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SECOND HANKEL DETERMINANT ESTIMATES VIA GEGENBAUER POLYNOMIALS

PRAMILA G. WAKHURE[1] and MALLIKARJUN G. SHRIGAN[2*]

ABSTRACT. In this paper, we introduce a novel subclass of analytic functions associated with the Mittag–Leffler-type Poisson distribution connected with the Gegenbauer polynomials. We establish the Fekete–Szegö inequalities for functions with sharp bounds and derive coefficient bounds for the initial Taylor–Maclaurin coefficients. Moreover, we obtain upper estimates for $H_2(2)$ for functions in the class.

Keywords. Analytic functions, Gegenbauer polynomial, Hankel Determinant, Mittag–Leffler-type Poisson Distribution.
2020 Mathematics Subject Classification. 30C45

1. INTRODUCTION

The class of functions f that are analytic in open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the criteria $f(0) = f'(0) - 1 = 0$ is indicated by the symbol $\mathcal{A}$. Equivalently, if $f \in \mathcal{A}$, the Taylor-Maclaurin series representation takes the form:

$$f(z) = z + \sum_{n=2}^{\infty} \delta_n z^n, \quad z \in \mathbb{U}. \tag{1.1}$$

Let $\mathcal{S}$ be a subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$. A function h is subordinate to g denoted by $h \prec g$, if there is a function η in $\mathbb{U}$ such that $\eta(0) = 0$, $|\eta(z)| < 1$, with

$$h(z) = g(\eta(z)).$$

If the function g is univalent in $\mathbb{U}$ then we know that [10]:

$$h \prec g \ (z \in \mathbb{U}) \quad \Longleftrightarrow \quad f(0) = g(0) \text{ and } h(\mathbb{U}) \subseteq g(\mathbb{U}).$$

The generalized Mittag-Leffler function was introduced by Wiman [26] as,

$$E_{\tau,\sigma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\tau k + \sigma)}, \qquad (z, \tau, \sigma \in \mathbb{C}, \Re(\tau) > 0, \ \Re(\sigma) > 0). \tag{1.2}$$

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* Corresponding author

The function $E_{\tau,\sigma}$ represents popular functions as special cases. Some of the particular cases are as follows:

$$E_{1,1}(-z) = e^{-z}, \qquad E_{2,3}(z^2) = \frac{\sinh z - z}{z^3}, \qquad E_{2,2}(z^2) = \frac{\sinh z}{z}.$$

Special functions, particularly Mittag-Leffler functions and their generalizations have also received considerable attention due to their applications in fractional calculus and geometric function theory. The classical Mittag-Leffler function was first introduced by Mittag-Leffler [11], while Wiman [26] investigated its zeros and analytic properties. Later, Kiryakova [8] developed generalized fractional calculus involving Mittag-Leffler-type operators. In the context of geometric function theory, Attiya investigated applications of Mittag-Leffler functions in the unit disk, whereas Bansal and Prajapat [3] studied geometric properties associated with Mittag-Leffler functions. Frasin [5] and Frasin et al. [6] further introduced linear operators involving generalized Mittag-Leffler functions and examined their applications to subclasses of analytic and univalent functions. Ahmad et al. [1] introduced an approach based on Mittag-Leffler-type Poisson distributions to study analytic function classes related to conic domains.

The Mittag-Leffler function (refere to FIGURE 1) is an important tool in fractional differential and integral equations. It also finds applications in generalized kinetic equations, random walk models, Lévy processes, anomalous transport phenomena and the modeling of complex dynamical systems. Various analytical and geometric properties of the Mittag–Leffler function and its generalized forms have been extensively explored in the literature for instance [3, 5, 6, 7, 8, 13, 23].

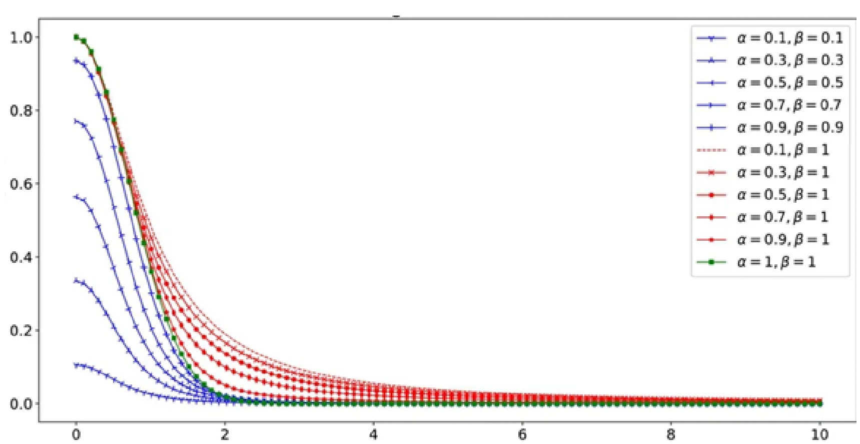

FIGURE 1. Graph of Mittag-Leffler Function

The function $E_{\tau,\sigma}(z)$ is not member of $\mathcal{A}$ so the normalization of Mittag-Leffler functions will be considered as,

$$E_{\tau,\sigma}(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(\sigma)}{\Gamma(\tau(k-1)+\sigma)} z^k.$$

The Mittag–Leffler-type Poisson distribution has the following probability mass function

$$p(r) = \frac{m^r}{\Gamma(\tau r+\sigma)E_{\tau,\sigma}(m)}, \quad r = 0, 1, 2, 3, \cdots,$$

where $m > 0$, $\tau > 0$ and $\sigma > 0$. We will restrict cases where τ, σ are real and $z \in \mathbb{U}$. Using normalized form of $E_{\tau,\sigma}(z)$ Frasin et al. [1] introduced the convolution operator as,

$$\psi^m_{\tau,\sigma} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\sigma)m^{n-1}}{\Gamma(\tau(n-1)+\sigma)E_{\tau,\sigma}(m)}\delta_n z^n, \quad z \in \mathbb{U}.$$

Hankel determinants are particularly useful in investigating the rationality of functions of bounded characteristic in the unit disk $\mathbb{U}$. The q^{th} Hankel determinant $H_q(n)$ of order n associated with $h \in \mathcal{A}$ is defined by [9, 14, 15],

$$H_q(n) := \begin{vmatrix} \delta_n & \delta_{n+1} & \delta_{n+2} & \cdots & \delta_{n+q-1} \\ \delta_{n+1} & \delta_{n+2} & \delta_{n+3} & \cdots & \delta_{n+q} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \delta_{n+q-1} & \delta_{n+q} & \delta_{n+q+1} & \cdots & \delta_{n+2q-2} \end{vmatrix} \tag{1.3}$$

with $\delta_1 = 1$ and for integer $q \geq 1$.

