

ON GENERALIZED DERIVATIONS AND COMMUTATIVITY OF PRIME Γ -RINGS

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ABSTRACT. Let M be a prime Γ -ring with center $Z(M)$, I a nonzero ideal of M and F be a generalized derivation with associated nonzero derivation d . In the present paper, our purpose is to produce commutativity results for prime Γ -rings M admitting a generalized derivation F satisfying any one of the properties: (i) $F(x\alpha y) \pm x\alpha y \in Z(M)$, (ii) $F(x\alpha y) \pm y\alpha x \in Z(M)$, (iii) $F(x)\alpha F(y) \pm x\alpha y \in Z(M)$, (iv) $F(x)\alpha F(y) \pm y\alpha x \in Z(M)$, (v) $F([x, y]_\alpha) = [F(x), y]_\alpha$, (vi) $[F(x), y]_\alpha \pm [x, F(y)]_\alpha = 0$, (vii) $F([x, y]_\alpha) \pm [d(x), d(y)]_\alpha = 0$, for all $x, y \in I$ and $\alpha \in \Gamma$. Also, some examples are given to show that the primeness of the various results is not superfluous.

1. INTRODUCTION

In 1964, Nobusawa [18] introduced the notion of a Γ -ring, an object more general than a ring. Barnes [7] slightly weakened the conditions in the definition of a Γ -ring in the sense of Nobusawa. Since then, many researchers have done a lot of work on Γ -rings and have obtained some generalizations of the corresponding results in ring theory (see [15], [20] and [24] for partial references). If M and Γ are additive abelian groups and there exists a mapping $(., ., .) : M \times \Gamma \times M \rightarrow M$ which satisfies the following conditions:

- (i) $(a, \beta, b) \in M$;
 - (ii) $(a + b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha + \beta)b = a\alpha b + a\beta b$, $a\alpha(b + c) = a\alpha b + a\alpha c$;
 - (iii) $(a\alpha b)\beta c = a\alpha(b\beta c)$, for $a, b, c \in M$ and $\alpha, \beta \in \Gamma$;
- then M is called a Γ -ring.

Following [9], every associative ring R is a Γ -ring with $\Gamma = U$ or Z , where Z is the ring of integers and U is an ideal of R . But the converse is in general not true. For example [8], Let V be a linear space with a basis $e_1, e_2, \dots, e_n$ over a field F of characteristic different from two. Define a multiplication $\cdot : V \times V \rightarrow V$ by the rule $vw = 0$ for all $v, w \in V$ with $v \neq e_1$, $v \neq -e_1$ and $e_1w = w$, $(-e_1)w = -w$. One can easily check that V is only a near-ring, not a ring. However, in [18], we know that V is a Γ -ring with $\Gamma = V$. Recall that a Γ -ring M is prime if

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$a\Gamma M\Gamma b = 0$ implies that $a = 0$ or $b = 0$; and is semiprime if $a\Gamma M\Gamma a = 0$ implies that $a = 0$. A Γ -ring M is said to be commutative if $x\alpha y = y\alpha x$ for all $x, y \in M$ and $\alpha \in \Gamma$. A Γ -ring M is said to be 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in M$. Moreover, the set $Z(M) = \{x \in M \mid x\alpha y = y\alpha x \ \forall \alpha \in \Gamma, y \in M\}$ is called the center of the Γ -ring M . We shall write $[x, y]_\alpha = x\alpha y - y\alpha x$ for all $x, y \in M; \alpha \in \Gamma$. An additive subgroup U of a Γ -ring M is called a left (resp. right) ideal of M if $M\Gamma U \subseteq U$ (resp. $U\Gamma M \subseteq U$). If U is both a left ideal and a right ideal, then we say that U is an ideal of M . An additive mapping $d : M \rightarrow M$ is called a derivation on M if $d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$. An additive mapping $F : M \rightarrow M$ is called a generalized derivation if there exists a derivation $d : M \rightarrow M$ such that $F(x\alpha y) = F(x)\alpha y + x\alpha d(y)$ holds for all $x, y \in M; \alpha \in \Gamma$, and d is called the associated derivation of F . Hence, the concept of generalized derivations covers both the concepts of a derivation and of a left multiplier (i.e. an additive mapping satisfying $F(x\alpha y) = F(x)\alpha y$ for all $x, y \in M; \alpha \in \Gamma$). Since the sum of two generalized derivations is a generalized derivation, every map of the form $F(x) = c\alpha x + d(x)$, where c is a fixed element of M and d is a derivation of M , is a generalized derivation. Basic examples are derivations and generalized inner derivations, namely, mappings of type $x \rightarrow a\alpha x + x\alpha b$ for some $a, b \in M; \alpha \in \Gamma$. We refer to call such mappings generalized inner derivations for the reason they present a generalization of the concept of inner derivations. Generalized derivations have been primarily studied on operator algebras. Therefore, any investigation from the algebraic point of view might be interesting.

