Gulf Journal of Mathematics
Vol 24, Issue 1 (2026) 1-13

ISSN: 2309-4966
https://doi.org/10.56947/hp9ghv59

BI-UNIVALENT FUNCTIONS WITH BOUNDED BOUNDARY ROTATION

AHMED AL-SHUJARY[1,2] and RAFID BUTI[3,4,*]

ABSTRACT. We introduce a new subclass of bi-univalent functions $\mathcal{B}_{\varrho,\gamma}^{\varkappa,\leftthreetimes}(\eta,\theta,\xi)$ associated with bounded boundary rotation by means of a linear operator defined via the convolution with the confluent hypergeometric Mittag–Leffler function. Sharp estimates are obtained for the initial coefficients $|a_2|$ and $|a_3|$, together with the Fekete–Szegö functional $|a_3-\Delta a_2^2|$. Several special cases are also discussed, showing that the obtained results unify and extend a number of known coefficient estimates for subclasses of bi-univalent functions.

Keywords. Bi-univalent functions, Mittag–Leffler function, Convolution (Hadamard product) , Fekete–Szegö inequality.
2020 Mathematics Subject Classification. Primary 30C45; Secondary 30C80

1. Introduction

Fractional calculus extends the concepts of classical calculus by allowing differentiation and integration of arbitrary real or complex orders instead of restricting them to integers. Although its origins date back more than four centuries, significant progress has been achieved during the past century due to its rich mathematical structure and its wide range of practical applications [13, 20]. Today, fractional calculus has become an effective tool for modeling phenomena with memory and hereditary effects arising in control theory, viscoelasticity, diffusion, signal processing, biology, and engineering.

The Riemann–Liouville fractional derivative is defined by

$$ {}_{a}^{RL}D_x^{\varrho}f(x)=\frac{1}{\Gamma(-\varrho)}\int_a^x (x-t)^{-\varrho-1}f(t)\,dt,\quad \Re(\varrho)<0, \tag{1.1}$$

while, for $\Re(\varrho)\geq 0$,

$$ {}_{a}^{RL}D_x^{\varrho}f(x)=\frac{d^m}{dx^m}\Big({}_{a}^{RL}D_x^{\varrho-m}f(x)\Big),\quad m=\lfloor\Re(\varrho)\rfloor+1. \tag{1.2}$$

Date: Received: Jul 6, 2026; Revised: Aug 4, 2026; Accepted: Aug 12, 2026.
* Corresponding author

Some useful formulas are

$$ {}_{0}^{RL}D_{x}^{\varrho}(x^{\gamma}) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\varrho+1)}x^{\gamma-\varrho}, \tag{1.3}$$

$$ {}_{-\infty}^{RL}D_{x}^{\varrho}(e^{\gamma x}) = \gamma^{\varrho}e^{\gamma x}. \tag{1.4}$$

The classical Mittag–Leffler function is defined by

$$E_{\varrho}(\zeta) = \sum_{n=0}^{\infty} \frac{\zeta^n}{\Gamma(\varrho n+1)}, \quad \Re(\varrho) > 0, \tag{1.5}$$

which generalizes the exponential function through the identity $E_1(\zeta) = e^{\zeta}$. Several important extensions have subsequently been introduced [25], including the two-parameter function

$$E_{\varrho,\varkappa}(\zeta) = \sum_{n=0}^{\infty} \frac{\zeta^n}{\Gamma(\varrho n+\varkappa)}, \quad \varrho, \varkappa \in \mathbb{C},\ \Re(\varrho) > 0, \tag{1.6}$$

the three-parameter generalization

$$E_{\delta,\varkappa}^{\gamma}(\zeta) = \sum_{n=0}^{\infty} \frac{(\gamma)_n\zeta^n}{n!\Gamma(\delta n+\varkappa)}, \quad \varkappa, \gamma, \delta \in \mathbb{C},\ \Re(\delta) > 0, \tag{1.7}$$

and the extension involving the generalized Beta function

$$E_{\varrho,\varkappa}^{\gamma;c}(\zeta;p) = \sum_{n=0}^{\infty} \frac{B_p(\gamma+n, c-\gamma)}{B(\gamma, c-\gamma)} \frac{(c)_n}{\Gamma(\varrho n+\varkappa)} \frac{\zeta^n}{n!}, \quad p \geq 0,\ \Re(c) > \Re(\gamma) > 0, \tag{1.8}$$

where

$$B_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{p}{t(1-t)}}\,dt, \quad \Re(p) > 0,\ \Re(x), \Re(y) > 0. \tag{1.9}$$

