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ON KIRCHHOFF-TYPE VARIATIONAL INEQUALITIES WITH NONLINEARITIES

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Abstract. This paper examines the existence of nontrivial radial solutions for a class of Kirchhoff-type variational inequalities involving concave nonlinearities. Our analysis relies on non-smooth critical point theory, employing Szulkin-type functionals as the primary analytical framework to address the nonsmooth and nonlocal characteristics of the problem.

*Keywords.*Mountain pass theorem; concave; Kirchhoff; Ekeland variational principle.
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1. Introduction

This paper is devoted to the study of a Kirchhoff-type variational inequality with concave nonlinearity. We establish the existence of multiple nontrivial solutions by using nonsmooth critical point theory and Szulkin's framework.

Consider the Kirchhoff-type problem

$$\begin{cases} -\big(a+b\displaystyle\int_\Omega |\nabla u|^2\big)\Delta u \geq \alpha(x)|u|^{q-2}u + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $0 \leq \alpha(x) \in L^\infty(\Omega)$, $\alpha \not\equiv 0$, $1 < q < 2$, Ω is a smooth bounded domain of $\mathbb{R}^N$ with $N = 1, 2, 3$, $a, b > 0$, and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. The restriction on the dimension is imposed to guarantee the required Sobolev embeddings in the analysis.

In order to obtain multiple solutions for problem (1.1), we assume that the following conditions hold:

(f_1) Subcritical growth condition: there exist $c > 0$ and $p \in (1, 2^*)$ ($2^* = 6$ if $N = 3$; $2^* = \infty$ if $N = 1, 2$) such that

$$|f(x,u)| \leq c(1 + |u|^{p-1}), \qquad \forall (x,u) \in \Omega \times \mathbb{R}.$$

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(f_2) $\dfrac{F(x,u)}{|u|^4} \to +\infty$ as $|u| \to +\infty$ uniformly in $x \in \Omega$, where $F(x,u) = \int_0^u f(x,s)\,ds$. In combination with ($f_1$), this superquartic growth condition forces the admissible power range to satisfy $p > 4$.

(f_3) There exists a constant $\beta < a\lambda_1$ such that

$$4F(x,u) \le f(x,u)u + \beta|u|^2, \quad \forall (x,u) \in \Omega \times \mathbb{R},$$

where $\lambda_1 > 0$ is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$.

(f_4) $\limsup_{u\to 0} \dfrac{2F(x,u)}{au^2} < \lambda_1$ uniformly in $x \in \Omega$.

(f_5) $f(x,-u) = -f(x,u)$ for all $(x,u) \in \Omega \times \mathbb{R}$.

Variational inequalities have been extensively studied because of their applications in nonlinear analysis, mechanics, and optimization. In particular, Kirchhoff-type problems have received considerable attention due to their nonlocal structure. Motivated by the results obtained in [6], where multiple solutions for Kirchhoff-type problems with concave nonlinearities were investigated, we extend this approach to the framework of variational inequalities.

Let $H := H_0^1(\Omega)$ be the Sobolev space endowed with the inner product and norm

$$\langle u, v\rangle = \int_\Omega \nabla u \nabla v\,dx, \qquad \|u\| = \langle u, u\rangle^{1/2}.$$

We denote by $|\cdot|_p$ the usual norm in $L^p(\Omega)$. Since Ω is bounded, the embedding

$$H \hookrightarrow L^p(\Omega)$$

is continuous for $p \in [1, 2^*]$ and compact for $p \in [1, 2^*)$. Moreover, there exists $\gamma_p > 0$ such that

$$|u|_p \le \gamma_p \|u\|. \tag{1.2}$$

Let K be a closed, convex, symmetric subset of H such that $0 \in K$. The corresponding variational inequality is formulated as follows:

$$\begin{aligned} &\Big(a + b\int_\Omega |\nabla u|^2 dx\Big)\int_\Omega \nabla u(\nabla v - \nabla u)dx \\ &\quad - \int_\Omega \big[\alpha(x)|u|^{q-2}u(v-u) + f(x,u)(v-u)\big]dx \ge 0, \end{aligned} \tag{1.3}$$

for all $v \in K$.

