

KERNEL-SPECIFIC STRUCTURES IN NON-SINGULAR FRACTIONAL DERIVATIVES

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ABSTRACT. This paper develops a unified analytical framework for evaluating a broad class of recent non-singular fractional derivatives acting on generalized Mittag–Leffler functions. Using the four-parameter Mittag–Leffler function as a representative function, kernel-specific representations are derived for different non-singular fractional operators. The proposed formulation expresses each operator through an associated kernel-specific function, enabling direct structural comparison. Kernel-specific bounds, a general uniform convergence theorem, and corresponding Laplace-transform representations are established. Numerical illustrations highlight the influence of different kernels on the resulting fractional derivatives and support the theoretical results.

Keywords. Fractional calculus, non-singular fractional operators, Mittag–Leffler functions, unified series representation, kernel-specific functions, Laplace transform.

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1. INTRODUCTION

The theory of fractional calculus has attracted much attention in recent years due to its ability to capture memory and hereditary properties inherent in a wide range of complex phenomena, including viscoelastic materials, anomalous diffusion, bioengineering applications, and econophysics [21, 24]. At the center of the analytical models in fractional calculus is the Mittag–Leffler function, a generalization of the exponential function, and is often seen to emerge as a solution of fractional differential equations (FDEs) [11, 9]. Various generalizations of the classical Mittag–Leffler function have been developed over the years in an attempt to improve modeling flexibility, such as the extension via the extended Beta function [15]. Recent studies have also established Riemann–Liouville fractional calculus results for generalized Mittag–Leffler functions, including the seven-parameter Mittag–Leffler function [19]. Among these, the four-parameter Mittag–Leffler (4P–ML) function introduced by Shukla and Prajapati [23] provides a practically useful level of generality, subsuming the classical one-, two-, and three-parameter Mittag–Leffler functions [27, 17] as special cases.

The 4P–ML function belongs to the broader family of Fox H-functions and Wright-type hypergeometric functions [23, 4, 12, 18], which have been studied

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extensively in the context of classical fractional operators. Analytical representations for fractional operators acting on generalized special functions have been extensively investigated, particularly for singular kernel operators of Riemann–Liouville and Caputo type [25, 6]. However, these frameworks are primarily oriented toward singular kernel operators of Riemann–Liouville and Caputo type, whose structural properties differ fundamentally from those of non-singular kernel operators.

Various non-singular fractional derivatives such as Atangana–Baleanu–Caputo (ABC) [3], Mittag–Leffler Caputo–Fabrizio (MLCF) [2] and others have been proposed concurrently with the development of generalized special functions, yet their analytical study has largely proceeded in a fragmented manner, focusing either on a specific operator or a particular function class. Although each operator has a distinct kernel and treatment of memory effects, their analytical response when applied to the same class of functions tends to differ in form, making direct comparison or unified analysis challenging.

In light of these challenges, it becomes essential to devise a unifying framework that can systematically incorporate generalized Mittag–Leffler functions together with diverse fractional operators. Such a framework would bridge the gap between isolated developments and provide a consistent basis for analytical investigation of non-singular fractional operators. In particular, a unified series-based treatment of non-singular kernel operators acting on the four-parameter Mittag–Leffler function has not been systematically developed. The choice of the 4P–ML function is motivated by its hierarchical structure, which encompasses the classical one-, two-, and three-parameter Mittag–Leffler functions while remaining analytically tractable. Although an extension to the broader Fox H-function class is theoretically possible, it would introduce substantial parametric complexity beyond the scope of the present work.

To address this gap, the present work develops the Unified Mittag–Leffler (UML) Series, an operator-independent analytical framework in which different non-singular fractional operators are represented through kernel-specific functions. This representation separates the operator kernel from the underlying Mittag–Leffler structure, enabling systematic comparison of kernel-dependent effects within a common mathematical setting. The resulting kernel-specific functions $\Phi_n(t)$ explicitly characterize the influence of different memory kernels and provide a unified basis for analytical comparison across a broad class of non-singular fractional operators. Although the unified formulation is developed explicitly for the four-parameter Mittag–Leffler function, its applicability is not restricted to a single special function, since a wide range of classical and generalized Mittag–Leffler functions arise as special cases of the 4P–ML function through parameter reduction.

The structure of this paper is as follows. In Sec. 2, we provide the necessary preliminaries, including the definition of the four-parameter Mittag–Leffler function $E_{\alpha,\beta}^{\gamma,\delta}(z)$ and the considered fractional derivatives. In Sec. 3, we present the main theorems that derive explicit expressions for the fractional derivatives under each operator. Sec. 4 introduces the Unified Mittag–Leffler (UML) Series and the

Unified Mittag–Leffler Integral (UML–Integral) Series, develops the associated convergence theory through kernel-specific bounds and a general uniform convergence theorem, and establishes a unified representation for the derived fractional derivatives. Sec. 5 provides a Laplace transform analysis of the derived expressions via the UML formulations, yielding closed-form results for each operator. Sec. 6 presents numerical illustrations, including graphical comparisons of unified fractional derivatives across different operators and an investigation of the numerical behavior of the kernel-specific functions appearing in the unified expressions. Sec. 7 discusses applications and implications of the unified expression. Finally, Sec. 8 concludes the paper and outlines potential directions for future research.

2. PRELIMINARIES

Definition 2.1. For $\alpha, \beta, \rho \in \mathbb{C}$ and $q \in (0, 1) \cup \mathbb{N}$, the Four-Parameter Mittag–Leffler Function (4P–ML) is defined as

$$E_{\alpha, \beta}^{\rho, q}(z) = \sum_{n=0}^{\infty} \frac{(\rho)_{qn} z^n}{\Gamma(\alpha n + \beta) n!}, \quad (2.1)$$

where $\operatorname{Re}(\alpha) > 0$, $\operatorname{Re}(\rho) > 0$, and $(\rho)_{qn}$ denotes the generalized Pochhammer symbol. This function includes several classical Mittag–Leffler functions as special cases [23].

Definition 2.2. The Atangana–Baleanu–Caputo (ABC) fractional derivative [3] of order $\nu \in (0, 1)$ is defined as

$${}^{ABC}D_t^\nu u(t) = \frac{B(\nu)}{1-\nu} \int_0^t E_\nu \left(-\frac{\nu}{1-\nu} (t-s)^\nu \right) u'(s) ds, \quad (2.2)$$

where $E_\nu(z)$ is the one-parameter Mittag–Leffler function.