- Let $q = 2$, $n = 1$, we obtain,

$$H_2(1) = \begin{vmatrix} \delta_1 & \delta_2 \\ \delta_2 & \delta_3 \end{vmatrix} = \left|\delta_3 - \delta_2^2\right|, \quad (\delta_1 = 1).$$

- Let $q = 2$, $n = 2$, we have,

$$H_2(2) = \begin{vmatrix} \delta_2 & \delta_3 \\ \delta_3 & \delta_4 \end{vmatrix} = \left|\delta_2\delta_4 - \delta_3^2\right|.$$

The study of Hankel determinants has become an important topic in geometric function theory because of their close relationship with coefficient estimates, growth problems and geometric properties of analytic and univalent functions [19]. The theory of univalent functions was established in a systematic manner by Duren [4], where several coefficient problems and geometric properties of analytic functions were discussed. Earlier Rogosinski [17] established important results on subordinate functions which later became fundamental tools in coefficient estimate problems involving Hankel determinants. Furthermore, Pommerenke [14, 15] initiated the study of Hankel determinants for univalent functions and obtained several important inequalities that motivated further investigations in this direction.

In recent years, subclasses of bi-univalent functions associated with orthogonal polynomials and special functions have attracted increasing interest. Among these, Gegenbauer polynomials have emerged as a powerful tool because of their generating functions and orthogonality properties. Wanas and Cotîrlă [25] applied Gegenbauer polynomials to a family of bi-Bazilevič functions associated with the q-Srivastava–Attiya operator and derived several coefficient inequalities. Later, Saravanan et al. [18] introduced a new subclass of bi-univalent functions defined by Gegenbauer polynomials and established bounds for the initial Taylor coefficients. Their work demonstrated the effectiveness of Gegenbauer polynomial techniques in the study of bi-univalent function classes.

The investigation of Hankel determinants has recently become a rapidly developing area of research. Riaz et al. [16] studied $H_2(2)$, $H_3(1)$ for starlike and convex functions associated with the three-leaf function. Similarly, Shrigan [21, 22] investigated upper bounds for the $H_2(2)$ for subclasses associated with q-differential operators and related univalent function classes.

Moreover, Tang et al. [24] derived sharp coefficient estimates and Hankel determinant inequalities for functions associated with symmetric domains. Zeyani and Hussen [27] studied the $H_2(2)$ for subclasses of bi-univalent functions associated with Gegenbauer (ultraspherical) polynomials and obtained several coefficient bounds using subordination. Furthermore, Alnajar et al. [2] studied Hankel determinants for analytic functions connected with Bell polynomials. Recently, Naz et al. [12] investigated the $H_3(1)$ for inverse functions associated with exponential mappings.

Inspired by the above-mentioned works, we study Hankel determinant estimates for subclasses of analytic and bi-univalent functions defined in terms of Gegenbauer polynomials. By employing generating function techniques, coefficient comparison methods, and the theory of differential subordinations developed by Miller and Mocanu [10], new bounds for the Hankel determinant are obtained.The findings presented in this work broaden and build upon several earlier results available in the literature.

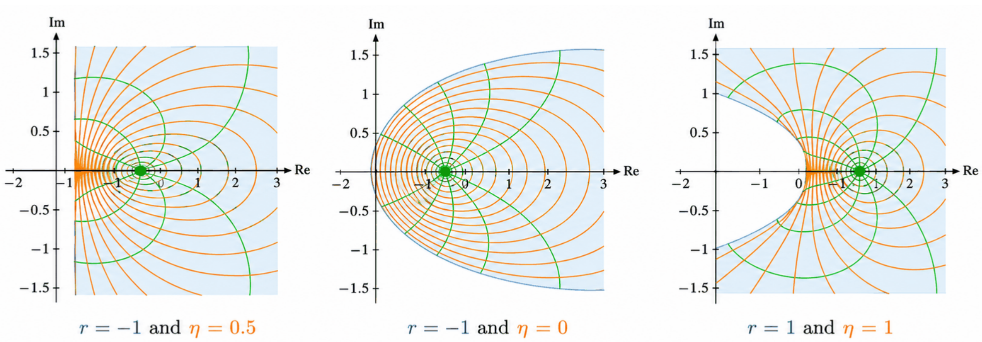

Figure 1: Image of $\mathbb{D}$ under $\mathcal{G}_\eta(r, z)$.

FIGURE 2. Image of $\mathbb{D}$ under $\mathcal{G}_\eta(r, z)$ and Graph of $\mathcal{G}_k^\eta(r)$

For a fixed parameter r, the function $\mathcal{G}_\eta(r, z)$ [18] (refer to FIGURE 2) is analytic throughout the unit disk $\mathbb{D}$ and can be represented by the series

$$\mathcal{G}_\eta(r, z) = \sum_{k=0}^{\infty} Q_k^\eta(r) z^k,$$

where $Q_k^\eta(r)$ is the Gegenbauer polynomial of order k (refer to FIGURE 3).

