \left|\frac{cos3\hbar}{5}\right|\left|\frac{|\vartheta|}{1+|\vartheta|}-\frac{|\varphi|}{1+|\varphi|}\right|\\
&\leq \frac{1}{5}\left|\frac{|\vartheta|+|\vartheta|\,|\varphi|-|\varphi|-|\vartheta|\,|\varphi|}{(1+|\vartheta|)(1+|\varphi|)}\right|\\
&\leq \frac{1}{5}\left|\frac{|\vartheta|-|\varphi|}{(1+|\vartheta|)(1+|\varphi|)}\right|\\
&\leq \frac{1}{5}\left|\vartheta-\varphi\right|\\
&= \mu(\vartheta,\varphi)\left|\vartheta(\hbar)-\varphi(\hbar)\right|.
\end{aligned}$$

And,

$$\begin{aligned}\sup_{\ell\in I}\int_0^1 \mathbb{K}(\ell,\hbar)\mu(\vartheta,\varphi)d\hbar &= \sup_{\ell\in J}\int_0^1 \frac{\hbar}{3+\ell}\times\frac{1}{5}d\hbar\\ &= \sup_{\ell\in I}\left(\frac{1}{10(3+\ell)}\right)\\ &= \frac{1}{30}<1.\end{aligned}$$

Hence, all the conditions of Theorem 3.1 are satisfied. Therefore, the integral equation (3.4) has exactly one solution in $C(J,\mathbb{R})$.

4. Conclusions

In this paper, we introduced a new contraction, which is called the $(\alpha_{qs^p},\beta_{qs^p})-\psi$-Geraghty rational contraction in b-metric space which extends the classical Banach and Geraghty contractions . Utilizing this $(\alpha_{qs^p},\beta_{qs^p})-\psi$-Geraghty rational contraction, we investigated the existence of unique fixed point in b-metric spaces. An example is provided to illustrate the validity of our findings. Further, we applied the fixed point results to prove the existence and uniqueness of solutions for a certain class of Fredholm integral equations. Our works extended and generalized numerous results already present in the literature. Also, we believe our results have built a strong analytical relation among fixed point theory and nonlinear analysis. In future work, our newly obtained contraction may be used to examine the existence of fixed points in various spaces.

Acknowledgement. The first author sincerely thanks Manipur University, Canchipur, for its research facilities and support.

References

1. Alam, K. H., Rohen, Y. & Saleem, N. (2023) Fixed points of (α,β,F^*) and (α,β,F^{**}) Weak Geraghty contractions with an Application. Symmetry, 15, 243. https://doi.org/10.3390/sym15010243
2. Alam, K. H., Rohen, Y. & Tomar, A. (2024). (α,F)-Geraghty-type generalized F-contractions on non-Archimedean fuzzy metric-unlike spaces. Demonstratio Mathematica, 57: 20240046. https://doi.org/10.1515/dema-2024-0046
3. Bakhtin, I. A. (1989). The contraction mapping principle in quasimetric spaces. Functional Analysis, Ulyanovsk State Pedagogical Institute, 30, 26-37.
4. Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations integrals. Fundamenta Mathematicae, 3, 133–181. DOI:10.4064/fm-3-1-133-181
5. Chand, D. & Rohen, Y. (2023). Fixed points of $\alpha s-\beta s-\psi$-contraction mappings in S-metric spaces. Nonlinear Functional Analysis and Application, 28(2),571-587. https://doi.org/10.22771/nfaa.2023.28.02.1
6. Chand, D., Rohen, Y. & Fabiano, N. (2023). Paired-Kannan contraction mappings and fixed point results. Gulf Journal of Mathematics, 14(1), 1-13. https://doi.org/10.56947/gjom.v14i1.000
7. Chandok, S. (2015). Some fixed point theorems for (α,β)-admissible Geraghty type contractive mappings and related results. Mathematical Sciences, 9, 127-135. DOI10.1007/s40096-015-0159-4