Our main results are stated as follows.

Theorem 1.1. *Assume that (f_1), (f_2), (f_3), and (f_5) hold. Then problem (1.1) admits at least one pair of distinct nontrivial solutions.*

Theorem 1.2. *Assume that (f_1), (f_2), (f_3), and (f_4) hold. Then there exists $\alpha_0 > 0$ such that problem (1.1) admits at least two distinct nontrivial solutions whenever $\|\alpha\|_\infty < \alpha_0$.*

2. Variational Framework and Technical Lemmas

Recall that a function $u \in H$ is called a weak solution of (1.1) if

$$\Big(a+b\int_\Omega |\nabla u|^2 dx\Big)\int_\Omega \nabla u(\nabla v-\nabla u)dx-\int_\Omega \big[\alpha(x)|u|^{q-2}u(v-u)+f(x,u)(v-u)\big]dx \geq 0 \quad \forall v \in K. \tag{2.1}$$

Definition 2.1. Let X be a real Banach space and X^* its dual. Let ϕ be a functional of class C^1 and let $\psi : X \to \mathbb{R} \cup \{+\infty\}$ be proper (i.e., $\psi \not\equiv +\infty$), convex and lower semicontinuous. We say that $I = \phi + \psi$ is a Szulkin-type functional. An element $u \in X$ is called a critical point of $I = \phi + \psi$ if

$$\phi'(u)(v - u) + \psi(v) - \psi(u) \geq 0 \quad \text{for all } v \in X.$$

Definition 2.2. The functional $I = \phi + \psi$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$, denoted by $(\mathrm{PSZ})_c$, if every sequence $\{u_n\} \subset X$ such that $\lim_{n\to\infty} I(u_n) = c$ and

$$\langle \phi'(u_n), v - u_n\rangle + \psi(v) - \psi(u_n) \geq -\epsilon_n\|v - u_n\| \quad \text{for all } v \in X,$$

where $\epsilon_n \to 0$, possesses a convergent subsequence.

Theorem 2.3. [10] *Let X be a Banach space, $I = \phi+\psi : X \longrightarrow \mathbb{R}$ a Szulkin-type functional and assume that*

(i) $I(0) = 0$ and there exist $\mu, \rho > 0$ such that $I|_{\partial B_\rho} \geq \mu$;
(ii) $I(e) \leq 0$ for some $e \notin \overline{B}_\rho$.

If I satisfies $(\mathrm{PSZ})_c$ for

$$c = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I(\gamma(t)), \quad \text{with} \quad \Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = 0,\ \gamma(1) = e\},$$

then c is a critical value of I and $c \geq \mu$.

Theorem 2.4. [8] *Let X be a Banach space, and let $\phi : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function bounded from below. Given $\epsilon > 0$ and $v_0 \in X$ with $\phi(v_0) \leq \inf_X \phi + \epsilon$, there exists $v \in X$ such that*

$$\phi(v) \leq \phi(v_0), \quad \|v - v_0\| \leq \sqrt{\epsilon},$$

and

$$\phi(x) \geq \phi(v) - \sqrt{\epsilon}\,\|x - v\| \quad \forall x \in X.$$

Theorem 2.5. [10] *Let X be a Banach space, $I : X \to (-\infty, +\infty]$ a Szulkin-type functional satisfying $(\mathrm{PSZ})_c$. Assume that $I(0) = 0$ and ϕ, ψ are even. Suppose further that*

(i) there exists a subspace X_1 of X of finite codimension and numbers $\alpha, \rho > 0$ such that $I|_{\partial B_\rho\cap X_1} \geq \alpha$;
(ii) there is a finite-dimensional subspace $X_2 \subset X$ with $\dim X_2 > \operatorname{codim} X_1$ such that $I(u) \to -\infty$ as $\|u\| \to \infty$, $u \in X_2$.

Then I has at least $\dim X_2 - \operatorname{codim} X_1$ distinct pairs of nontrivial critical points.