Definition 2.3. The Mittag–Leffler Caputo–Fabrizio (MLCF) fractional derivative [2] for $\nu \in (0, 1)$ and $0 < \eta \leq 1$ is defined as

$${}^{MLCF}\mathcal{D}_t^{\nu, \eta} u(t) = \frac{M(\nu)}{1-\nu} \cdot \left[E_\eta \left(\frac{\nu}{1-\nu} t^\eta \right) \right]^{-1} \cdot \int_0^t E_\eta \left(\frac{\nu}{1-\nu} s^\eta \right) u'(s) ds, \quad (2.3)$$

where $E_\eta(z)$ is the one-parameter Mittag–Leffler function and $M(\nu)$ is a normalization constant satisfying $M(0) = M(1) = 1$.

Definition 2.4. The Hattaf-type Generalized fractional derivative [8] in Caputo sense of order $\sigma \in [0, 1)$, with parameters $\theta, \gamma > 0$, for a function $u(t) \in H^1(0, b)$, is defined as

$${}^H D_{0, t, w}^{\sigma, \theta, \gamma} u(t) = \frac{N(\sigma)}{1-\sigma} \cdot \frac{1}{w(t)} \int_0^t E_\theta [-\mu_\sigma (t-x)^\gamma] \frac{d}{dx} (w(x)u(x)) dx, \quad (2.4)$$

where $\mu_\sigma = \frac{\sigma}{1-\sigma}$, $w(t) \in C^1(0, b)$ with $w(t), w'(t) > 0$, and $N(\alpha)$ is a normalization function.

Definition 2.5. Let $0 < \mu \leq 1$, $u(t) \in H^1((0, b))$, and $t > 0$. The sigmoidal fractional derivative [20] of a function $u(t)$ is defined by

$$\sigma D^\mu u(t) = C_1(\mu) \int_0^t \operatorname{sech}^2\left(\frac{s-t}{1-\mu}\right) u'(s) ds, \quad (2.5)$$

where $C_1(\mu)$ is a normalization constant.

Definition 2.6. The Mutaz–Saadaoui derivative [13] of order $0 < \mu \leq 1$ is defined as

$$D_t^\mu u(t) = \frac{t^{1-\mu}}{\mu} u'(t), \quad t > 0. \quad (2.6)$$

This operator reduces to the classical derivative as $\mu \rightarrow 1$ and acts as a rescaled first derivative for $\mu < 1$. Although it is a local fractional derivative, it is included to illustrate that the proposed unified series representation may also extend to local fractional derivatives.

3. MAIN RESULTS

In this section, we derive the main analytical results for the fractional derivatives of the 4P–ML function under different fractional operators. These results are primarily illustrative, while the underlying structural insights are consolidated in Section 4.

Theorem 3.1. Let $u(t) = t^{\beta-1} E_{\alpha,\beta}^{\rho,q}(\lambda t^\alpha)$. Then, under the Atangana–Baleanu–Caputo (ABC) fractional derivative of order $\nu \in (0, 1)$, we have the exact representation

$${}^{ABC}D_t^\nu u(t) = t^{\beta-1} \sum_{n=0}^{\infty} \frac{(\rho)_{qn} \lambda^n t^{\alpha n}}{n!} E_{\nu, \alpha n + \beta} \left(-\frac{\nu}{1-\nu} t^\nu \right). \quad (3.1)$$

Proof. Let $u(t) = t^{\beta-1} E_{\alpha,\beta}^{\rho,q}(\lambda t^\alpha)$. Then,

$$\frac{d}{dt} u(t) = \sum_{n=0}^{\infty} \frac{(\rho)_{qn} \lambda^n}{\Gamma(\alpha n + \beta) n!} \cdot \frac{d}{dt} (t^{\alpha n + \beta - 1}) = \sum_{n=0}^{\infty} \frac{(\rho)_{qn} \lambda^n (\alpha n + \beta - 1)}{\Gamma(\alpha n + \beta) n!} t^{\alpha n + \beta - 2}. \quad (3.2)$$

Substituting Eq. (3.2) into the definition of the ABC derivative given in Eq. (2.2) and interchanging the order of summation and integration using the convergence of the four-parameter Mittag–Leffler function, we obtain

$${}^{ABC}D_t^\nu u(t) = \sum_{n=0}^{\infty} \frac{(\rho)_{qn} \lambda^n}{\Gamma(\alpha n + \beta - 1) n!} \cdot \frac{B(\nu)}{1-\nu} \int_0^t E_\nu \left(-\frac{\nu}{1-\nu} (t-s)^\nu \right) s^{\alpha n + \beta - 2} ds. \quad (3.3)$$

It is known [16] that

$$\int_0^t E_\nu(-a(t-s)^\nu) s^{\mu-1} ds = \Gamma(\mu) t^\mu E_{\nu, \mu+1}(-at^\nu), \quad (3.4)$$

for $\Re(\mu) > 0$, $\Re(\nu) > 0$, and $a > 0$. Setting $a = \frac{\nu}{1-\nu}$ and $\mu = \alpha n + \beta - 1$ in Eq. (3.4) gives

$$\int_0^t E_\nu \left(-\frac{\nu}{1-\nu} (t-s)^\nu \right) s^{\alpha n + \beta - 2} ds = \Gamma(\alpha n + \beta - 1) t^{\alpha n + \beta - 1} E_{\nu, \alpha n + \beta} \left(-\frac{\nu}{1-\nu} t^\nu \right). \quad (3.5)$$

Substituting Eq. (3.5) into Eq. (3.3) yields Eq. (3.1). $\square$

Theorem 3.1 provides an explicit fractional derivative of the 4P–ML function under the ABC operator. The complementary result for the Atangana–Baleanu Riemann–Liouville (AB–RL) derivative of the same function was established in [10]. Following a similar approach, we now derive the corresponding results for other operators.

Theorem 3.2. *If $u(t) = t^{\beta-1} E_{\alpha, \beta}^{\rho, q}(\lambda t^\alpha)$, then, under the Mittag–Leffler Caputo–Fabrizio (MLCF) fractional derivative of order $\nu \in (0, 1)$ with kernel index $\eta \in (0, 1]$, we have*

$$\begin{aligned} {}^{MLCF} \mathcal{D}_t^{\nu, \eta} u(t) &= \frac{M(\nu)}{1-\nu} \cdot [E_\eta \left(\frac{\nu}{1-\nu} t^\eta \right)]^{-1} \\ &\times \sum_{n=0}^{\infty} \frac{(\rho)_{qn} \lambda^n}{\Gamma(\alpha n + \beta - 1) n!} \sum_{k=0}^{\infty} \frac{\left(\frac{\nu}{1-\nu} \right)^k t^{\eta k + \alpha n + \beta - 1}}{\Gamma(\eta k + 1) (\eta k + \alpha n + \beta - 1)}. \end{aligned} \quad (3.6)$$