8. Chidume, C. E., Adamu, A., Minjibir, M. S.& Nnyaba, U. V. (2020). On the strong convergence of the proximal point algorithm with an application to Hammerstein equations. Journal of Fixed Point Theory and Applications, 22:61 . https://doi.org/10.1007/s11784-020-00793-6
9. Chidume, C. E., Adamu, A. & Nnakwe, M. O.(2021). An Inertial Algorithm for Solving Hammerstein Equations. Symmetry, 13, 376. .https://doi.org/10.3390/sym13030376
10. Czerwik, S. (1993). Contraction mappings in b-metric spaces. Acta Mathematica et Informatica Universitatis Ostraviensis, 1, 5-11. http://dml.cz/dmlcz/120469
11. Geraghty, M. (1973) On contractive mappings. Proceedings of the American Mathematical Society, 40, 604-608. https://doi.org/10.2307/2039421
12. Khan, M. S., Priyobarta, N., & Rohen, Y. (2019). Fixed point of generalised rational $\alpha^* - \psi$-Geraghty contraction for multivalued mappings. Journal of Advanced Mathematical Studies, 12, 156-169. http://journal.fairpartners.ro
13. Monika, T. & Rohen, Y. (2026). Fixed point results for interpolative F-contraction in S-metric spaces. Boletim da Sociedade Paranaense de Matematica, 44(2), 1-11. doi:10.5269/bspm.79260
14. Pradeep, M., Panday, S. & Rohen, Y. (2026). Fixed points of α_s- interpolative contractions in S-metric spaces. Boletim da Sociedade Paranaense de Matematica, 44(2), 1-10. doi:10.5269/bspm.79259
15. Priyobarta, N., Khomdram, B., Rohen, Y. & Saleem, N. (2021). On generalized rational α-Geraghty contraction mapping in G-metric spaces. Journal of Mathematics, 2021, Article ID 6661045. https://doi.org/10.1155/2021/6661045
16. Priyobarta, N., Rohen, Y., Stephen, T. & Stojan, R. (2022). Some remarks on α-admissibility in S-metric spaces. Journal of Inequalities and Applications, 2022:34. https://doi.org/10.1186/s13660-022-02767-3
17. Rohen, Y., Guru, D. Surndra, S. & Saleem, N. (2025). Fixed point and common fixed point theorems for Pata-Type contractions in b-metric spaces. Journal of Nonlinear and Convex Analysis, 26(8), 2563-2582. https://www.scopus.com/pages/publications/105032798729
18. Samet, B., Vetro, P., & Vetro,C. (2012). Fixed point theorems for α–ψ-contractive type mappings. Nonlinear Analysis: Theory, Methods & Applications, 75(4), 2154-2165. https://doi.org/10.1016/j.na.2011.10.014
19. Shanu, P. & Rohen, Y. (2021). Generalised rational αs- Meir-Keeler contraction mapping in S-metric Spaces. American Journal of Applied Mathematics and Statistics, 9(2), 48-52. DOI:10.12691/ajams-9-2-2
20. Singh, L. M. & Rohen, Y. (2026). On fixed points of $(\alpha_{qs^p}, \beta_{qs^p}) - \phi$-hybrid contraction in b-metric spaces. Applications of Fixed-Point Theorem, Springer,Singapore, 273-286. DOI:10.1007/978-981-95-5160-6_16
21. Zoto, K., Rhoades, B. E., & Radenovic, S. (2017). Some generalizations for $(\alpha$-$\psi, \phi)$-contractions in b-metric-like spaces and an application. Fixed Point Theory and Applications, 2017:26. DOI10.1186/s13663-017-0620-1

[1] Department of Mathematics, Manipur University, Canchipur-795003, Imphal, Manipur, India

Email address: meleiloitongbam@gmai.com

Email address: melei.phd.math@manipuruniv.ac.in

[2] Department of Mathematics, Manipur University, Canchipur-795003, Imphal, Manipur, India

Email address: ymnehor2008@yahoo.com

Email address: yumnam.rohen@manipuruniv.ac.in

[3] Department of Mathematics, Jadonang Memorial College, Noney, Manipur, India
Email address: btsalun29@gmail.com

[2] Department of Mathematics, Dhanamanjuri University, Imphal-795001, Manipur, India
Email address: ningthoujampriyo9@gmail.com