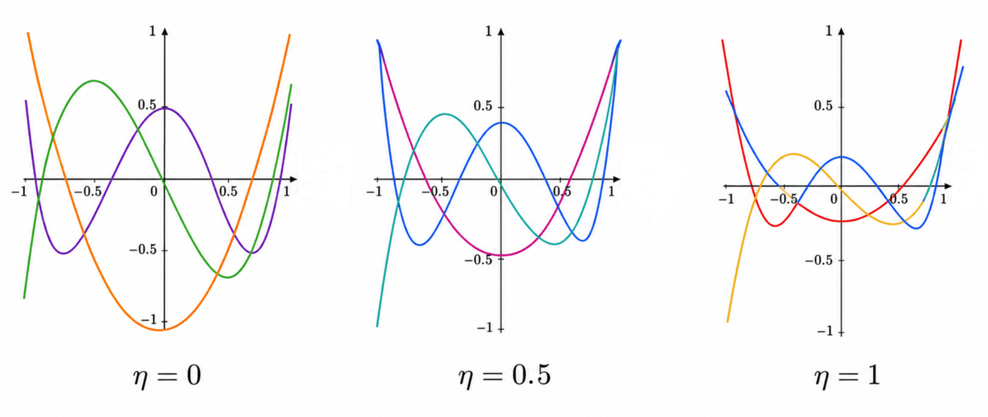

FIGURE 3. Graph of $\mathcal{Q}_k^{\eta}(r)$

Definition 1.1. A function $f(z)$ given by (1.1) belong to $\mathbb{M}_{\epsilon}^{n}(\lambda,\tau,\sigma,x)$ if,

$$1+\frac{1}{\eta}\left[(1-\epsilon)\frac{\psi_{\tau,\sigma}^{m}f(z)}{z}+\epsilon\left(\psi_{\tau,\sigma}^{m}f(z)\right)'-1\right] \prec \mathcal{G}(x,z),$$

where $\eta\in\mathbb{C}^*$; $0\leq\epsilon\leq 1$; $|x|<1$; $0<\lambda\leq 1$; $z\in\mathbb{U}$ and $\mathcal{G}(x,z)$ is Gegenbauer polynomial given by [18]:

$$\frac{1}{(1-2xz+z^2)^{\lambda}}=1+2\lambda xz+\left[2\lambda(\lambda+1)x^2-\lambda\right]z^2 + \left[\frac{4\lambda(\lambda+1)(\lambda+2)x^3}{3}-2\lambda(\lambda+1)x\right]z^3+\cdots. \tag{1.4}$$

We assume that $\eta\in\mathbb{C}^*$, $\tau,\sigma\in\mathbb{C}$, $\Re(\tau)>0$, $\Re(\sigma)>0$, $0\leq\epsilon\leq 1$, $0<\lambda\leq 1$, $|x|<1$ and $z\in\mathbb{U}$. We will use lemma mentioned below.

Lemma 1.2. [4] *Let*

$$q(z)=1+\sum_{n=1}^{\infty}u_n z^n \prec 1+\sum_{n=1}^{\infty}U_n z^n=\mathcal{Q}(z), \quad (z\in\mathbb{U}).$$

Suppose that $\mathcal{Q}$ is a univalent function on $\mathbb{U}$ with $\mathcal{Q}(\mathbb{U})$ being a convex set, then

$$|u_n|\leq|U_1|.$$

Lemma 1.3. [4] *Consider a function $p\in\mathcal{P}$ represented as,*

$$p(z)=1+u_1z+u_2z^2+\cdots, \quad (z\in\mathbb{U})$$

then for $n\in\mathbb{N}$ we have $|u_n|\leq 2$.

Lemma 1.4. [4] *Let $p\in\mathcal{P}$ For every complex number ν we have,*

$$\left|u_2-\nu u_1^2\right|\leq 2\max\{1,\ |2\nu-1|\}.$$

Lemma 1.5. [4] *Let a function $p\in\mathcal{P}$ then,*

$$2u_2=u_1^2+\kappa(4-u_1^2),$$

for some $|\kappa| \leq 1$. *And*

$$4u_3 = u_1^3 + 2(4 - u_1^2)u_1\kappa - u_1(4 - u_1^2)\kappa^2 + 2(4 - u_1^2)(1 - |\kappa|^2)z,$$

for some $|z| \leq 1$.

Lemma 1.6. [4] *The series* $p(z)$ *represents a function belonging to the Carathéodory class* $\mathcal{P}$ *in the unit disk* $\mathbb{U}$ *if and only if the Toeplitz determinants*

$$T_n = \begin{vmatrix} 2 & u_1 & u_2 & \cdots & u_n \\ u_{-1} & 2 & u_1 & \cdots & u_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{-n} & u_{-n+1} & u_{-n+2} & \cdots & 2 \end{vmatrix}, \qquad n \geq 1,$$

with $u_{-k} = \overline{u_k}$ *are non negative. Moreover, these determinants are positive unless* p *admits the representation,*

$$p(z) = \sum_{k=1}^{m} \rho_k \, p_0\left(e^{it_k} z\right),$$

where $\rho_k > 0$ *and the parameters* $t_1, t_2, \ldots, t_m$ *are distinct. In this exceptional situation,*

$$T_n > 0 \quad (n < m-1), \qquad \text{while} \qquad T_n = 0 \quad (n \geq m).$$

Motivated and stimulated by above works, here we investigate a novel subclass $\mathcal{M}_\eta^\epsilon(\lambda, \tau, \sigma, x)$ of analytic functions associated with the Mittag–Leffler-type Poisson distribution and Gegenbauer polynomials. We establish sharp Fekete–Szegö inequalities, derive bounds for the initial Taylor–Maclaurin coefficients, and obtain upper estimates for the second Hankel determinant.