We define the functional $\phi : H \to \mathbb{R}$ by

$$\phi(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega \Big[\frac{\alpha(x)}{q}|u|^q + F(x,u)\Big]dx. \tag{2.2}$$

Using (f_1) and the Sobolev embedding theorem, one proves easily that $\phi \in C^1(H,\mathbb{R})$. We define the indicator functional of the set K by

$$\psi_K(u) = \begin{cases} 0, & u \in K, \\ +\infty, & u \notin K. \end{cases}$$

The functional ψ_K is convex, proper and lower semicontinuous. Hence $I = \phi + \psi_K$ is a Szulkin-type functional.

Lemma 2.6. *If $u \in H$ is a critical point of $I = \phi + \psi_K$, then u is a solution of problem (1.1).*

Proof. Let $u \in H$ be a critical point of I. Then for all $v \in H$,

$$\phi'(u)(v-u) + \psi_K(v) - \psi_K(u) \geq 0.$$

We first prove that $u \in K$. If not, then $\psi_K(u) = +\infty$; taking $v = 0 \in K$ in the inequality gives a contradiction. Hence $u \in K$.

Now fix $v \in K$. Since $\psi_K(v) = \psi_K(u) = 0$, we have

$$0 \leq \phi'(u)(v-u) = (a+b\|u\|^2)\int_\Omega \nabla u(\nabla v - \nabla u)dx - \int_\Omega \big[\alpha(x)|u|^{q-2}u(v-u) + f(x,u)(v-u)\big]dx,$$

which is exactly the variational inequality (2.1). □

Lemma 2.7. *Assume that f satisfies (f_1) and (f_3). Then $I = \phi + \psi_K$ satisfies the $(\mathrm{PSZ})_c$ condition for every $c \in \mathbb{R}$.*

Proof. Let $\{u_n\} \subset H$ be such that

$$I(u_n) = \phi(u_n) + \psi_K(u_n) \to c, \qquad c \in \mathbb{R}, \tag{2.3}$$

and

$$\phi'(u_n)(v-u_n) + \psi_K(v) - \psi_K(u_n) \geq -\epsilon_n\|v-u_n\|, \quad \forall v \in H, \tag{2.4}$$

with $\epsilon_n \to 0$. By (2.3), $\{u_n\} \subset K$. We show that $\{u_n\}$ is bounded in H. From (f_3) and (2.4) we obtain

$$\begin{aligned} c + 1 + \|u_n\| &\geq \phi(u_n) - \frac{1}{4}\langle \phi'(u_n), u_n\rangle \\ &= \frac{a}{4}\|u_n\|^2 - \int_\Omega \Big[\Big(\frac{1}{q} - \frac{1}{4}\Big)\alpha(x)|u_n|^q + \frac{1}{4}f(x,u_n)u_n - F(x,u_n)\Big]dx \\ &\geq \frac{a}{4}\|u_n\|^2 - \Big(\frac{1}{q} - \frac{1}{4}\Big)\|\alpha\|_\infty \gamma_q^q \|u_n\|^q - \frac{\beta}{4}\int_\Omega |u_n|^2 dx \\ &\geq \frac{a}{4}\|u_n\|^2 - \Big(\frac{1}{q} - \frac{1}{4}\Big)\|\alpha\|_\infty \gamma_q^q \|u_n\|^q - \frac{\beta}{4\lambda_1}\|u_n\|^2 \\ &= \frac{1}{4}\Big(a - \frac{\beta}{\lambda_1}\Big)\|u_n\|^2 - \Big(\frac{1}{q} - \frac{1}{4}\Big)\|\alpha\|_\infty \gamma_q^q \|u_n\|^q. \end{aligned}$$

Since $\beta < a\lambda_1$, we have $a - \beta/\lambda_1 > 0$. Thus $\{u_n\}$ is bounded in H. Hence, passing to a subsequence,

$$u_n \rightharpoonup u \quad \text{in } H, \qquad u_n \to u \quad \text{in } L^p(\Omega)\ (1 \le p < 2^*), \qquad u_n(x) \to u(x) \text{ a.e. in } \Omega.$$