A further multi-parameter extension is given by [15]

$$E_{\varrho,\gamma}^{\varkappa,\curlywedge}(\zeta) = \sum_{n=0}^{\infty} \frac{\Gamma(\varkappa n+\curlywedge)}{\Gamma(\curlywedge)\Gamma(\varrho n+\gamma)n!}\zeta^n. \tag{1.10}$$

Owing to their remarkable analytical properties, Mittag–Leffler-type functions have found numerous applications in diffusion theory, optics, nanotechnology, chaotic systems, biology, and epidemiology. They have also become useful tools in geometric function theory for investigating coefficient estimates, Fekete–Szegö inequalities, and subclasses of starlike and convex functions [2, 11].

The confluent hypergeometric function $\mathbb{F}(\curlywedge;\gamma;\zeta)$ satisfies the differential equation [21]

$$\zeta\frac{d^2\mathbb{F}}{d\zeta^2} + (\gamma-\zeta)\frac{d\mathbb{F}}{d\zeta} - \curlywedge\mathbb{F} = 0, \quad (\curlywedge, \gamma, \zeta \in \mathbb{C}), \tag{1.11}$$

and, whenever $\gamma \notin \mathbb{Z}$, a second linearly independent solution is

$$W(\curlywedge;\gamma;\zeta) = \zeta^{1-\gamma}\mathbb{F}(\curlywedge-\gamma+1; 2-\gamma; \zeta). \tag{1.12}$$

Its series representation is

$$\mathbb{F}(\leftthreetimes;\gamma;\zeta)=\sum_{n=0}^{\infty}\frac{\Gamma(\gamma)\Gamma(n+\leftthreetimes)}{\Gamma(\leftthreetimes)\Gamma(n+\gamma)n!}\zeta^{n}, \tag{1.13}$$

and it satisfies the identities

$$\frac{d}{d\zeta}\mathbb{F}(\leftthreetimes;\gamma;\zeta)=\frac{\leftthreetimes}{\gamma}\mathbb{F}(\leftthreetimes+1;\gamma+1;\zeta),$$
$$\leftthreetimes\mathbb{F}(\leftthreetimes+1;\gamma+1;\zeta)=(\leftthreetimes-\gamma)\mathbb{F}(\leftthreetimes;\gamma+1;\zeta)+\gamma\mathbb{F}(\leftthreetimes;\gamma;\zeta).$$

A further extension, called the confluent hypergeometric Mittag–Leffler function, is defined by

$$\mathbb{F}_{\varrho,\gamma}^{\varkappa,\leftthreetimes}(\zeta)=\sum_{n=0}^{\infty}\frac{\Gamma(\gamma)\Gamma(\varkappa n+\leftthreetimes)}{\Gamma(\leftthreetimes)\Gamma(\varrho n+\gamma)n!}\zeta^{n},\quad(\varrho,\gamma,\varkappa,\leftthreetimes\in\mathbb{C};\ \Re(\varrho)>0). \tag{1.14}$$

Some representative special cases of $\mathbb{F}_{\varrho,\gamma}^{\varkappa,\leftthreetimes}(\zeta)$ are listed in Table 1.

Function	**Special Case**
$\mathbb{F}_{0,\gamma}^{0,\leftthreetimes}(\zeta)$	e^{ζ}
$\mathbb{F}_{0,1}^{1,1}(\zeta)$	$\frac{1}{1-\zeta}$
$\mathbb{F}_{2,1}^{1,1}(-\zeta^2)$	$\cos\zeta$
$\mathbb{F}_{\varrho,1}^{1,1}(\zeta)$	$E_{\varrho}(\zeta)$

Table 1. Some special cases of $\mathbb{F}_{\varrho,\gamma}^{\varkappa,\leftthreetimes}(\zeta)$.