As is well known, a classical problem in ring theory is to study and generalize conditions under which a ring is commutative. One of the most effective tools found for this purpose is the derivations on rings and also on their modules. During the past few decades, there has been an ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain special types of derivations. The first result in this topic is due to Posner [23] who proved that if a prime ring R admits a derivation d such that $[d(x), x] \in Z(R)$ for all $x \in R$, then R is commutative or $d = 0$. This classical result indicates that the global structure of a ring R is often tightly connected to the behavior of additive mappings defined on R , and it is subsequently refined and extended by a lot of algebraists. Recently, some authors have obtained commutativity of prime and semiprime rings with derivations, automorphisms, generalized derivations et al. satisfying certain polynomial constraints (cf.; [2, 6, 12, 14, 16, 17] and references therein). In [10], Daif and Bell showed that if in a semiprime ring R there exists a nonzero ideal I of R and a derivation d such that $d([x, y]) \pm [x, y] = 0$ for all $x, y \in I$, then $I \subseteq Z(R)$. In particular, in the prime case, R must be commutative. Later, Hogan [13] generalized the above result to the central case. Ashraf and Rehman [3] explored the commutativity in prime and semiprime rings with derivations satisfying $d(xy) \pm xy \in Z(R)$; $d(xy) \pm yx \in Z(R)$; $d(x)d(y) \pm xy \in Z(R)$ for all $x, y \in R$. In [4], Ashraf et al. extended these results in the setting of generalized derivations. On the other hand, in [21] Quadri et al. proved that if

R is a prime ring, I a nonzero ideal of R and F a generalized derivation associated with a nonzero derivation d such that $F([x, y]) = \pm[x, y]$ for all $x, y \in I$, then R is commutative. Ali, Kumar and Miyan [1] obtained the commutativity of prime rings by studying the central identities: $F([x, y]) + [x, y] \in Z(R)$; $F([x, y]) - [x, y] \in Z(R)$, for all x, y in some nonzero right ideal. Recently, Ashraf and Jamal [5] investigated the commutativity of prime Γ -rings satisfying certain differential identities. Motivated by the above observation, we will continue this line of investigation and prove some commutativity results for prime Γ -rings involving a generalized derivation, which are of independent interest. In fact, our results unify, extend and complement many related results previously obtained in literature.

In order to prove our theorems, we recall some well known results which will be helpful in the sequel.

Lemma 1.1. [22, Lemma 2] (i) Let M be a prime Γ -ring, U a nonzero right (left) ideal of M and $a \in M$. If $U\Gamma a = 0$ ($a\Gamma U = 0$), then $a = 0$. (ii) Let U be a nonzero ideal of M and $a, b \in M$. If $a\Gamma U\Gamma b = 0$, then $a = 0$ or $b = 0$.

Lemma 1.2. [19, Lemma 3] Let M be a 2-torsion free semi-prime Γ -ring, I a nonzero ideal of M and $a, b \in M$. Then the following are equivalent:

- (i) $a\alpha x\beta b = 0$ for all $x, y \in I$ and $\alpha, \beta \in \Gamma$;
- (ii) $b\alpha x\beta a = 0$ for all $x, y \in I$ and $\alpha, \beta \in \Gamma$;
- (iii) $a\alpha x\beta b + b\alpha x\beta a = 0$ for all $x, y \in I$ and $\alpha, \beta \in \Gamma$.