Because K is weakly closed, $u \in K$. Setting $v = u$ in (2.4) yields

$$(a+b\|u_n\|^2)\int_\Omega \nabla u_n(\nabla u - \nabla u_n)dx + \int_\Omega \big[\alpha(x)|u_n|^{q-2}u_n + f(x,u_n)\big](u-u_n)dx \ge -\epsilon_n\|u-u_n\|.$$

Hence

$$\begin{aligned}(a+b\|u_n\|^2)\|u-u_n\|^2 &\le (a+b\|u_n\|^2)\int_\Omega \nabla u(\nabla u - \nabla u_n)dx \\ &\quad + \int_\Omega \big[\alpha(x)|u_n|^{q-2}u_n + f(x,u_n)\big](u-u_n)dx + \epsilon_n\|u-u_n\|.\end{aligned}$$

By (f_1) and the strong convergence in L^p, the second term tends to 0. Also, $(a+b\|u_n\|^2)\langle u, u-u_n\rangle \to 0$. Since $\epsilon_n \to 0$, we conclude $\|u-u_n\| \to 0$. Thus $\{u_n\}$ converges strongly to $u \in H$. □

3. Proof of Main Results

3.1. **Proof of Theorem 1.1.** Let $\{e_j\}$ be an orthonormal basis of H and define $X_j = \mathbb{R}e_j$,

$$Y_l = \bigoplus_{j=1}^{l} X_j, \qquad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}, \qquad l,k \in \mathbb{N}. \tag{3.1}$$

Lemma 3.1 ([11], Lemma 3.8)**.** *If $1 \le p \le 2^*$, then*

$$\beta_k(p) := \sup_{\substack{u\in Z_k \\ \|u\|=1}} |u|_p \to 0 \quad \text{as } k \to \infty.$$

Lemma 3.2. *Suppose that (f_1) holds. Then there exist constants $\rho, \alpha > 0$ and $m \in \mathbb{N}$ such that $I|_{\partial B_\rho \cap Z_m} \ge \alpha$.*

Proof. By Lemma 3.1, we can choose m so large that

$$|u|_1 \le \beta_m(1)\|u\|, \quad |u|_p \le \beta_m(p)\|u\|, \quad |u|_q \le \beta_m(q)\|u\| \quad \forall u \in Z_m.$$

For $u \in Z_m \cap K$ (otherwise $I(u) = +\infty$), using (f_1) we have

$$\begin{aligned}I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega \Big(\frac{\alpha(x)}{q}|u|^q + F(x,u)\Big)dx \\ &\ge \frac{a}{2}\|u\|^2 - c|u|_1 - \frac{c}{p}|u|_p^p - \frac{\|\alpha\|_\infty}{q}|u|_q^q \\ &\ge \frac{a}{2}\|u\|^2 - c\beta_m(1)\|u\| - \frac{c}{p}\beta_m(p)^p\|u\|^p - \frac{\|\alpha\|_\infty}{q}\beta_m(q)^q\|u\|^q.\end{aligned}$$

Setting $\|u\| = \rho$ and using $\beta_m(\cdot) \to 0$, for sufficiently small ρ and large m the right-hand side is positive. □

Lemma 3.3. *Suppose that* (f_1) *and* (f_2) *hold. Then for any finite-dimensional subspace* $\widetilde{H} \subset H_0^1$*, there exists* $R = R(\widetilde{H}) > 0$ *such that*

$$\phi(u) \leq 0 \quad \forall u \in \widetilde{H} \setminus B_R.$$

Proof. Let $\widetilde{H} \subset Y_n$ for some n. On a finite-dimensional space all norms are equivalent, so there exists $c_4 > 0$ with $|u|_4 \geq c_4\|u\|$ for all $u \in \widetilde{H}$. By (f_2), for any $M > \frac{b}{4c_4^4}$ there is $C_M > 0$ such that

$$F(x,u) \geq M|u|^4 - C_M|u|^2, \quad \forall (x,u) \in \Omega \times \mathbb{R}.$$