The Laplace transform of the confluent hypergeometric Mittag–Leffler function is given by [26]

$$\mathcal{L}\{t^{\gamma-1}\mathbb{F}_{\varrho,\gamma}^{\varkappa,\leftthreetimes}(\mu t^{\varrho})\}=\rho^{-\gamma}(1-\mu\rho^{-\varrho})^{-\leftthreetimes},\quad\Re(\rho)>0,\ \Re(\gamma)>0. \tag{1.15}$$

It is closely connected with convolution-type integral equations arising in fractional integration [5, 8, 10, 14], including

$$\int_0^x(x-t)^{b-1}\Gamma(b)^{-1}\,{}_1F_1(a;b;c(x-t))f(t)\,dt=g(x),\quad\Re(b)>0, \tag{1.16}$$

and

$$\int_a^x(t-x)^{\gamma-1}\mathbb{F}_{\varrho,\gamma}^{\varkappa,\leftthreetimes}(\mu(t-x)^{\varrho})f(t)\,dt=g(x),\quad\Re(\gamma)>0,\ a>0. \tag{1.17}$$

Furthermore, several relations between $\mathbb{F}_{\varrho,\gamma}^{\varkappa,\leftthreetimes}$ and the Riemann–Liouville fractional derivative were established in [12]:

- $$\mathbb{F}_{\varrho,\gamma}^{\varkappa,\leftthreetimes}(\zeta)=\Gamma(\gamma)\Gamma(\leftthreetimes)\,{}_0^{RL}D_{\zeta}^{n\varkappa+\leftthreetimes-n-1}\left[\zeta^{\leftthreetimes-1}E_{\varrho,\gamma}(\zeta^{\varkappa})\right],$$
- $$\mathbb{F}_{\varrho}^{\varkappa,\leftthreetimes}(\zeta)=\Gamma(\gamma)\Gamma(\leftthreetimes)\,{}_0^{RL}D_{\zeta}^{n\varkappa+\leftthreetimes-n-1}\left[\zeta^{\leftthreetimes-1}E_{\varrho}(\zeta^{\varkappa})\right],$$

- $$\mathbb{F}^{\varkappa,\leftthreetimes}_{\varrho,\gamma}(\tau\zeta^{\varrho}) = \frac{\varrho\sin(\pi\leftthreetimes)B(1-\leftthreetimes, n-n\varkappa)}{\pi\Gamma(n-n\varkappa)\Gamma(\varrho n+\gamma-n\varkappa)}\int_0^{\zeta} w_{\varrho,\gamma,\leftthreetimes}(x;\zeta)E_{\varrho,\gamma}(\tau x^{\varrho\varkappa})\,dx,$$
  where
  $$w_{\varrho,\gamma,\leftthreetimes}(x;\zeta) = \int_x^{\zeta}(\zeta^{\varrho}-u^{\varrho})^{-n\varkappa+n-\leftthreetimes}u^{\varrho(n\varkappa+\leftthreetimes-n)-\gamma}(u-x)^{\varrho n+\gamma-n\varkappa-1}\,du.$$

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n\zeta^n, \quad \zeta\in\mathbb{D}, \tag{1.18}$$

and let $\mathcal{S}\subset\mathcal{A}$ be the class of normalized univalent functions.

Each $f\in\mathcal{S}$ possesses an inverse f^{-1} in $|\varpi| < r_0(f) \geq 1/4$ with expansion

$$g(\varpi) = \varpi - a_2\varpi^2 + (2a_2^2 - a_3)\varpi^3 - (5a_2^3 - 5a_2a_3 + a_4)\varpi^4 + \cdots. \tag{1.19}$$

A function is called bi-univalent if both f and f^{-1} are univalent in $\mathbb{D}$; the corresponding class is denoted by Σ. Some examples are listed in Table 2.

Function	**Expression**	**Inverse**
f_1	$\dfrac{\zeta}{1-\zeta}$	$\dfrac{\varpi}{1+\varpi}$
f_2	$\dfrac{1}{2}\log\left(\dfrac{1+\zeta}{1-\zeta}\right)$	$\dfrac{e^{2\varpi}-1}{e^{2\varpi}+1}$
f_3	$-\log(1-\zeta)$	$\dfrac{e^{\varpi}-1}{e^{\varpi}}$

TABLE 2. Examples of bi-univalent functions and their inverses.

The subordination relation $f\prec g$ means that $f(\zeta) = g(w(\zeta))$ for some Schwarz function w satisfying $w(0) = 0$ and $|w(\zeta)| < 1$. If g is univalent, this is equivalent to $f(0) = g(0)$ and $f(\mathbb{D})\subset g(\mathbb{D})$.