Lemma 1.3. [11, Lemma 2.3] If a prime Γ -ring M contains a nonzero commutative right ideal, then M is commutative.

Lemma 1.4. [5, Theorem 2.3] Let M be a prime Γ -ring and U a nonzero ideal of M . If d is a nonzero derivation on M satisfying $[d(x), x]_\gamma = 0$ for all $x \in U$; $\gamma \in \Gamma$, then M is commutative.

Throughout the paper, we shall assume that $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$; $\alpha, \beta \in \Gamma$ and in this case we have some basic identities: $[x\beta y, z]_\alpha = [x, z]_\alpha \beta y + x\beta[y, z]_\alpha$; $[x, y\beta z]_\alpha = [x, y]_\alpha \beta z + y\beta[x, z]_\alpha$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

2. RESULTS

Theorem 2.1. Let M be a prime Γ -ring and I a nonzero ideal of M . If M admits a generalized derivation F with associated nonzero derivation d such that $F(x\alpha y) - x\alpha y \in Z(M)$ for all $x, y \in I$ and $\alpha \in \Gamma$, then M is commutative.

Proof. If $F = 0$, then $x\alpha y \in Z(M)$ for all $x, y \in I$ and $\alpha \in \Gamma$. Using the same techniques in the beginning of proof of [5, Theorem 2.9], we get the required result. Hence, in the sequel we assume that $F \neq 0$. By the given hypothesis we have $F(x\alpha y) - x\alpha y \in Z(M)$, which can be rewritten as

$$F(x)\alpha y + x\alpha d(y) - x\alpha y \in Z(M) \text{ for all } x, y \in I; \alpha \in \Gamma. \quad (2.1)$$

Replacing y by $y\beta z$ in (2.1), we get $F(x)\alpha y\beta z + x\alpha d(y)\beta z + x\alpha y\beta d(z) - x\alpha y\beta z \in Z(M)$ for all $x, y, z \in I$ and $\alpha, \beta \in \Gamma$. Thus, $[(F(x)\alpha y + x\alpha d(y) - x\alpha y)\beta z + x\alpha y\beta d(z), z]_\gamma = 0$. This can reduce to $[x\alpha y\beta d(z), z]_\gamma = 0$ for all $x, y, z \in I$ and $\alpha, \beta, \gamma \in \Gamma$. Expanding the last equation, we get

$$x\alpha y\beta[d(z), z]_\gamma + x\alpha[y, z]_\gamma + [x, z]_\gamma\alpha y\beta d(z) = 0 \text{ for all } x, y, z \in I; \alpha, \beta, \gamma \in \Gamma. \quad (2.2)$$

Replacing x by $w\delta x$ in (2.2), we obtain

$$w\delta x\alpha y\beta[d(z), z]_\gamma + w\delta x\alpha[y, z]_\gamma\beta d(z) + w\delta[x, z]_\gamma\alpha y\beta d(z) + [w, z]_\gamma\delta x\alpha y\beta d(z) = 0. \quad (2.3)$$

Combining (2.2) and (2.3), we find that

$$[w, z]_\gamma\delta x\alpha y\beta d(z) = 0 \text{ for all } x, y, z, w \in I; \alpha, \beta, \gamma \in \Gamma. \quad (2.4)$$

In view of Lemma 1.1, for each $z \in I$, either $[w, z]_\gamma\delta x = 0$ or $d(z) = 0$. Let $I_1 = \{z \in I \mid [w, z]_\gamma\delta x = 0\}$ and $I_2 = \{z \in I \mid d(z) = 0\}$. Then, I_1 and I_2 are both additive subgroups of I such that $I = I_1 \cup I_2$. Since a group can't be a union of its two proper subgroups, we have either $I = I_1$ or $I = I_2$. If $I = I_1$, then $[I, I]_\gamma\Gamma I = 0$. The primeness of I forces that $[I, I]_\gamma = 0$, that is, I is commutative and so M by Lemma 1.3. If $I = I_2$, then $d(I) = 0$ and hence $0 = d(I\Gamma M) = d(I)\Gamma M + I\Gamma d(M) = I\Gamma d(M)$. The primeness of I yields that $d = 0$, a contradiction.