Proof. Using Eq. (3.2), we obtain

$$\int_0^t E_\eta \left(\frac{\nu}{1-\nu} s^\eta \right) u'(s) ds = \sum_{n=0}^{\infty} \frac{(\rho)_{qn} \lambda^n}{\Gamma(\alpha n + \beta - 1) n!} \int_0^t E_\eta \left(\frac{\nu}{1-\nu} s^\eta \right) s^{\alpha n + \beta - 2} ds. \quad (3.7)$$

Evaluating the integral gives

$$\int_0^t E_\eta \left(\frac{\nu}{1-\nu} s^\eta \right) s^{\alpha n + \beta - 2} ds = \sum_{k=0}^{\infty} \frac{\left(\frac{\nu}{1-\nu} \right)^k t^{\eta k + \alpha n + \beta - 1}}{\Gamma(\eta k + 1) (\eta k + \alpha n + \beta - 1)}. \quad (3.8)$$

Substituting the above expression into the definition of the MLCF derivative given in Eq. (2.3) immediately yields Eq. (3.6). $\square$

Theorem 3.3. *If $u(t) = t^{\beta-1} E_{\alpha, \beta}^{\rho, q}(\lambda t^\alpha)$, then, under the Hattaf-type fractional derivative in Caputo sense with parameters $\sigma \in [0, 1)$, $\theta, \gamma > 0$, and a weight function $w(t) = t^\delta$, the following representation holds.*

$$\begin{aligned} {}^H D_{0, t, w}^{\sigma, \theta, \gamma} u(t) &= \frac{N(\sigma)}{1-\sigma} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\rho)_{qn} \lambda^n (-\mu_\sigma)^k}{\Gamma(\alpha n + \beta) \Gamma(\theta k + 1) n!} \\ &\times \frac{\Gamma(\gamma k + 1) \Gamma(\alpha n + \beta + \delta)}{\Gamma(\gamma k + \alpha n + \beta + \delta)} t^{\alpha n + \beta - 1 + \gamma k}. \end{aligned} \quad (3.9)$$

where $\mu_\sigma = \frac{\sigma}{1-\sigma}$.

Proof. Let $w(t) = t^\delta$. Then

$$\frac{d}{dt} [w(t) u(t)] = \sum_{n=0}^{\infty} \frac{(\rho)_{qn} \lambda^n (\alpha n + \beta - 1 + \delta)}{\Gamma(\alpha n + \beta) n!} t^{\alpha n + \beta - 2 + \delta}.$$

Substituting the above expression into the definition of the Hattaf-type derivative given in Eq. (2.4), we obtain

$${}^H D_{0,t,w}^{\sigma,\theta,\gamma} u(t) = \frac{N(\sigma)}{1-\sigma} \frac{1}{t^\delta} \sum_{n=0}^{\infty} \frac{(\rho)_{qn} \lambda^n (\alpha n + \beta - 1 + \delta)}{\Gamma(\alpha n + \beta) n!} I_n, \quad (3.10)$$

where

$$I_n = \int_0^t E_\theta(-\mu_\sigma(t-x)^\gamma) x^{\alpha n + \beta - 2 + \delta} dx.$$

Expanding the Mittag–Leffler function into its defining series and interchanging the order of summation and integration, justified by convergence, give

$$I_n = \sum_{k=0}^{\infty} \frac{(-\mu_\sigma)^k}{\Gamma(\theta k + 1)} \frac{\Gamma(\gamma k + 1) \Gamma(\alpha n + \beta - 1 + \delta)}{\Gamma(\gamma k + \alpha n + \beta + \delta)} t^{\alpha n + \beta - 1 + \delta + \gamma k}.$$

Substituting this expression into Eq. (3.10) yields Eq. (3.9). $\square$

Theorem 3.4. For $u(t) = t^{\beta-1} E_{\alpha,\beta}^{\rho,q}(\lambda t^\alpha)$, the Sigmoidal fractional derivative of order $\mu \in (0, 1]$ is given by

$$\sigma D^\mu u(t) = \sum_{n=0}^{\infty} \frac{(\rho)_{qn} \lambda^n}{\Gamma(\alpha n + \beta - 1) n!} \cdot \int_0^t C_1(\mu) \cdot \operatorname{sech}^2\left(\frac{s-t}{1-\mu}\right) s^{\alpha n + \beta - 2} ds. \quad (3.11)$$

Proof. Substituting Eq. (3.2) into the definition of the Sigmoidal fractional derivative given in Eq. (2.5) immediately yields Eq. (3.11). $\square$

The explicit integral representation obtained in Theorem 3.4 also enables us to derive useful estimates. By bounding the kernel of the Sigmoidal operator, we arrive at the following inequality.

Corollary 3.5. Let $\alpha, \beta, \rho, q, \lambda \in (0, \infty)$ and $u(t) = t^{\beta-1} E_{\alpha,\beta}^{\rho,q}(\lambda t^\alpha)$. Then, for $\mu \in (0, 1]$, the Sigmoidal fractional derivative satisfies the upper bound:

$$\sigma D^\mu u(t) \leq C_1(\mu) \cdot t^{\beta-1} \cdot E_{\alpha,\beta}^{\rho,q}(\lambda t^\alpha). \quad (3.12)$$

Proof. Using the inequality $\operatorname{sech}^2\left(\frac{s-t}{1-\mu}\right) \leq 1$ in Eq. (3.11) yields Eq. (3.12). $\square$

Corollary 3.5 extends the technique used in [20], where a similar upper bound was derived for the two-parameter Mittag–Leffler function.

Theorem 3.6. For $u(t) = t^{\beta-1} E_{\alpha,\beta}^{\rho,q}(\lambda t^\alpha)$, the Mutaz–Saadaoui derivative of order $\mu \in (0, 1]$ and $t > 0$ is given by

$$D_t^\mu u(t) = t^{\beta-1} \sum_{n=0}^{\infty} \frac{(\rho)_{qn} \lambda^n t^{\alpha n}}{n!} \cdot \left(\frac{1}{\mu t^\mu} \cdot \frac{1}{\Gamma(\alpha n + \beta - 1)} \right). \quad (3.13)$$

Proof. Substituting Eq. (3.2) into the definition of the Mutaz–Saadaoui derivative given in Eq. (2.6) immediately yields Eq. (3.13). $\square$

The results obtained for the various fractional derivatives considered in this study can be interpreted within a common framework. In the next section, we unify these results by presenting them in a single generalized form, which not only streamlines the analysis but also highlights the underlying connections among the different operators.

4. UNIFIED REPRESENTATION

In this section, we present two general formulations for computing fractional derivatives of the 4P–ML function under various operators with non-singular kernels.