2. Main Results

Theorem 2.1. *Let* $f(z) \in \mathbb{M}_\eta^\epsilon(\lambda, \tau, \sigma, x)$ *and* $\eta \in \mathbb{C}^*$*,then the coefficient bounds are given by,*

$$|\delta_{k+1}| \leq \frac{2|\lambda x||\eta|\Gamma(\tau k + \sigma)\, E_{\tau,\sigma}(m)}{\Gamma(\sigma)\, m^k (1 + k\epsilon)}.$$

Proof. Let $f(z) \in \mathbb{M}_\eta^\epsilon(\lambda, \tau, \sigma, x)$ then, we have

$$
\begin{aligned}
&1+\frac{1}{\eta}\left[(1-\epsilon)\frac{\psi^m_{\tau,\sigma}f(z)}{z}+\epsilon\left(\psi^m_{\tau,\sigma}f(z)\right)'-1\right]\\
&\quad=1+\frac{1}{\eta}\left\{(1-\epsilon)\left[1+\sum_{n=2}^{\infty}\frac{\Gamma(\sigma)\,m^{n-1}\delta_n z^{n-1}}{\Gamma(\tau(n-1)+\sigma)\,E_{\tau,\sigma}(m)}\right]\right.\\
&\quad\left.+\epsilon\left[1+\sum_{n=2}^{\infty}\frac{\Gamma(\sigma)\,m^{n-1}n\delta_n z^{n-1}}{\Gamma(\tau(n-1)+\sigma)\,E_{\tau,\sigma}(m)}\right]-1\right\}\\
&\quad=1+\frac{1}{\eta}\left\{\sum_{n=2}^{\infty}\frac{\Gamma(\sigma)\,m^{n-1}\delta_n z^{n-1}}{\Gamma(\tau(n-1)+\sigma)\,E_{\tau,\sigma}(m)}[1-\epsilon+n\epsilon]\right\}
\end{aligned}
$$

substitute $n-1=k \quad \Rightarrow \quad n=k+1$

$$
=1+\frac{1}{\eta}\left\{\sum_{k=1}^{\infty}\frac{\Gamma(\sigma)\,m^{k}\delta_{k+1} z^{k}}{\Gamma(\tau k+\sigma)\,E_{\tau,\sigma}(m)}[1+k\epsilon]\right\}
$$

By definition, we get

$$
=1+\frac{1}{\eta}\left\{\sum_{k=1}^{\infty}\frac{\Gamma(\sigma)\,m^{k}\delta_{k+1} z^{k}}{\Gamma(\tau k+\sigma)\,E_{\tau,\sigma}(m)}[1+k\epsilon]\right\}\prec 1+2\lambda xz+\left[2\lambda(\lambda+1)x^2-\lambda\right]z^2+\cdots.
$$

By Lemma 1.2 we have,

$$
\left|\frac{1}{\eta}\right|\left|\frac{\Gamma(\sigma)\,m^{k}\delta_{k+1}}{\Gamma(\tau k+\sigma)\,E_{\tau,\sigma}(m)}[1+k\epsilon]\right|\le|2\lambda x|.
$$

$$
\Rightarrow|\delta_{k+1}|\le\frac{2|\lambda x\eta|\Gamma(\tau k+\sigma)\,E_{\tau,\sigma}(m)}{\Gamma(\sigma)\,m^k(1+k\epsilon)}.
$$

□

Theorem 2.2. *If $f(z)\in\mathbb{M}^{\epsilon}_{\eta}(\lambda,\tau,\sigma,x)$ and $\eta\in\mathbb{C}^*$ then,*

$$
|\delta_2|\le\frac{2|\eta\lambda x|E_{\tau,\sigma}(m)\,\Gamma(\tau+\sigma)}{\Gamma(\sigma)\,m(1+\epsilon)}.
$$

$$
|\delta_3|\le\frac{|\eta\lambda x|\Gamma(2\tau+\sigma)\,E_{\tau,\sigma}(m)}{\Gamma(\sigma)m^2(1+2\epsilon)}\times 2\max\left\{1,\left|\frac{1}{2x}-\frac{1}{2}(\lambda+1)x\right|\right\}.
$$

$$
\begin{aligned}
|\delta_3-\mu\delta_2^2|\le{}&\frac{E_{\tau,\sigma}(m)|\eta\lambda x\Gamma(2\tau+\sigma)|}{\Gamma(\sigma)m^2(1+2\epsilon)}\\
&\times 2\max\left\{1,\frac{2\mu(1+2\epsilon)\lambda x\eta(\Gamma(\tau+\sigma))^2E_{\tau,\sigma}(m)}{\Gamma(2\tau+\sigma)\,\Gamma(\sigma)(1+\epsilon)^2}+\frac{(1-(\lambda+1)2x^2}{2x}\right\}.
\end{aligned}
$$

Proof. Let $f(z)\in\mathbb{M}^{n}_{\epsilon}(\lambda,\tau,\sigma,x)$, then

$$
1+\frac{1}{\eta}\left[(1-\epsilon)\frac{\psi^m_{\tau,\sigma}f(z)}{z}+\epsilon\left(\psi^m_{\tau,\sigma}f(z)\right)'-1\right]=\Phi(w(z)),\quad(z\in\mathbb{U})\tag{2.1}
$$

where,

$$\begin{aligned}\phi(z) &= \frac{1}{(1-2xz+z^2)^{\lambda}} \\ &= 1+2\lambda xz+\left[2\lambda(\lambda+1)x^2-\lambda\right]z^2+ \\ &\quad+\left[\frac{4\lambda(\lambda+1)(\lambda+2)x^3}{3}-2\lambda(\lambda+1)x\right]z^3+\ldots \qquad (2.2) \\ &= 1+P_1(x)z+P_2(x)z^2+P_3(x)z^3+P_4(x)z^4+\cdots . \qquad (2.3)\end{aligned}$$

Assume that $p_1(z)\in\mathcal{P}$ is analytic in $\mathbb{U}$ satisfies $\mathrm{Re}(p_1(z))>0$ and $p_1(0)=1$. Then,

$$p_1(z)=\frac{1+w(z)}{1-w(z)}=1+u_1z+u_2z^2+u_3z^3+\cdots,\qquad (z\in\mathbb{U}). \qquad (2.4)$$

Since $w(z)$ is a Schwartz function, we have

$$\begin{aligned}1+\frac{1}{\eta}&\left[(1-\epsilon)\frac{\psi^m_{\tau,\sigma}f(z)}{z}+\epsilon\left(\psi^m_{\tau,\sigma}f(z)\right)'-1\right] \\ &= 1+d_1z+d_2z^2+d_3z^3+\cdots,\qquad (z\in\mathbb{U}). \qquad (2.5)\end{aligned}$$