Theorem 2.2. Let M be a prime Γ -ring and I a nonzero ideal of M . If M admits a generalized derivation F with associated nonzero derivation d such that $F(x\alpha y) + x\alpha y \in Z(M)$ for all $x, y \in I$ and $\alpha \in \Gamma$, then M is commutative.

Proof. If F is a generalized derivation such that $F(x\alpha y) + x\alpha y \in Z(M)$ for all $x, y \in I$ and $\alpha \in \Gamma$, then $-F$ is also a generalized derivation and satisfies $(-F)(x\alpha y) - x\alpha y \in Z(M)$. It follows from Theorem 2.1 that M is commutative.

Theorem 2.3. Let M be a prime Γ -ring and I a nonzero ideal of M . If M admits a generalized derivation F with associated nonzero derivation d such that $F(x\alpha y) - y\alpha x \in Z(M)$ for all $x, y \in I$ and $\alpha \in \Gamma$, then M is commutative.

Proof. If $F = 0$, then $y\alpha x \in Z(M)$ for all $x, y \in I$ and $\alpha \in \Gamma$. So we are done in this case. If $F \neq 0$, we are given that

$$F(x)\alpha y + x\alpha d(y) - y\alpha x \in Z(M) \text{ for all } x, y \in I; \alpha \in \Gamma. \quad (2.5)$$

Thus, $[F(x)\alpha y + x\alpha d(y) - y\alpha x, z]_\beta = 0$ for all $x, y \in I$ and $\alpha, \beta \in \Gamma$. Expand this equation to get

$$[F(x), z]_\beta \alpha y + F(x)\alpha[y, z]_\beta + [x, z]_\beta \alpha d(y) + x\alpha[d(y), z]_\beta = [y, z]_\beta \alpha x + y\alpha[x, z]_\beta. \quad (2.6)$$

Replacing y by $y\delta z$ in (2.6) and using (2.6), we obtain

$$[y, z]_\beta \alpha[z, x]_\delta + y\alpha[z, [x, z]_\beta]_\delta = [x, z]_\beta \alpha y \delta d(z) + x\alpha y \delta[d(z), z]_\beta + x\alpha[y, z]_\beta \delta d(z). \quad (2.7)$$

Replace y by $x\gamma y$ in (2.7) and use (2.7) to get

$$[x, z]_\beta \gamma y \alpha[z, x]_\beta = [x, z]_\beta \alpha x \gamma y \delta d(z) \text{ for all } x, y, z \in I; \alpha, \beta, \gamma, \delta \in \Gamma. \quad (2.8)$$

Substituting $z + x$ for x in (2.8) and using (2.8), we have

$$[x, z]_\beta \alpha x \gamma y \delta d(x) = 0 \text{ for all } x, y, z \in I; \alpha, \beta, \gamma, \delta \in \Gamma. \quad (2.9)$$

Now the primeness of I , for each $x \in I$, gives either $[x, z]_\beta \alpha x = 0$ or $d(x) = 0$. Suppose that $[x, z]_\beta \alpha x = 0$. Replacing z by $z\gamma w$, we have $0 = [x, z\gamma w]_\beta \alpha x = [x, z]_\beta \gamma w \alpha x + z\gamma[x, w]_\beta \alpha x = [x, z]_\beta \gamma w \alpha x$. By the primeness of I , either $[x, z]_\beta = 0$ or $x = 0$. But $x = 0$ also implies that $[x, z]_\beta = 0$. Thus, it remains only to dispose of the case when for each $x \in I$, either $[x, z]_\beta = 0$ or $d(x) = 0$. Repeating the same arguments as in equation (2.4), we complete the proof.