Form 1 (Unified Mittag–Leffler Integral Series). *Let $u(t) = t^{\beta-1} E_{\alpha,\beta}^{\rho,q}(\lambda t^\alpha)$ be defined in terms of the four-parameter Mittag–Leffler function. Then, for a broad class of non-singular fractional operators $\mathcal{D}^*$, each associated with a kernel $\mathcal{K}(t, s)$, the derivative can be expressed in the Unified Mittag–Leffler Integral Series representation:*

$$\mathcal{D}^* u(t) = \sum_{n=0}^{\infty} \frac{(\rho)_{qn} \lambda^n}{\Gamma(\alpha n + \beta - 1) n!} \int_0^t \mathcal{K}(t, s) s^{\alpha n + \beta - 2} ds, \quad (4.1)$$

where $\mathcal{K}(t, s)$ corresponds to the following kernels:

Atangana–Baleanu–Caputo ${}^{ABC}D_t^\nu u(t)$:

$$\mathcal{K}(t, s) = \frac{B(\nu)}{1 - \nu} E_\nu \left(-\frac{\nu}{1 - \nu} (t - s)^\nu \right). \quad (4.2)$$

MLCF ${}^{MLCF}D_t^{\nu,\eta} u(t)$:

$$\mathcal{K}(t, s) = \frac{M(\nu)}{1 - \nu} \frac{E_\eta \left(\frac{\nu}{1 - \nu} s^\eta \right)}{E_\eta \left(\frac{\nu}{1 - \nu} t^\eta \right)}. \quad (4.3)$$

Hattaf-type ${}^H D_{0,t}^{\sigma,\theta,\gamma} u(t)$:

$$\mathcal{K}(t, s) = \frac{N(\sigma)}{1 - \sigma} \cdot \frac{(\alpha n + \beta - 1 + \delta)}{(\alpha n + \beta - 1)} E_\theta (-\mu_\sigma (t - s)^\gamma) t^{-\delta} s^\delta. \quad (4.4)$$

Sigmoidal $\sigma D^\mu u(t)$:

$$\mathcal{K}(t, s) = C_1(\mu) \cdot \operatorname{sech}^2 \left(\frac{s-t}{1-\mu} \right). \quad (4.5)$$

Mutaz–Saadaoui $D^\mu u(t)$:

$$\mathcal{K}(t, s) = (\alpha n + \beta - 1) \cdot \frac{1}{\mu t^\mu}. \quad (4.6)$$

The proofs of the above representations follow directly from the definitions given in Sec. 2.

Form 2 (Unified Mittag–Leffler Series). *Let $u(t) = t^{\beta-1} E_{\alpha,\beta}^{\rho,q}(\lambda t^\alpha)$ be a function defined in terms of the 4P–ML function. Then, for a wide class of non-singular*

fractional derivatives $\mathcal{D}^\alpha$, the derivative admits a Unified Mittag–Leffler Series representation:

$$\mathcal{D}^\alpha u(t) = t^{\beta-1} \sum_{n=0}^{\infty} \frac{(\rho)_{qn} \lambda^n t^{\alpha n}}{n!} \cdot \Phi_n(t), \quad (4.7)$$

where $\Phi_n(t)$ (kernel-specific function) encapsulates the kernel-specific contribution of the fractional operator. Representative forms include:

Atangana–Baleanu–Caputo ${}^{ABC}D_t^\nu u(t)$:

$$\Phi_n(t) = E_{\nu, \alpha n + \beta} \left(-\frac{\nu}{1-\nu} t^\nu \right). \quad (4.8)$$

Derived from Eq. (3.1).

MLCF ${}^{MLCF}D_t^{\nu, \eta} u(t)$:

$$\Phi_n(t) = \frac{M(\nu)}{1-\nu} \cdot \frac{1}{E_\eta \left(\frac{\nu}{1-\nu} t^\eta \right)} \sum_{k=0}^{\infty} \frac{\left(\frac{\nu}{1-\nu} \right)^k \cdot t^{\eta k}}{\Gamma(\alpha n + \beta - 1) \Gamma(\eta k + 1) (\eta k + \alpha n + \beta - 1)}. \quad (4.9)$$

Derived from Eq. (3.6).

Hattaf-type ${}^H D_{0,t}^{\sigma, \theta, \gamma} u(t)$:

$$\Phi_n(t) = \frac{N(\sigma)}{1-\sigma} \sum_{k=0}^{\infty} \frac{(-\mu_\sigma)^k}{\Gamma(\alpha n + \beta) \Gamma(\theta k + 1)} \cdot \frac{\Gamma(\gamma k + 1) \Gamma(\alpha n + \beta + \delta)}{\Gamma(\gamma k + \alpha n + \beta + \delta)} \cdot t^{\gamma k}. \quad (4.10)$$

Derived from Eq. (3.9).

Sigmoidal $\sigma D^\mu u(t)$: admits an upper-bound kernel estimate

$$\Phi_n(t) \lesssim \frac{C_1(\mu)}{\Gamma(\alpha n + \beta)}. \quad (4.11)$$

Derived from Eq. (3.12).

Mutaz–Saadaoui $D^\mu u(t)$:

$$\Phi_n(t) = \frac{1}{\mu t^\mu} \cdot \frac{1}{\Gamma(\alpha n + \beta - 1)}. \quad (4.12)$$

Derived from Eq. (3.13).

The kernel-based series representation for the 4P–ML function naturally includes several important special cases through parameter reduction. Setting $q = 1$ gives the three-parameter Mittag–Leffler (3P–ML), or Prabhakar function [17], while additionally setting $\rho = 1$ gives the two-parameter Mittag–Leffler (2P–ML), or Wiman function [27]. Finally, setting $\beta = 1$ recovers the classical one-parameter Mittag–Leffler function $E_\alpha(t^\alpha)$. Thus, the proposed series provides a unified representation for a broad class of Mittag–Leffler functions and their associated fractional operators, while also facilitating the analysis of various integral transforms.

The following propositions establish that the kernel-specific functions associated with the fractional operators considered in this work satisfy the uniform bounds required in the subsequent convergence theorem.