In view of (2.1) and (2.4) we have,

$$p(z)=\Phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right).$$

$$\frac{p_1(z)-1}{p_1(z)+1}=\frac{1}{2}\left[u_1z+\left(u_2-\frac{u_1^2}{2}\right)z^2+\left(u_3+\frac{u_1^3}{4}-u_1u_2\right)z^3+\cdots\right]. \qquad (2.6)$$

Therefore, we have

$$\begin{aligned}\Phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) &= 1+\frac{1}{2}P_1(x)u_1z+\left[\frac{1}{2}P_1(x)\left(u_2-\frac{u_1^2}{2}\right)+\frac{1}{4}P_2(x)u_1^2\right]z^2 \\ &+\left(\frac{P_1(x)}{2}\left(u_3-u_1u_2+\frac{u_1^3}{4}\right)+\frac{P_2(x)u_1}{2}\left(u_2-\frac{u_1^2}{2}\right)+\frac{P_3(x)u_1^3}{8}\right)z^3+\cdots . \qquad (2.7)\end{aligned}$$

From (2.5) and (2.7), we get

$$d_1=\frac{1}{2}P_1(x)u_1,\qquad d_2=\frac{1}{2}P_1(x)\left(u_2-\frac{u_1^2}{2}\right)+\frac{1}{4}P_2(x)u_1^2. \qquad (2.8)$$

$$d_3=\frac{P_1(x)}{2}\left(u_3-u_1u_2+\frac{u_1^3}{4}\right)+\frac{P_2(x)u_1}{2}\left(u_2-\frac{u_1^2}{2}\right)+\frac{P_3(x)u_1^3}{8}. \qquad (2.9)$$

Then, from (2.2), we obtain

$$d_1=\frac{1}{\eta}\frac{\Gamma(\sigma)\ m\delta_2(1+\epsilon)}{\Gamma(\tau+\sigma)E_{\tau,\sigma}(m)}. \qquad (2.10)$$

$$d_2=\frac{1}{\eta}\frac{\Gamma(\sigma)\ m^2\delta_3(1+2\epsilon)}{\Gamma(2\tau+\sigma)E_{\tau,\sigma}(m)}. \qquad (2.11)$$

$$d_3 = \frac{1}{\eta}\frac{\Gamma(\sigma)\, m^3\delta_4(1+3\epsilon)}{\Gamma(3\tau+\sigma)E_{\tau,\sigma}(m)}. \tag{2.12}$$

Now from (2.2), (2.5) and (2.10) we get,

$$\frac{1}{\eta}\frac{\Gamma(\sigma)\, m\delta_2(1+\epsilon)}{\Gamma(\tau+\sigma)E_{\tau,\sigma}(m)} = \lambda x u_1.$$

Hence

$$\delta_2 = \frac{\lambda x u_1 \eta \Gamma(\tau+\sigma)E_{\tau,\sigma}(m)}{\Gamma(\sigma)m(1+\epsilon)}. \tag{2.13}$$

Therefore

$$|\delta_2| \leq \frac{2|\lambda x\eta|\Gamma(\tau+\sigma)E_{\tau,\sigma}(m)}{\Gamma(\sigma)m(1+\epsilon)}.$$

Similarly, we obtain

$$\frac{1}{\eta}\frac{\Gamma(\sigma)\ m^2\delta_3(1+2\epsilon)}{\Gamma(2\tau+\sigma)E_{\tau,\sigma}(m)} = \frac{1}{2}\lambda x\left(u_2 - \frac{u_1^2}{2}\right) + \frac{1}{4}\left[2\lambda(\lambda+1)x^2 - \lambda\right]u_1^2.$$

Hence

$$\delta_3 = \frac{\eta\Gamma(2\tau+\sigma)\, E_{\tau,\sigma}(m)}{\Gamma(\sigma)\, m^2(1+2\epsilon)}\left\{\lambda x\left(u_2 - \frac{u_1^2}{2}\right) + \frac{1}{4}\left[2\lambda(\lambda+1)x^2 - \lambda\right]u_1^2\right\}.$$

Equivalently

$$\delta_3 = \frac{\eta\Gamma(2\tau+\sigma)\, E_{\tau,\sigma}(m)}{\Gamma(\sigma)\, m^2(1+2\epsilon)}\lambda x\left\{u_2 - \frac{u_1^2}{2}\left[1 - \frac{1}{2}\left(2(\lambda+1)x - \frac{1}{x}\right)\right]\right\}. \tag{2.14}$$

Now using Lemma 1.4, we get

$$|\delta_3| \leq \frac{|\eta\lambda x|\Gamma(2\tau+\sigma)\, E_{\tau,\sigma}(m)}{\Gamma(\sigma)\, m^2(1+2\epsilon)} 2\times max\left\{1, \left|\frac{1}{2x} - (\lambda+1)x\right|\right\}.$$

Similarly, we get

$$\begin{aligned}\delta_4 = \frac{\Gamma(3\tau+\sigma)\, E_{\tau,\sigma}(m)\eta}{\Gamma(\sigma)\, m^3(1+3\epsilon)}&\left\{\lambda x u_3 + u_1u_2\left[\frac{-2\lambda x + 2\lambda(\lambda+1)x^2 - \lambda}{2}\right]\right.\\ &\left.+ u_1^3\left[\frac{\lambda x - 2\lambda(\lambda+1)x^2 + \lambda - \lambda(\lambda+1)x}{4} + \frac{\lambda(\lambda+1)(\lambda+2)x^3}{6}\right]\right\}.\end{aligned} \tag{2.15}$$