Lewin [16] initiated the study of Σ by proving $|a_2| < 1.51$. Later, Brannan and Clunie [6, 7], Netanyahu [23], and many subsequent authors [9, 18] investigated several subclasses of Σ and obtained coefficient estimates for $|a_2|$ and $|a_3|$, whereas the general coefficient problem remains open.

The classes of starlike and convex functions of order μ are characterized by

$$\frac{\zeta f'(\zeta)}{f(\zeta)} \prec \frac{1+(1-2\mu)\zeta}{1-\zeta}, \tag{1.20}$$

and

$$1+\frac{\zeta f''(\zeta)}{f'(\zeta)} \prec \frac{1+(1-2\mu)\zeta}{1-\zeta}, \tag{1.21}$$

respectively.

Ma and Minda [19] unified these subclasses by introducing an analytic function Φ satisfying $\Re(\Phi(\zeta)) > 0$, $\Phi(0) = 1$, and $\Phi'(0) > 0$. They defined

$$\mathcal{S}^*(\Phi) = \left\{ f \in \mathcal{A} : \frac{\zeta f'(\zeta)}{f(\zeta)} \prec \Phi(\zeta) \right\}, \quad \mathcal{C}(\Phi) = \left\{ f \in \mathcal{A} : 1 + \frac{\zeta f''(\zeta)}{f'(\zeta)} \prec \Phi(\zeta) \right\}, \tag{1.22}$$

and some representative choices of Φ are listed in Table 3.

Function	**Subclass**
$\zeta + \sqrt{1+\zeta^2}$	$\mathcal{S}_l^*$ (Raina & Sokoł, 2015 [27])
$1 + \frac{4}{3}\zeta + \frac{2}{3}\zeta^2$	$\mathcal{S}_C^*$ (Sharma et al., 2016 [29])
$1 + \zeta - \frac{1}{3}\zeta^2 + \frac{1}{9}\zeta^3$	$\mathcal{S}_H^*$ (Tayyah and Atshan, 2025 [31])
e^ζ	$\mathcal{S}_e^*$ (Mendiratta et al., 2015 [22])
$\cosh(\zeta)$	$\mathcal{S}_{\cosh}^*$ (Alotaibi et al., 2020 [3])

TABLE 3. Examples of subclasses defined through Ma–Minda functions Φ.

For

$$f(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n \zeta^n, \qquad h(\zeta) = \zeta + \sum_{n=2}^{\infty} b_n \zeta^n,$$

their Hadamard product is

$$(f * h)(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n b_n \zeta^n.$$

Using this product, we define

$$\mathcal{J}_{\varrho,\gamma}^{\varkappa,\curlywedge} f(\zeta) = \zeta \mathbb{F}_{\varrho,\gamma}^{\varkappa,\curlywedge}(\zeta) * f(\zeta),$$

or equivalently,

$$\mathcal{J}_{\varrho,\gamma}^{\varkappa,\curlywedge} f(\zeta) = \zeta + \sum_{n=2}^{\infty} \mathcal{L}_{\varrho}^{\varkappa}(n, \curlywedge, \gamma) a_n \zeta^n,$$

where

$$\mathcal{L}_{\varrho}^{\varkappa}(n, \curlywedge, \gamma) = \frac{\Gamma(\gamma)\Gamma(\varkappa(n-1) + \curlywedge)}{\Gamma(\curlywedge)\Gamma(\varrho(n-1) + \gamma) n!}.$$

Padmanabhan and Parvatham [1] introduced in 1975 the class $\mathcal{T}_\theta(\xi)$ of analytic functions associated with bounded boundary rotation. A function p, analytic in $\mathbb{D}$, belongs to $\mathcal{T}_\theta(\xi)$ if

$$\int_0^{2\pi} \left| \Re(p(\zeta)) - \frac{\xi}{1-\xi} \right| dt \le \theta\pi, \qquad \zeta = re^{it} \in \mathbb{D}.$$

Equivalently, $p \in \mathcal{T}_\theta(\xi)$ if and only if there exists a non-decreasing function ϕ on $[0, 2\pi]$ satisfying

$$\int_0^{2\pi} d\phi(t) = 2, \qquad \int_0^{2\pi} |d\phi(t)| \leq \theta, \qquad \theta \geq 2,$$

such that

$$p(\zeta) = \int_0^{2\pi} \frac{1 + (1 - 2\xi)\zeta e^{-it}}{1 - \zeta e^{-it}} \, d\phi(t). \tag{1.23}$$

Related classes of functions with bounded boundary and radius rotation, namely $\mathcal{V}_\theta$ and $\mathcal{R}_\theta$, were later investigated by Pinchuk [24]. The notion of order for these classes was introduced by Robertson [28] and subsequently generalized by Padmanabhan and Parvatham [1]; further developments can be found in Li *et al.* [17].