Proposition 4.1. *Let $\alpha, \beta, \rho, q, \lambda \in (0, \infty)$, $\nu \in (0, 1)$, and $\beta \geq \nu$. Then the kernel-specific function for the Atangana–Baleanu–Caputo operator satisfies*

$$\Phi_n(t) = E_{\nu, \alpha n + \beta} \left(-\frac{\nu}{1 - \nu} t^\nu \right) \leq \frac{1}{\Gamma(\alpha n + \beta)}, \quad \forall n \geq 0, t \geq 0. \quad (4.13)$$

Proof. Since $\alpha > 0$, we have $\alpha n \geq 0$ for every $n \geq 0$, and hence $\alpha n + \beta \geq \beta \geq \nu$ under the hypothesis $\beta \geq \nu$. Also, since $0 < \nu < 1$, we have $\frac{\nu}{1 - \nu} > 0$, so $z = \frac{\nu}{1 - \nu} t^\nu \geq 0$ for every $t \geq 0$. Consequently, the function $z \mapsto E_{\nu, \alpha n + \beta}(-z)$ is completely monotone on $[0, \infty)$ [22, 1]. In particular, it is non-increasing, so

$$E_{\nu, \alpha n + \beta}(-z) \leq E_{\nu, \alpha n + \beta}(0) = \frac{1}{\Gamma(\alpha n + \beta)}, \quad z \geq 0, \quad (4.14)$$

which is the required bound, uniformly in n . $\square$

Proposition 4.2. *Let $\alpha, \beta, \rho, q, \lambda \in (0, \infty)$, $\nu \in (0, 1)$, $\eta \in (0, 1]$, and $\beta > 1$. Then the kernel-specific function for the Mittag–Leffler Caputo–Fabrizio operator satisfies*

$$\Phi_n(t) \leq \frac{M(\nu)}{1 - \nu} \cdot \frac{1}{\Gamma(\alpha n + \beta)}, \quad \forall n \geq 0, t \geq 0. \quad (4.15)$$

Proof. Under the given parameter ranges, $a = \frac{\nu}{1 - \nu} > 0$, and $\eta k + \alpha n + \beta - 1 \geq \alpha n + \beta - 1 > 0$ for every $k \geq 0$ and $n \geq 0$. It follows that

$$\sum_{k=0}^{\infty} \frac{a^k t^{\eta k}}{\Gamma(\eta k + 1) (\eta k + \alpha n + \beta - 1)} \leq \frac{1}{\alpha n + \beta - 1} \sum_{k=0}^{\infty} \frac{a^k t^{\eta k}}{\Gamma(\eta k + 1)} = \frac{E_\eta(at^\eta)}{\alpha n + \beta - 1}. \quad (4.16)$$

Substituting this into Eq. (4.9) and simplifying yields Eq. (4.15). $\square$

Proposition 4.3. *Let $\alpha, \beta, \rho, q, \lambda, \delta \in (0, \infty)$, $\sigma \in [0, 1)$, $\theta, \gamma > 0$, with $\theta > \gamma$ and $\beta + \delta \geq x_0$, where x_0 is the point at which Γ attains its minimum on $(0, \infty)$. Then, for every $T > 0$, there exists $G_T < \infty$ such that the kernel-specific function for the Hattaf-type operator satisfies*

$$|\Phi_n(t)| \leq \frac{N(\sigma)}{1 - \sigma} \cdot \frac{G_T}{\Gamma(\alpha n + \beta)}, \quad \forall n \geq 0, t \in [0, T]. \quad (4.17)$$

Proof. Write $\mu = \alpha n + \beta$ and

$$P_n(t) = \sum_{k=0}^{\infty} (-\mu_\sigma)^k C_{n,k} t^{\gamma k}, \quad C_{n,k} = \frac{\Gamma(\gamma k + 1) \Gamma(\mu + \delta)}{\Gamma(\theta k + 1) \Gamma(\gamma k + \mu + \delta)}, \quad (4.18)$$

so that $\Phi_n(t) = \frac{N(\sigma)}{(1 - \sigma)\Gamma(\mu)} P_n(t)$. Since $\mu + \delta \geq \beta + \delta \geq x_0$ for every $n \geq 0$ and $\gamma k + \mu + \delta \geq \mu + \delta$, both arguments lie in $[x_0, \infty)$, where Γ is increasing, so $\Gamma(\gamma k + \mu + \delta) \geq \Gamma(\mu + \delta)$, which implies

$$C_{n,k} \leq \frac{\Gamma(\gamma k + 1)}{\Gamma(\theta k + 1)} = A_k, \quad \forall n \geq 0, k \geq 0. \quad (4.19)$$

Therefore

$$|P_n(t)| \leq \sum_{k=0}^{\infty} A_k |\mu_\sigma|^k t^{\gamma k}, \quad (4.20)$$

which converges for every $t \geq 0$, since $\theta > \gamma$ makes A_k decay exponentially in k [14, 26, 5]. Setting

$$G_T = \max_{t \in [0, T]} \sum_{k=0}^{\infty} A_k (|\mu_\sigma| t^\gamma)^k < \infty \quad (4.21)$$

gives $|\Phi_n(t)| \leq \frac{N(\sigma)}{1-\sigma} \cdot \frac{G_T}{\Gamma(\alpha n + \beta)}$ for all $n \geq 0$, $t \in [0, T]$. $\square$

Proposition 4.4. *Let $\alpha, \beta, \rho, q, \lambda \in (0, \infty)$, $\mu \in (0, 1]$, and $\beta > 1$. Then for every $t_0 > 0$, the kernel-specific function for the Mutaz–Saadaoui operator satisfies*

$$\Phi_n(t) \leq \frac{1}{\mu t_0^\mu} \cdot \frac{1}{\Gamma(\alpha n + \beta - 1)}, \quad \forall n \geq 0, t \geq t_0. \quad (4.22)$$

Proof. Under the given parameter ranges, the stated bound follows directly from Eq. (4.12). $\square$

Theorem 4.5. *Let $u(t) = t^{\beta-1} E_{\alpha, \beta}^{\rho, q}(\lambda t^\alpha)$, and suppose that a fractional operator D^* admits a Unified Mittag–Leffler Series (UML) representation of the form*

$$D^* u(t) = t^{\beta-1} \sum_{n=0}^{\infty} \frac{(\rho)_{qn} \lambda^n t^{\alpha n}}{n!} \Phi_n(t), \quad (4.23)$$

where $\Phi_n(t)$ denotes the kernel-specific function associated with the operator D^* . Assume that, for any finite $T > 0$, there exists a positive constant C_T such that

$$|\Phi_n(t)| \leq \frac{C_T}{\Gamma(\alpha n + \beta)}, \quad \forall n \geq 0, t \in [0, T]. \quad (4.24)$$

Furthermore, assume that the parameters satisfy the standard convergence conditions for the four-parameter Mittag–Leffler function, namely $q < \Re(\alpha) + 1$, or $q = \Re(\alpha) + 1$ with $|\lambda| T^\alpha < 1$. Then the series in (4.23) converges uniformly on $[0, T]$.