Therefore, we have

$$\begin{aligned}\delta_3 - \mu\delta_2^2 =& \frac{\eta E_{\tau,\sigma}(m)\lambda x\Gamma(2\tau+\sigma)}{\Gamma(\sigma)m^2(1+2\epsilon)}\\ &\times\left\{u_2 - u_1^2\left[\frac{2x - (\lambda+1)(2x)^2 + 1}{4x} + \frac{\mu(1+2\epsilon)}{\Gamma(2\tau+\sigma)}\frac{\lambda x\eta\Gamma(\tau+\sigma)^2E_{\tau,\sigma}(m)}{\Gamma(\sigma)(1+\epsilon)^2}\right]\right\}\end{aligned} \tag{2.16}$$

$$\delta_3 - \mu\delta_2^2 = \frac{\eta E_{\tau,\sigma}(m)\lambda x\Gamma(2\tau+\sigma)}{\Gamma(\sigma)m^2(1+2\epsilon)}\left(u_2 - \nu' u_1^2\right), \tag{2.17}$$

where

$$\nu^{'} = \left[\frac{2x-(\lambda+1)(2x)^2+1}{4x} + \frac{\mu(1+2\epsilon)}{\Gamma(2\tau+\sigma)}\frac{\lambda x\eta\Gamma(\tau+\sigma)^2 E_{\tau,\sigma}(m)}{\Gamma(\sigma)(1+\epsilon)^2}\right]. \tag{2.18}$$

Using Lemma 1.3, we get

$$\begin{aligned}|\delta_3-\mu\delta_2^2| \le & \frac{E_{\tau,\sigma}(m)|\eta\lambda x|\Gamma(2\tau+\sigma)}{\Gamma(\sigma)m^2} \\ & \times 2\max\left\{1, \frac{2\mu(1+2\epsilon)\lambda x\eta(\Gamma(\tau+\sigma))^2 E_{\tau,\sigma}(m)}{\Gamma(2\tau+\sigma)\,\Gamma(\sigma)(1+\epsilon)^2} + \frac{(1-(\lambda+1)2x^2}{2x}\right\}.\end{aligned}$$

$\square$

The result is sharp for the functions

$$1+\frac{1}{\eta}\left[(1-\epsilon)\frac{\psi^m_{\tau,\sigma}f(z)}{z} + \epsilon\left(\psi^m_{\tau,\sigma}f(z)\right)' - 1\right] = \Phi(z^2) \qquad (z\in\mathbb{U}). \tag{2.19}$$

where

$$\phi(z^2) = \frac{1}{(1-2xz^2+z^4)^\lambda} = 1+2\lambda xz^2+\left[2\lambda(\lambda+1)x^2-\lambda\right]z^4+\cdots,$$

or equivalently

$$\Phi(z^2) = 1+R_1(x)z^2+R_2(x)z^4+R_3(x)z^6+\cdots, \qquad (z\in\mathbb{U}).$$

Here $d_1 = 0 \Rightarrow \delta_2 = 0$. Also, we obtain $u_1 = 0$ and

$$d_2 = R_1(x) \quad\Longrightarrow\quad \delta_3 = \frac{2\eta\lambda x\Gamma(2\tau+\sigma)E_{\tau,\sigma}(m)}{\Gamma(\sigma)\ m^2\delta_3(1+2\epsilon)}. \tag{2.20}$$

Thus, by (2.16)

$$|\delta_3-\mu\delta_2^2| \le |\frac{2|\eta\lambda x|E_{\tau,\sigma}(m)\Gamma(2\tau+\sigma)}{\Gamma(\sigma)m^2(1+2\epsilon)}. \tag{2.21}$$

Since this is $|\delta_3|$, we have

$$|a_3-\mu a_2^2| = |a_3|.$$

Hence, the estimate is sharp for the extremal function corresponding to $\Phi(z^2)$.

Theorem 2.3. *If $f(z)\in\mathbb{M}^{\epsilon}_{\eta}(\lambda,\tau,\sigma,x)$ then,*

$$|\delta_2\delta_4-\delta_3^2| \le \left(\frac{2x\lambda\eta\Gamma(2\tau+\sigma)E_{\tau,\sigma}(m)}{\Gamma(\sigma)m^2(1+2\epsilon)}\right)^2. \tag{2.22}$$

Proof. Let $f(z) \in \mathcal{M}_\eta^\epsilon(\lambda, \tau, \sigma, x)$, then from (2.13),(2.14) and (2.15), we have

$$\begin{aligned}\delta_2\delta_4 - \delta_3^2 =& \frac{\lambda x u_1 \eta \Gamma(\tau+\sigma)E_{\tau,\sigma}(m)}{\Gamma(\sigma)m(1+\epsilon)} \cdot \frac{\Gamma(3\tau+\sigma)E_{\tau,\sigma}(m)\eta}{\Gamma(\sigma)m^3(1+3\epsilon)} \\ &\times \left\{ \lambda x u_3 + u_1 u_2 \left[\frac{-2\lambda x + 2\lambda(\lambda+1)x^2 - \lambda}{2} \right] \right. \\ &\left. + u_1^3 \left[\frac{\lambda x - 2\lambda(\lambda+1)x^2 + \lambda - \lambda(\lambda+1)x}{4} + \frac{\lambda(\lambda+1)(\lambda+2)x^3}{6} \right] \right\} \\ &- \left(\frac{\eta\Gamma(2\tau+\sigma)E_{\tau,\sigma}(m)}{\Gamma(\sigma)m^2(1+2\epsilon)} \lambda x \left\{ u_2 - \frac{u_1^2}{2}\left[1 - \frac{1}{2}\left(2(\lambda+1)x - \frac{1}{x} \right) \right] \right\} \right)^2.\end{aligned} \tag{2.23}$$

For the sake of brevity, we consider

$$M = \frac{\lambda x u_1 \eta^2 \Gamma(\tau+\sigma)E_{\tau,\sigma}^2(m)}{[\Gamma(\sigma)]^2 m^4(1+\epsilon)} \cdot \frac{\Gamma(3\tau+\sigma)}{(1+3\epsilon)} > 0 \tag{2.24}$$

and

$$N = \left(\frac{\eta\Gamma(2\tau+\sigma)E_{\tau,\sigma}(m)}{\Gamma(\sigma)m^2(1+2\epsilon)} \right)^2 > 0. \tag{2.25}$$