Consequently,

$$\begin{aligned} I(u) &\leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - M|u|_4^4 + C_M|u|_2^2 - \int_\Omega \frac{\alpha(x)}{q}|u|^q dx \\ &\leq \frac{a}{2}\|u\|^2 - (Mc_4^4 - \tfrac{b}{4})\|u\|^4 + C_M\gamma_2^2\|u\|^2, \end{aligned}$$

and the right-hand side becomes negative for $\|u\|$ large enough. □

Now set $X = H$, $X_1 = Z_m$ (from Lemma 3.2), and $X_2 = Y_n$ with $n > m$. Clearly $I(0) = 0$ and (f_5) implies I is even. Lemmas 2.7, 3.2 and 3.3 verify the hypotheses of the multiple critical points theorem 2.5. Hence I possesses $\dim X_2 - \operatorname{codim} X_1 = n - m + 1 > 1$ distinct pairs of positive solutions.

3.2. **Proof of Theorem 1.2.** We show that I satisfies the geometry conditions of the Mountain Pass Theorem 2.3.

There exist $\beta, \rho > 0$ such that $I(u) \geq \beta$ for all u with $\|u\| = \rho$. For small $\epsilon > 0$, by (f_1) and (f_4) there exists a constant $A > 0$ such that

$$F(x,u) \leq \frac{a}{2}(\lambda_1 - \epsilon)|u|^2 + A|u|^p, \quad \forall x \in \Omega. \tag{3.1}$$

Hence

$$\begin{aligned} I(u) &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\|\alpha\|_\infty}{q}|u|_q^q - \frac{a}{2}(\lambda_1 - \epsilon)|u|_2^2 - A|u|_p^p \\ &\geq \frac{a}{2}\|u\|^2 - \frac{\gamma_q^q}{q}\|\alpha\|_\infty\|u\|^q - \frac{a}{2}\frac{\lambda_1 - \epsilon}{\lambda_1}\|u\|^2 - A\gamma_p^p\|u\|^p \\ &= \|u\|^2\Big(\frac{a\epsilon}{2\lambda_1} - \frac{\gamma_q^q}{q}\|\alpha\|_\infty\|u\|^{q-2} - A\gamma_p^p\|u\|^{p-2}\Big). \end{aligned}$$

Define

$$g(t) = \frac{\gamma_q^q}{q}\|\alpha\|_\infty t^{q-2} + A\gamma_p^p t^{p-2}, \quad t > 0.$$

Since $1 < q < 2$ and $5 \leq p < 2^*$, we have $\lim_{t\to 0^+} g(t) = \lim_{t\to\infty} g(t) = +\infty$. The minimum of g occurs at

$$t_0 = \Big[\frac{(2-q)\|\alpha\|_\infty\gamma_q^q}{q(p-2)A\gamma_p^p}\Big]^{\frac{1}{p-q}}. \tag{3.2}$$

Then

$$g(t_0) = A\gamma_p^p\Big[\frac{(2-q)\gamma_q^q\|\alpha\|_\infty}{q(p-2)A\gamma_p^p}\Big]^{\frac{p-2}{p-q}} + \frac{\gamma_q^q}{q}\|\alpha\|_\infty\Big[\frac{(2-q)\gamma_q^q\|\alpha\|_\infty}{q(p-2)A\gamma_p^p}\Big]^{\frac{q-2}{p-q}}. \tag{3.3}$$

If $\|\alpha\|_\infty < \alpha_0$ as defined in Theorem 1.2, then $g(t_0) < \frac{a\epsilon}{2\lambda_1}$. Taking $\rho = t_0$ we obtain

$$I(u) \geq \Big(\frac{a\epsilon}{2\lambda_1} - g(\rho)\Big)\rho^2 =: \beta > 0 \quad \text{for } \|u\| = \rho. \tag{3.4}$$