Similarly, we can prove the following:

Theorem 2.4. Let M be a prime Γ -ring and I a nonzero ideal of M . If M admits a generalized derivation F with associated nonzero derivation d such that $F(x\alpha y) + y\alpha x \in Z(M)$ for all $x, y \in I$ and $\alpha \in \Gamma$, then M is commutative.

Theorem 2.5. Let M be a prime Γ -ring and I a nonzero ideal of M . If M admits a generalized derivation F with associated nonzero derivation d such that $F(x)\alpha F(y) - x\alpha y \in Z(M)$ for all $x, y \in I$ and $\alpha \in \Gamma$, then M is commutative.

Proof. Assume that $F = 0$, then $x\alpha y \in Z(M)$ and we are done. Hence, onward we assume that $F \neq 0$. By hypothesis, we have

$$F(x)\alpha F(y) - x\alpha y \in Z(M) \text{ for all } x, y \in I; \alpha \in \Gamma. \quad (2.10)$$

Replacing y by $y\beta s$ in (2.10), we get $(F(x)\alpha F(y) - x\alpha y)\beta s + F(x)\alpha y\beta d(s) \in Z(M)$ for all $x, y \in I$, $s \in M$ and $\alpha, \beta \in \Gamma$. This implies that $[F(x)\alpha y\beta d(s), s]_\gamma = 0$, which can be rewritten as

$$F(x)\alpha[y\beta d(s), s]_\gamma + [F(x), s]_\gamma \alpha y\beta d(s) = 0 \text{ for all } x, y \in I; s \in M; \alpha, \beta, \gamma \in \Gamma. \quad (2.11)$$

Replacing y by $F(x)\delta y$ in (2.11) and using (2.11), we arrive at

$$[F(x), s]_\gamma \alpha F(x)\delta y\beta d(s) = 0 \text{ for all } x, y \in I; s \in M; \alpha, \beta, \gamma, \delta \in \Gamma. \quad (2.12)$$

The primeness of I gives that for each $s \in M$, either $[F(x), s]_\gamma \alpha F(x) = 0$ or $d(s) = 0$. The set of $s \in M$ for which these two properties hold, are additive subgroups of M whose union is M . Then, either $[F(x), s]_\gamma \alpha F(x) = 0$ or $d(s) = 0$ for all $x \in I; s \in M$. In the case of $d(s) = 0$ for all $s \in M$,

then $d = 0$, a contradiction. Suppose that $[F(x), s]_\gamma \alpha F(x) = 0$. Replace s by $s\beta t$ to get $[F(x), s\beta t]_\gamma \alpha F(x) = 0$. This implies that $0 = s\beta[F(x), t]_\gamma \alpha F(x) + [F(x), s]_\gamma \beta t \alpha F(x) = [F(x), s]_\gamma \beta t \alpha F(x)$. The primeness of M implies that either $[F(x), s]_\gamma = 0$ or $F(x) = 0$. But $F(x) = 0$ also give that $[F(x), s]_\gamma = 0$. Thus in each case we have $[F(x), s]_\gamma = 0$, which implies that $F(x) \in Z(M)$. In this case, the hypothesis yields that $x\alpha y \in Z(M)$ since $F(x)\alpha F(y) \in Z(M)$ and we are done.

Arguing as above we can prove the following:

Theorem 2.6. Let M be a prime Γ -ring and I a nonzero ideal of M . If M admits a generalized derivation F with associated nonzero derivation d such that $F(x)\alpha F(y) + x\alpha y \in Z(M)$ for all $x, y \in I$ and $\alpha \in \Gamma$, then M is commutative. Using the same techniques with necessary variations we get the following:

Theorem 2.7. Let M be a prime Γ -ring and I a nonzero ideal of M . If M admits a generalized derivation F with associated nonzero derivation d such that $F(x)\alpha F(y) \pm y\alpha x \in Z(M)$ for all $x, y \in I$ and $\alpha \in \Gamma$, then M is commutative.

Theorem 2.8. Let M be a prime Γ -ring and I a nonzero ideal of M . If M admits a generalized derivation F with associated nonzero derivation d such that $F([x, y]_\alpha) = [F(x), y]_\alpha$ for all $x, y \in I$ and $\alpha \in \Gamma$, then M is commutative.