Proof. Let $t \in [0, T]$. Using the bound (4.24), we obtain

$$\left| \frac{(\rho)_{qn} \lambda^n t^{\alpha n}}{n!} \Phi_n(t) \right| \leq C_T \frac{(\rho)_{qn} |\lambda|^n T^{\alpha n}}{\Gamma(\alpha n + \beta) n!}.$$

Recall the series representation of the four-parameter Mittag–Leffler function given in Definition 2.1, which converges absolutely for all z when $q < \Re(\alpha) + 1$, and for $|z| < 1$ when $q = \Re(\alpha) + 1$ (see [23]). In particular, for fixed $T > 0$ and admissible parameters, the series

$$\sum_{n=0}^{\infty} \frac{(\rho)_{qn} (|\lambda| T^\alpha)^n}{\Gamma(\alpha n + \beta) n!} = E_{\alpha, \beta}^{\rho, q}(|\lambda| T^\alpha)$$

is convergent. Therefore, the sequence of functions

$$M_n(t) = C_T \frac{(\rho)_{qn} (|\lambda| T^\alpha)^n}{\Gamma(\alpha n + \beta) n!}$$

forms a convergent majorant for the terms of the series (4.23) on $[0, T]$. By the Weierstrass M -test, the Unified Mittag–Leffler Series (4.23) converges uniformly on $[0, T]$. $\square$

Remark 4.6. For the Mutaz–Saadaoui operator, the bound established above shows that hypothesis (4.24) of Theorem 4.5 holds on $[t_0, T]$ for any $0 < t_0 < T$, with $\alpha n + \beta$ replaced by $\alpha n + \beta - 1$. Hence, by the same Weierstrass M -test argument, the Unified Mittag–Leffler Series for the Mutaz–Saadaoui operator converges uniformly on $[t_0, T]$ for every $0 < t_0 < T$.

5. LAPLACE TRANSFORMS VIA UNIFIED FRAMEWORK

In the previous section, we established a unified series representation of the fractional derivatives of the 4P–ML function under various operators. Building on this unified form, we now examine their Laplace transforms. This step is important, as many analytical and computational studies of fractional differential equations are conveniently carried out in the frequency domain. For some operators, closed-form Laplace transforms are derived, while others, owing to their kernel complexity, are expressed in symbolic form.

For $u(t) = t^{\beta-1} E_{\alpha, \beta}^{\rho, q}(\lambda t^\alpha)$, let $\mathcal{D}^*$ be a fractional operator admitting the Unified Mittag–Leffler Series representation in Eq. (4.7). Then the Laplace transform can be written as

$$\mathcal{L}\{\mathcal{D}^* u(t)\}(s) = \sum_{n=0}^{\infty} \frac{(\rho)_{qn} \lambda^n}{n!} \cdot \mathcal{L}\{t^{\alpha n + \beta - 1} \Phi_n(t)\}(s). \quad (5.1)$$

We now compute this transform for the various kernel-specific functions $\Phi_n(t)$ corresponding to different fractional operators as presented in the following lemmas.

Lemma 5.1. *For the kernel-specific function $\Phi_n(t)$ defined in Eq. (4.8) for the Atangana–Baleanu–Caputo (ABC) fractional derivative, the Laplace transform of the product $t^{\alpha n + \beta - 1} \Phi_n(t)$ is given by*

$$\mathcal{L}\{t^{\alpha n + \beta - 1} \Phi_n(t)\}(s) = \frac{s^{\nu - \alpha n - \beta}}{s^\nu + \frac{\nu}{1-\nu}}, \quad \text{for } \left| \frac{\nu}{(1-\nu)s^\nu} \right| < 1. \quad (5.2)$$

Proof. From Eq. (4.8) we have

$$t^{\alpha n + \beta - 1} \Phi_n(t) = t^{\alpha n + \beta - 1} E_{\nu, \alpha n + \beta} \left(-\frac{\nu}{1-\nu} t^\nu \right). \quad (5.3)$$

We use the known Laplace transform formula for the two-parameter Mittag–Leffler function from [9].

$$\mathcal{L}\{t^{\sigma-1} E_{\nu, \sigma}(-at^\nu)\}(s) = \frac{s^{\nu-\sigma}}{s^\nu + a}, \quad \Re(s) > 0, \ a > 0. \quad (5.4)$$

This formula is valid under the condition $\left| \frac{a}{s^\nu} \right| < 1$, which ensures convergence of the Laplace integral representation of the Mittag–Leffler function. This condition is typically satisfied when $s \in \mathbb{C}$ with $\Re(s) > 0$ and $a > 0$. Taking the Laplace transform of Eq. (5.3) and applying Eq. (5.4) with $\sigma = \alpha n + \beta$ and $a = \frac{\nu}{1-\nu}$ yields Eq. (5.2). $\square$

Lemma 5.2. *For the kernel-specific function $\Phi_n(t)$ defined in Eq. (4.9) for the MLCF fractional derivative, the Laplace transform of $t^{\alpha n + \beta - 1} \Phi_n(t)$ is given by*

$$\begin{aligned} \mathcal{L} \{ t^{\alpha n + \beta - 1} \Phi_n(t) \} (s) &= \frac{M(\nu)}{1 - \nu} \cdot \mathcal{L} \left\{ \frac{1}{E_\eta \left(\frac{\nu}{1 - \nu} t^\eta \right)} \right. \\ &\quad \times \left. \sum_{k=0}^{\infty} \frac{\left(\frac{\nu}{1 - \nu} \right)^k t^{\eta k + \alpha n + \beta - 1}}{\Gamma(\alpha n + \beta - 1) \Gamma(\eta k + 1) (\eta k + \alpha n + \beta - 1)} \right\} (s). \end{aligned} \quad (5.5)$$

Remark 5.3. The expression in Eq. (5.5) is retained in symbolic form for theoretical analysis. For practical computation, it can be approximated by truncating the series and applying a finite-series expansion of the reciprocal Mittag–Leffler function $\frac{1}{E_\eta \left(\frac{\nu}{1 - \nu} t^\eta \right)}$, which is valid under suitable conditions on the parameter η .

This approach follows the work of Hanneken and Achar [7], who presented a finite series representation of the inverse Mittag–Leffler function.

Lemma 5.4. *For the kernel-specific function $\Phi_n(t)$ defined in Eq. (4.10) for the Hattaf-type fractional derivative, the Laplace transform of $t^{\alpha n + \beta - 1} \Phi_n(t)$ is given by*

$$\begin{aligned} \mathcal{L} \{ t^{\alpha n + \beta - 1} \Phi_n(t) \} (s) &= \frac{N(\sigma)}{1 - \sigma} \sum_{k=0}^{\infty} \frac{(-\mu_\sigma)^k \Gamma(\gamma k + 1) \Gamma(\alpha n + \beta + \delta)}{\Gamma(\alpha n + \beta) \Gamma(\theta k + 1) \Gamma(\gamma k + \alpha n + \beta + \delta)} \\ &\quad \times \frac{\Gamma(\alpha n + \beta + \gamma k)}{s^{\alpha n + \beta + \gamma k}}. \end{aligned} \quad (5.6)$$