Thus, we have

$$\begin{aligned}|\delta_2\delta_4 - \delta_3^2| = &\left| M u_1 \left\{ \lambda x u_3 + u_1 u_2 \left[\frac{-2\lambda x + 2\lambda(\lambda+1)x^2 - \lambda}{2} \right] \right. \right. \\ &\left. + u_1^3 \left[\frac{\lambda x - 2\lambda(\lambda+1)x^2 + \lambda - \lambda(\lambda+1)x}{4} + \frac{\lambda(\lambda+1)(\lambda+2)x^3}{6} \right] \right\} \\ &\left. - N \left(\lambda x u_2 - \frac{u_1^2}{2}\left[\lambda x - \frac{1}{2}\left(2\lambda(\lambda+1)x^2 - \lambda \right) \right] \right)^2 \right|.\end{aligned} \tag{2.26}$$

Let $u_1 = c$ where $c \in [0, 2]$. To derive the bound, we employ Lemma 1.5. Furthermore, we assume that $u_1 > 0$. Then using Lemma 1.6, we have

$$T_2 = \begin{vmatrix} 2 & u_1 & u_2 \\ u_1 & 2 & u_1 \\ u_2 & u_1 & 2 \end{vmatrix} = 8 + 2\Re\{u_1^2 u_2\} - 2|u_2|^2 - 4u_1^2 \geq 0 \tag{2.27}$$

which is equivalent to

$$2u_2 = u_1^2 + \kappa(4 - u_1^2). \tag{2.28}$$

for some κ, $|\kappa| \leq 1$. Then $D_3 \geq 0$ is equivalent to

$$|(4u_3 - 4u_1u_2 + u_1^3)(4 - u_1^2) + u_1(2u_2 - u_1^2)^2| \leq 2(4 - u_1^2) - 2|2u_2 - u_1^2|^2. \tag{2.29}$$

Using Lemma 1.4 and (2.23), we get

$$4u_3 = u_1^3 + 2(4 - u_1^2)u_1\kappa - u_1(4 - u_1^2)\kappa^2 + 2(4 - u_1^2)(1 - |\kappa|^2)z, \tag{2.30}$$

where $|z| \leq 1$.

Using (2.22) along with (2.24) and (2.23), we have

$$
\begin{aligned}
|\delta_2\delta_4-\delta_3^2| \leq & \Bigg| M u_1 \Bigg\{ \lambda x u_3 + u_1 u_2 \left[\frac{-2\lambda x + 2\lambda(\lambda+1)x^2 - \lambda}{2}\right] \\
& + u_1^3 \left[\frac{\lambda x - 2\lambda(\lambda+1)x^2 + \lambda - \lambda(\lambda+1)x}{4} + \frac{\lambda(\lambda+1)(\lambda+2)x^3}{6}\right] \Bigg\} \Bigg| \\
& + \left| N \left(\lambda x u_2 - \frac{u_1^2}{2}\left[\lambda, x - \frac{1}{2}\left(2\lambda(\lambda+1)x^2 - \lambda\right)\right]\right)^2 \right|. \qquad (2.31)
\end{aligned}
$$

$$
\begin{aligned}
|\delta_2\delta_4-\delta_3^2| \leq & \Bigg| M \Bigg\{ \lambda x u_1 u_3 + u_1^2 u_2 \left[\frac{-2\lambda x + 2\lambda(\lambda+1)x^2 - \lambda}{2}\right] \\
& + u_1^4 \left[\frac{\lambda x - 2\lambda(\lambda+1)x^2 + \lambda - \lambda(\lambda+1)x}{4} + \frac{\lambda(\lambda+1)(\lambda+2)x^3}{6}\right] \Bigg\} \Bigg| \\
& + \left| N \left(\lambda x u_2 - \frac{u_1^2}{2}\left[\lambda, x - \frac{1}{2}\left(2\lambda(\lambda+1)x^2 - \lambda\right)\right]\right)^2 \right|. \qquad (2.32)
\end{aligned}
$$

By using Lemma 1.4, we have

$$
\begin{aligned}
|\delta_2\delta_4-\delta_3^2| \leq \ & M \Bigg| \frac{u_1^4}{4} \Bigg\{ -\lambda(\lambda+1)x + \frac{2\lambda(\lambda+1)(\lambda+2)x^3}{3} \Bigg\} - \frac{\lambda x}{4} c^2 k^2 (4-c^2) \\
& + \frac{c^2 k}{2} \Bigg\{ \lambda x (4-c^2) + (4-c^2) \frac{-2\lambda x + 2\lambda(\lambda+1)x^2 - \lambda}{2} \Bigg\} \\
& + \frac{\lambda x c}{2}(4-c^2)(1-|k|^2) z \Bigg| \\
& + N \Bigg| \frac{c^4}{4} \Bigg\{ \frac{1}{4}\left[2\lambda(\lambda+1)x^2 - \lambda\right]^2 \Bigg\} + \frac{\lambda^2 x^2 k^2}{4}(4-c^2)^2 \\
& + \frac{k c^2 (4-c^2)}{2} \left[\lambda^2 x^2 - \lambda x \left(\lambda x - \frac{1}{2}\left(2\lambda(\lambda+1)x^2 - \lambda\right)\right)\right] \Bigg|.
\end{aligned}
$$

$$|\delta_2\delta_4-\delta_3^2| \leq \; M\left|\frac{u_1^4}{4}\left\{-\lambda(\lambda+1)x+\frac{2\lambda(\lambda+1)(\lambda+2)x^3}{3}\right\}-\frac{\lambda x}{4}c^2\rho^2(4-c^2)\right.$$
$$\left.+\frac{c^2\rho(4-c^2)}{2}\left\{\lambda x+\frac{-2\lambda x+2\lambda(\lambda+1)x^2-\lambda}{2}\right\}+\frac{\lambda xc}{2}(4-c^2)(1-\rho^2)\right|$$
$$+N\left|\frac{c^4}{4}\left\{\frac{1}{4}\left[2\lambda(\lambda+1)x^2-\lambda\right]^2\right\}+\frac{\lambda^2x^2\rho^2}{4}(4-c^2)^2\right.$$
$$\left.+\frac{\rho, c^2(4-c^2)}{2}\left[\lambda^2x^2-\lambda x\left(\lambda x-\frac{1}{2}\big(2\lambda(\lambda+1)x^2-\lambda\big)\right)\right]\right|=\mathbb{F}(\rho,c),$$

where $\rho=|k|\leq 1$ and $|z|<1$. Suppose that the maximum is attained at an interior point of the region $\{(\rho,c):\rho\in[0,1],\ c\in[0,2]\}$.