Proof. We are given that

$$F([x, y]_\alpha) = [F(x), y]_\alpha \text{ for all } x, y \in I; \alpha \in \Gamma. \quad (2.13)$$

Replacing x by $z\beta x$ in (2.13), we get

$$z\beta d([x, y]_\alpha) + F([x, y]_\alpha)\beta x = z\beta[d(x), y]_\alpha + [F(x), y]_\alpha\beta x \text{ for all } x, y, z \in I; \alpha, \beta \in \Gamma. \quad (2.14)$$

Comparing (2.13) and (2.14), we can find that $z\beta(d([x, y]_\alpha) - [d(x), y]_\alpha) = 0$, that is, $I\Gamma(d([x, y]_\alpha) - [d(x), y]_\alpha) = 0$. Making use of Lemma 1.1 gives

$$d([x, y]_\alpha) - [d(x), y]_\alpha = 0 \text{ for all } x, y \in I; \alpha \in \Gamma. \quad (2.15)$$

Replacing y by $y\beta z$ in (2.15) and using (2.15), we obtain $[d(x), y]_\alpha\beta z = 0$ for all $x, y, z \in I; \alpha, \beta \in \Gamma$. Again using Lemma 1.1, $[d(x), y]_\alpha = 0$ in particular $[d(x), x]_\alpha = 0$ for all $x \in I; \alpha \in \Gamma$. In light of Lemma 1.4, M is commutative.

Theorem 2.9. Let M be a prime Γ -ring and I a nonzero ideal of M . If M admits a generalized derivation F with associated nonzero derivation d such that $[F(x), y]_\alpha = [x, F(y)]_\alpha$ for all $x, y \in I$ and $\alpha \in \Gamma$, then M is commutative.

Proof. By assumption, we have

$$[F(x), y]_\alpha = [x, F(y)]_\alpha \text{ for all } x, y \in I; \alpha \in \Gamma. \quad (2.16)$$

Replacing y by $y\beta x$ in the above expression and using (2.16) we obtain

$$y\beta[F(x), x]_\alpha = y\beta[x, d(x)]_\alpha + [x, y]_\alpha\beta d(x) \text{ for all } x, y \in I; \alpha, \beta \in \Gamma. \quad (2.17)$$

Substituting $z\delta y$ for y in (2.17) and using (2.17), we find that $[x, z]_\alpha \delta y \beta d(x) = 0$ for all $x, y, z \in I$ and $\alpha, \beta, \delta \in \Gamma$. Thus using similar approach as used for equation (2.4), we get the required result.

Application of similar arguments yields the following:

Theorem 2.10. Let M be a prime Γ -ring and I a nonzero ideal of M . If M admits a generalized derivation F with associated nonzero derivation d such that $[F(x), y]_\alpha + [x, F(y)]_\alpha = 0$ for all $x, y \in I$ and $\alpha \in \Gamma$, then M is commutative.

Theorem 2.11. Let M be a 2-torsion free prime Γ -ring and I a nonzero ideal of M . If M admits a generalized derivation F with associated nonzero derivation d such that $F([x, y]_\alpha) = [d(x), d(y)]_\alpha$ for all $x, y \in I$ and $\alpha \in \Gamma$, then M is commutative.

Proof. By assumption, we have

$$F([x, y]_\alpha) = [d(x), d(y)]_\alpha \text{ for all } x, y \in I; \alpha \in \Gamma. \quad (2.18)$$

Replacing y by $y\beta x$ in (2.18) we obtain

$$F([x, y]_\alpha) \beta x + [x, y]_\alpha \beta d(x) = [d(x), d(y)]_\alpha \beta x + d(y) \beta [d(x), x]_\alpha + [d(x), y]_\alpha \beta d(x). \quad (2.19)$$

Using (2.18) and (2.19), we find that

$$[x, y]_\alpha \beta d(x) = d(y) \beta [d(x), x]_\alpha + [d(x), y]_\alpha \beta d(x) \text{ for all } x, y \in I; \alpha, \beta \in \Gamma. \quad (2.20)$$