Proof. From Eq. (4.10) we have

$$t^{\alpha n + \beta - 1} \Phi_n(t) = \sum_{k=0}^{\infty} C_{n,k} \cdot t^{\alpha n + \beta - 1 + \gamma k}, \quad (5.7)$$

where

$$C_{n,k} = \frac{N(\sigma)}{1 - \sigma} \cdot \frac{(-\mu_\sigma)^k \Gamma(\gamma k + 1) \Gamma(\alpha n + \beta + \delta)}{\Gamma(\alpha n + \beta) \Gamma(\theta k + 1) \Gamma(\gamma k + \alpha n + \beta + \delta)}.$$

Taking the Laplace transform of Eq. (5.7), we obtain Eq. (5.6). $\square$

Lemma 5.5. *The Laplace transform of the Sigmoidal fractional derivative operator $\mathcal{D}^* u(t)$ expressed in the Unified Mittag–Leffler Integral Series form given in Eq. (4.1), where $\mathcal{K}(t, s) = C_1(\mu) \cdot \text{sech}^2 \left(\frac{s-t}{1-\mu} \right)$, is given by*

$$\mathcal{L} \{ \mathcal{D}^* u(t) \} (s) = \sum_{n=0}^{\infty} \frac{(\rho)_{qn} \lambda^n}{\Gamma(\alpha n + \beta - 1) n!} \cdot \mathcal{L} \left\{ \int_0^t \text{sech}^2 \left(\frac{s-t}{1-\mu} \right) s^{\alpha n + \beta - 2} ds \right\} (s). \quad (5.8)$$

Remark 5.6. The expression in Eq. (5.8) is retained in symbolic form for theoretical clarity. In practice, the Laplace transform of the Sigmoidal kernel admits a closed-form expression involving the digamma function, as demonstrated in [20], which can be used to make Eq. (5.8) explicit.

Lemma 5.7. *For the kernel-specific function $\Phi_n(t)$ defined in Eq. (4.12) for the Mutaz–Saadaoui fractional derivative, the Laplace transform of $t^{\alpha n + \beta - 1} \Phi_n(t)$ is given by*

$$\mathcal{L} \{ t^{\alpha n + \beta - 1} \Phi_n(t) \} (s) = \frac{\Gamma(\alpha n + \beta - \mu)}{\mu \Gamma(\alpha n + \beta - 1)} \cdot s^{-(\alpha n + \beta - \mu)}, \quad (5.9)$$

Proof. From Eq. (4.12) we have

$$t^{\alpha n + \beta - 1} \Phi_n(t) = \frac{1}{\mu \Gamma(\alpha n + \beta - 1)} \cdot t^{\alpha n + \beta - 1 - \mu}. \quad (5.10)$$

Taking the Laplace transform of Eq. (5.10), we obtain Eq. (5.9). $\square$

The Unified-series formulation provides a systematic framework for computing the Laplace transforms of a wide class of fractional derivatives with non-singular kernels. It yields expressions that are directly applicable to fractional differential equations, stability analysis, and related problems. This approach accommodates both series-based and integral-based kernel structures, depending on the nature of the operator. In the subsequent section, we present numerical illustrations to demonstrate the behavior of the fractional derivatives obtained in this study.

6. NUMERICAL ILLUSTRATION

To illustrate the numerical behavior of the proposed framework, we consider the 4P–ML function with parameters

$$(\alpha, \beta, \rho, q, \lambda) = (0.9, 1.2, 1.5, 1, 0.5).$$

The fractional derivatives $D^*u(t)$ corresponding to the five operators derived in the previous sections are evaluated over the interval $t \in [0.1, 5]$. The infinite series are truncated at $n = 20$. The operator parameters are chosen as $\nu = 0.6$ for the ABC derivative, $(\nu, \eta) = (0.6, 0.9)$ for the MLCF derivative, $\mu = 0.6$ for the Mutaz–Saadaoui derivative, and $(\sigma, \theta, \gamma, \delta) = (0.6, 1, 0.9, 0.5)$ for the Hattaf-type fractional derivative in the Caputo sense, with $w(t) = t^{0.5}$. For the Sigmoidal fractional derivative, we set $\mu = 0.6$ and evaluate the integral involving the kernel $\text{sech}^2\left(\frac{s-t}{1-\mu}\right)$ numerically using the trapezoidal rule. All the above operators except for the sigmoidal case are implemented using the unified series expression involving $\Phi_n(t)$. For the sigmoidal operator, although the exact form is computed through numerical integration, the series representation is still used to estimate an upper bound for comparative purposes. As seen in Fig. 1, the graph clearly illustrates the distinctive non-local and memory-driven dynamics induced by each operator. The different growth rates and temporal evolution patterns observed across operators highlight the flexibility and generality of the proposed unified series framework. These results confirm that even under the same base function $u(t)$, the choice of operator significantly influences the qualitative temporal behavior and magnitude of the fractional derivative for the chosen parameter set, with Mutaz–Saadaoui showing the steepest growth, followed by Hattaf-type, MLCF, and ABC operators respectively.

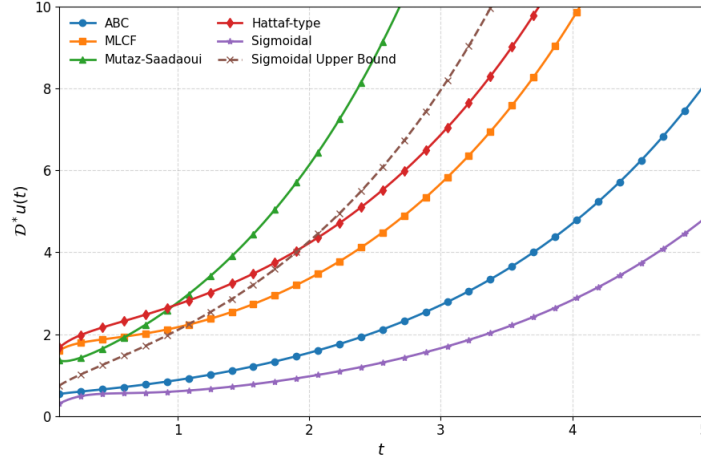


FIGURE 1. Graphical comparison of unified fractional derivatives for different operators over the interval $t \in [0.1, 5]$.