$$\frac{\partial\mathbb{F}}{\partial\rho}= \; M\left[-\frac{\lambda x}{4}c^2 2\rho(4-c^2)+\frac{\lambda xc}{2}(4-c^2)(1-2\rho)\right.$$
$$\left.+\frac{c^2(4-c^2)}{2}\left\{\lambda x+\frac{-2\lambda x+2\lambda(\lambda+1)x^2-\lambda}{2}\right\}\right]$$
$$+N\left[\frac{\lambda^2x^2 2\rho}{4}(4-c^2)^2+\frac{c^2(4-c^2)}{2}\left[\lambda^2x^2-\lambda x\left(\lambda x-\frac{1}{2}\big(2\lambda(\lambda+1)x^2-\lambda\big)\right)\right]\right].$$

It is readily seen that

$$\frac{\partial\mathbb{F}(\rho,c)}{\partial\rho}>0.$$

Thus, F increases as ρ increase and hence

$$\max_{\rho\in[0,1]}\mathbb{F}(\rho,c)=\mathbb{F}(1,c).$$

This contradicts the hypothesis that the maximum is achieved at an interior point of $[0,1]$. Now let $\mathcal{V}(c) = \mathbb{F}(1,c)$.

$$\begin{aligned}\mathcal{V}(c) = M\Bigg[&\frac{c^4}{4}\Bigg\{-\lambda^2 x+\lambda+2\lambda(\lambda+1)x^2\left(\frac{(\lambda+2)x}{3}-1\right)\Bigg\}\\&+c^2\Bigg\{2\lambda(\lambda+1)x^2-\lambda-\lambda x\Bigg\}\Bigg]\\&+N\Bigg[\frac{1}{4}\Big(2\lambda(\lambda+1)x^2-\lambda\Big)^2-\lambda x\Bigg(2\lambda(\lambda+1)x^2-\lambda+\lambda^2x^2\Bigg)\Bigg]\frac{c^4}{4}\\&+c^2\Bigg\{\lambda x\Big(2\lambda(\lambda+1)x^2-\lambda\Big)-2\lambda^2x^2\Bigg\}+4\lambda^2x^2\Bigg]\end{aligned} \tag{2.33}$$

$$\begin{aligned}\mathcal{V}'(c) = M\Bigg[&c^3\Bigg\{-\lambda^2 x+\lambda+2\lambda(\lambda+1)x^2\big((\lambda+2)x-3\big)\Bigg\}\\&+2c\Bigg\{2\lambda(\lambda+1)x^2-\lambda-\lambda x\Bigg\}\Bigg]\\&+N\Bigg[c^3\Bigg\{\frac{1}{4}\Big(2\lambda(\lambda+1)x^2-\lambda\Big)^2-\lambda x\Big(2\lambda(\lambda+1)x^2-\lambda\Big)+\lambda^2x^2\Bigg\}\\&+2c\Bigg\{\lambda x\Big(2\lambda(\lambda+1)x^2-\lambda\Big)-2\lambda^2x^2\Bigg\}\Bigg]\\&=0.\end{aligned}$$

Equation (2.29) implies that c=0, resulting in a contradiction. Thus, it follows that

$$\begin{aligned}\mathcal{V}''(c) = M\Bigg[&3c^2\Bigg\{-\lambda^2 x+\lambda+2\lambda(\lambda+1)x^2\left(\frac{(\lambda+2)x}{3}-1\right)\Bigg\}\\&+2\Bigg\{2\lambda(\lambda+1)x^2-\lambda-\lambda, x\Bigg\}\Bigg]\\&+N\Bigg[3c^2\Bigg\{\frac{1}{4}\Big(2\lambda(\lambda+1)x^2-\lambda\Big)^2-\lambda x\Bigg(2\lambda(\lambda+1)x^2-\lambda+\lambda^2x^2\Bigg)\Bigg\}\\&+2\Bigg\{\lambda x\Big(2\lambda(\lambda+1)x^2-\lambda\Big)-2\lambda^2x^2\Bigg\}\Bigg]\\&<0\end{aligned}$$

Therefore, any point at which $\mathcal{V}$ attains its maximum must be located on the boundary of the interval $c \in [0, 2]$. Since $\mathcal{V}(c) \geq \mathcal{V}(2)$, it follows that the maximum value of $\mathcal{V}$ occurs at $c = 0$. Hence, the upper bound is achieved when $\rho = 1$ and $c = 0$, which yields,

$$|\delta_2\delta_4 - \delta_3^2| \leq 4\lambda^2 x^2 N = \left(\frac{2x\lambda\eta\Gamma(2\tau+\sigma)E_{\tau,\sigma}(m)}{\Gamma(\sigma)m^2(1+2\epsilon)}\right)^2.$$

□

3. Conclusion

In this paper, we introduced a novel subclass of analytic functions related to the Mittag–Leffler-type Poisson distribution and Gegenbauer polynomials. Using subordination techniques and coefficient comparison methods, we obtained bounds for the initial Taylor–Maclaurin coefficients, established the Fekete–Szegö inequality, and derived upper estimates for the second Hankel determinant $H_2(2)$. The obtained results generalize several earlier works in the literature and provide further applications of Gegenbauer polynomials in geometric function theory.

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[1] Department of mathematics, School of Computational Sciences, Faculty of Science and Technology, JSPM University, Pune, India.

Email address: pramila.wakhure@gmail.com

[2] Department of mathematics, School of Computational Sciences, Faculty of Science and Technology, JSPM University, Pune, India.

Email address: mgshrigan@gmail.com