There exists $e \in H$ with $\|e\| > \rho$ such that $I(e) \leq 0$. From (f_2) we have

$$\lim_{|u|\to\infty} \frac{F(x,u)}{|u|^4} = +\infty \quad \text{uniformly in } x.$$

Hence for any $M > 0$ there exist $T_M > 0$ and $C_M > 0$ such that $F(x,u) \geq Mu^4 - C_M u$ for $u \geq T_M$. Choose $\phi \in H$ with $\phi > 0$ a.e. and $|\phi|_4 = 1$. Then

$$\lim_{t\to\infty} \frac{1}{t^4}\int_\Omega F(x,t\phi)dx \geq M - \lim_{t\to\infty}\frac{C_M|\phi|_1}{t^3} = M,$$

and by the arbitrariness of M,

$$\lim_{t\to\infty} \frac{1}{t^4}\int_\Omega F(x,t\phi)dx = +\infty.$$

Consequently,

$$\frac{I(t\phi)}{t^4} = \frac{a\|\phi\|^2}{2t^2} + \frac{b\|\phi\|^4}{4} - \frac{t^{q-4}}{q}\int_\Omega \alpha(x)|\phi|^q dx - \frac{1}{t^4}\int_\Omega F(x,t\phi)dx \to -\infty.$$

Thus we can pick $e = t\phi$ with t large enough so that $\|e\| > \rho$ and $I(e) \leq 0$.

Now Lemma 2.7 guarantees the Palais-Smale condition. By the Mountain Pass Theorem 2.3, there exists a critical point $u_1 \in H$ with $I(u_1) = c \geq \beta > 0$. Moreover,

$$I|_{\partial B_\rho} \geq \beta, \quad I(e) \leq 0 \quad (e \notin \overline{B}_\rho),$$

and therefore

$$c_2 := \inf_{u\in B_\rho} I(u) < 0 < \inf_{u\in\partial B_\rho} I(u). \tag{3.5}$$

Second solution via Ekeland principle. Choose $n \in \mathbb{N}^*$ such that

$$\frac{1}{n} < \inf_{u\in\partial B_\rho} I(u) - \inf_{u\in B_\rho} I(u).$$

Applying the Ekeland variational principle 2.4 to the lower semicontinuous functional I on $\overline{B}_\rho$, there exists $u_n \in B_\rho$ with

$$I(u_n) \leq \inf_{u\in B_\rho} I(u) + \frac{1}{n}, \qquad I(w) \geq I(u_n) - \frac{1}{n}\|w - u_n\| \ \forall w \in \overline{B}_\rho.$$

From (2.4) and the boundedness of ρ, the sequence $\{u_n\}$ satisfies the hypotheses of the $(\mathrm{PSZ})_{c_2}$ condition. Passing to a subsequence, $u_n \to u_2$ in H. By lower semicontinuity of ψ_K, $u_2 \in K$ and $I(u_2) = c_2 < 0$. Thus u_2 is a nontrivial critical point of I, hence a solution of (1.1) different from u_1.

Examples

Let $f : [-1,1] \times \mathbb{R} \to \mathbb{R}$ be defined by $f(x,u) = h(x)\dfrac{u^7}{1+|u|^2}$, where $h \in L^\infty([-1,1])$ and $h(x) > 0$ a.e. Then:

- $|f(x,u)| \le C|u|^5$ – (f_1) holds with $p = 6$ (since $2^* = \infty$ for $N = 1$).
- $F(x,u) \approx \frac{h(x)|u|^6}{6} \to \infty$ as $|u| \to \infty$ – (f_2) holds.
- For large $|u|$, $4F(x,u) - f(x,u)u \approx -\frac{1}{3}h(x)u^6 \le 0 \le \beta|u|^2$ – (f_3) holds with $\beta = 0 < a\lambda_1$.
- For small $|u|$, $\frac{2F(x,u)}{au^2} \approx \frac{h(x)u^6}{4a} \to 0 < \lambda_1$ – (f_4) holds.
- $f(x,-u) = -f(x,u)$ – (f_5) holds.

Thus all conditions are satisfied.

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