Substituting $x\delta y$ for y in (2.20) and using (2.20), we get

$$d(x) \delta y \beta [d(x), x]_\alpha + [d(x), x]_\alpha \delta y \beta d(x) = 0 \text{ for all } x, y \in I; \alpha, \beta, \delta \in \Gamma. \quad (2.21)$$

Applying Lemma 1.2, $d(x) \delta y \beta [d(x), x]_\alpha = 0$. Again primeness of I gives that $d(x) = 0$ or $[d(x), x]_\alpha = 0$. In the former case, we also have $[d(x), x]_\alpha = 0$. Consequently, we must have $[d(x), x]_\alpha = 0$ for all $x \in I$ and $\alpha \in \Gamma$. Hence application of Lemma 1.4 completes the proof of the theorem.

Repeating the same arguments as above, we can prove the following:

Theorem 2.12. Let M be a 2-torsion free prime Γ -ring and I a nonzero ideal of M . If M admits a generalized derivation F with associated nonzero derivation d such that $F([x, y]_\alpha) + [d(x), d(y)]_\alpha = 0$ for all $x, y \in I$ and $\alpha \in \Gamma$, then M is commutative.

We provide an example to demonstrate that the primeness in the hypothesis of Theorem 2.1 and Theorem 2.5 is essential.

Example 2.13. Let Z , R and C be the set of all integers, real numbers and complex numbers, respectively. Let H be the ring of real quaternions, that is, $H = \{h = a + bi + cj + dk \mid a, b, c, d \in R\}$. We define a map $f : H \rightarrow H$ by $f(h) = h + [i, h]_\alpha$ for some fixed $\alpha \in \Gamma$, then f is a generalized derivation with

associated nonzero derivation d of H , where $d(h) = [i, h]_\alpha$ for all $h \in H$. Let $M = M_{1,2}(H)$, $I = M_{1,2}(C)$ and $\Gamma = \left\{ \begin{pmatrix} n & \\ & 0 \end{pmatrix} \mid n \in Z \right\}$. Then it is easy to see that M is a Γ -ring and I is a nonzero ideal of M . It follows from $0 \neq (0, 1)$ and $(0, 1)\Gamma M\Gamma(0, 1) = (0, 0)$ that M is not a prime Γ -ring. Define the map $F : M \rightarrow M$ by $F(x, y) = (f(x), f(y))$, then F is a generalized derivation of M associated with a nonzero derivation D , where $D(x, y) = (d(x), d(y))$ for all $x, y \in M$. Moreover, it can be easily seen that both $F(x\alpha y) - x\alpha y \in Z(M)$, and $F(x)\alpha F(y) - x\alpha y \in Z(M)$, for all $x, y \in I$ and $\alpha \in \Gamma$, but M is not commutative.

Also, the following example illustrates that M to be prime is essential in other theorems.

Example 2.14. Let D be a division ring and $M = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in D \right\}$. According to [18], M is a Γ -ring ($\Gamma = M$). It is obvious that $I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in D \right\}$ is a nonzero ideal of M . The fact that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Gamma M \Gamma \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ implying M is not prime. We define a map $F : M \rightarrow M$ by $F \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 2b \\ 0 & 0 \end{pmatrix}$. Then F is a generalized derivation associated with nonzero derivation d defined by $d \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & 0 \end{pmatrix}$. It is straightforward to check that (i) $F(x\alpha y) + x\alpha y \in Z(M)$, (ii) $F(x\alpha y) \pm y\alpha x \in Z(M)$, (iii) $F(x)\alpha F(y) + x\alpha y \in Z(M)$, (iv) $F(x)\alpha F(y) \pm y\alpha x \in Z(M)$, (v) $F([x, y]_\alpha) = [F(x), y]_\alpha$, (vi) $[F(x), y]_\alpha \pm [x, F(y)]_\alpha = 0$, (vii) $F([x, y]_\alpha) \pm [d(x), d(y)]_\alpha = 0$, for all $x, y \in I$ and $\alpha \in \Gamma$. However, M is not commutative.

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