6.1. Numerical Behavior of Kernel-Specific Functions $\Phi_n(t)$. To understand the effect of different fractional operators within the unified series formulation, we investigate the numerical behavior of the kernel-specific functions $\Phi_n(t)$ that appear in the general representation of fractional derivatives applied to the 4P-ML function. We consider all fractional derivatives derived in earlier section. These operators exhibit a variety of kernel structures, each encoding distinct memory and nonlocal effects. To study the numerical characteristics, we conduct two experiments. In the first experiment, we fix the time variable at $t = 0.1, 0.5, 1.0, 2.0$ and compute $\Phi_n(t)$ for $n = 0$ through 9. This reveals how the kernel-specific function decays or oscillates with increasing series index n , thus providing insight into the relative weight of higher-order terms in the expansion for each operator. In the second experiment, we fix the index $n = 0, 1, 2, 3$ and analyze the variation of $\Phi_n(t)$ as t ranges from 0.1 to 5. All computations are performed using the same parameter values specified earlier in the main section, thereby ensuring consistency and comparability across the different fractional models. For the Sigmoidal operator, the lack of a closed-form kernel-specific function necessitates an upper-bound approximation. This estimate, based on the maximum value of the underlying sigmoidal weight, aligns well with the unified framework and provides a consistent numerical interpretation. Fig. 2 illustrates the behavior of the contributions of kernel-specific function $\Phi_n(t)$ arising from the unified series representation of the Mittag-Leffler derivative. As n increases, the contributions decay, showing that higher-order terms have a diminishing effect. The ABC and MLCF operators display a relatively smoother decay pattern, whereas the Hattaf-type and Mutas-Saadaoui operators exhibit a steeper decay, indicating a faster suppression of higher-order terms. In particular, the Mutas-Saadaoui kernel shows an initial peak before decaying. Fig. 3 shows the variation of the kernel-specific function with respect to time for a fixed index. All kernels exhibit a monotone decay in time, but with distinct attenuation patterns. The ABC operator shows the slowest decay, retaining kernel contributions over longer

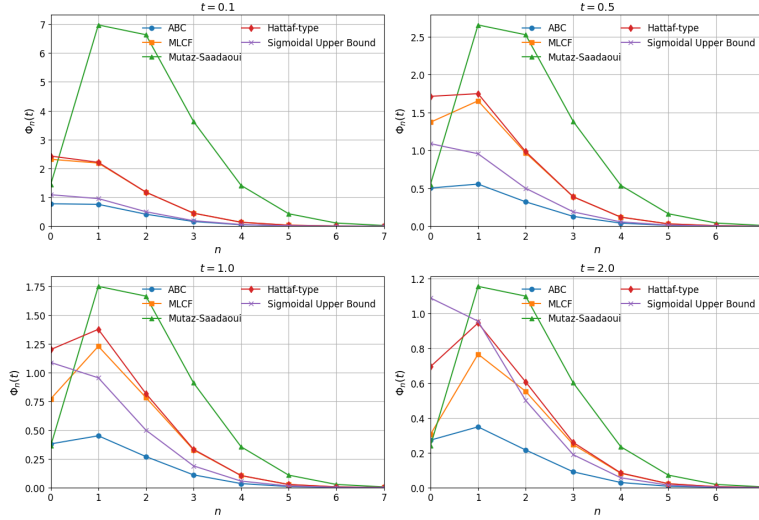


FIGURE 2. Variation of $\Phi_n(t)$ with $n \in [0, 9]$ for fixed $t = 0.1, 0.5, 1.0, 2.0$, across different fractional derivatives.

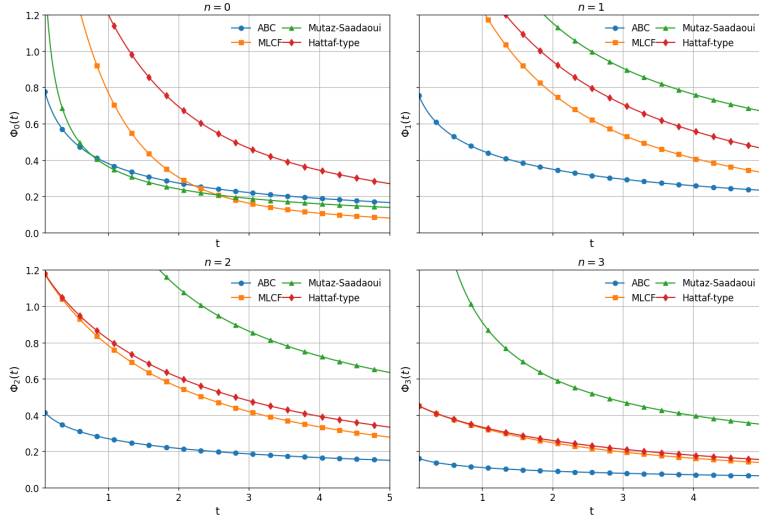


FIGURE 3. Variation of $\Phi_n(t)$ with $t \in [0.1, 5]$ for fixed $n = 0, 1, 2, 3$, across different fractional derivatives.

intervals, whereas the MLCF operator decays more rapidly. The Mutaz–Saadaoui and Hattaf–type operators display stronger initial values followed by a relatively faster suppression, especially for higher n . These differences highlight the varying distribution of weight across the series terms depending on the chosen operator.

These visual insights demonstrate how the choice of operator modulates the kernel structure and, consequently, the dynamic behavior of fractional models based on the Unified Mittag–Leffler Series formulation. It is emphasized that these numerical experiments are intended solely to illustrate the structural differences among operators within the proposed framework. They do not correspond to the solution of any specific applied fractional model.

7. APPLICATIONS AND IMPLICATIONS

The Unified Mittag–Leffler (UML) Series representation provides a common analytical framework for studying non-singular fractional derivatives acting on Mittag–Leffler–type functions. By expressing different operators in the unified form of Eq. (4.7), with the kernel-specific function $\Phi_n(t)$ capturing the operator dependence, their analytical structures can be compared directly. The framework also facilitates symbolic analysis, numerical implementation, and asymptotic investigations. An important consequence of the proposed formulation is its generality. Since the classical one-, two-, and three-parameter Mittag–Leffler functions are special cases of the four-parameter Mittag–Leffler function, their corresponding fractional derivatives follow immediately by parameter reduction, without separate derivations. Furthermore, the UML Series enables direct comparison of the kernel-specific functions $\Phi_n(t)$, providing a unified basis for analyzing and selecting non-singular fractional operators in applications such as anomalous diffusion and other memory-dependent systems.

8. CONCLUSION

In this work, we introduced the Unified Mittag–Leffler (UML) Series for representing a broad class of non-singular fractional operators through kernel-specific functions. The proposed representation eliminates the need for separate derivations and provides a systematic way to study different operators within a single formulation. The framework was developed for the four-parameter Mittag–Leffler function, from which several classical and generalized Mittag–Leffler functions follow by parameter reduction. Corresponding Laplace-transform representations were obtained, yielding either closed-form expressions or convergent series depending on the underlying operator. Kernel-specific bounds and a general uniform convergence theorem establish the analytical validity of the proposed UML Series. The numerical results illustrate the influence of different kernel structures on the resulting fractional derivatives and highlight the flexibility of the unified framework. Future work includes extending the proposed framework to the Fox H -function, variable-order fractional operators, multidimensional settings, and application-oriented fractional differential equations.

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