If $t \in \mathcal{T}_\theta(\xi)$ with $t(0) = 1$, then

$$\int_0^{2\pi} \frac{\Re(t(\zeta)) - \xi}{1 - \xi} \, d\zeta \leq \theta\pi, \qquad \theta \geq 2,\ 0 \leq \xi < 1.$$

Lemma 1.1. *[[17]] Let*

$$t(\zeta) = 1 + t_1\zeta + t_2\zeta^2 + \cdots \in \mathcal{T}_\theta(\xi).$$

Then

$$|t_n| \leq \theta(1 - \xi), \qquad n \geq 1,$$

and the estimate is sharp.

Classes associated with bounded boundary rotation have become an important topic in geometric function theory due to their applications in studying boundary behavior, rotation, and distortion properties of analytic and bi-univalent functions. They also provide a natural framework for constructing and investigating new operator-defined subclasses. For further developments in this direction, we refer the reader to [32, 33].

2. Coefficient Estimates and the Fekete-Szegö Problem with Bounded Boundary Rotation

Coefficient estimates play a central role in geometric function theory since they describe the growth, distortion, and geometric behavior of analytic and univalent functions. In recent years, subclasses of bi-univalent functions with bounded boundary rotation have received considerable attention because of their close connections with starlike and Bazilevič functions, as well as their relevance in complex and fractional analysis. In this work, we investigate sharp bounds for the initial coefficients $|a_2|$, $|a_3|$, and for the Fekete–Szegö functional $|a_3 - \Delta a_2^2|$.

The class $\mathcal{B}_\eta(\mu)$ of η-pseudo-starlike functions of order μ $(0 \leq \mu < 1)$, introduced by Babalola [4], consists of functions $f \in \mathcal{A}$ that satisfy

$$\frac{\zeta(f'(\zeta))^\eta}{f(\zeta)} \prec \frac{1 + (1 - 2\mu)\zeta}{1 - \zeta}.$$

Functions belonging to this class are univalent and belong to the family of Bazilevič functions of type $(1 - 1/\eta)$ and order ϱ/η.

Definition 2.1. Let $f \in \mathcal{A}$ be as in (1.18). We say f is in $\mathcal{B}_{\varrho,\gamma}^{\varkappa,\curlywedge}(\eta,\theta,\xi)$ if

$$\frac{\zeta^{1-\eta}\left(\mathcal{J}_{\varrho,\gamma}^{\varkappa,\curlywedge} f(\zeta)\right)'}{[\mathcal{J}_{\varrho,\gamma}^{\varkappa,\curlywedge} f(\zeta)]^{1-\eta}} \in \mathcal{T}_\theta(\xi) \quad \text{and} \quad \frac{\varpi^{1-\eta}\left(\mathcal{J}_{\varrho,\gamma}^{\varkappa,\curlywedge} g(\varpi)\right)'}{[\mathcal{J}_{\varrho,\gamma}^{\varkappa,\curlywedge} g(\varpi)]^{1-\eta}} \in \mathcal{T}_\theta(\xi)$$

where $\varrho,\gamma,\varkappa,\curlywedge \in \mathbb{C}$ with $\Re(\varrho) > 0$, $\eta \geq 0$, and $g(\varpi)$ is as in (1.19).

Remark 2.2. Setting $\eta = 1$ in Definition 2.1 gives $\mathcal{B}_{\varrho,\gamma}^{\varkappa,\curlywedge}(1,\theta,\xi) \equiv \mathcal{N}_{\varrho,\gamma}^{\varkappa,\curlywedge}(\theta,\xi)$. That is, f as in (1.18) belongs to $\mathcal{N}_{\varrho,\gamma}^{\varkappa,\curlywedge}(\theta,\xi)$ if

$$\left(\mathcal{J}_{\varrho,\gamma}^{\varkappa,\curlywedge} f(\zeta)\right)' \in \mathcal{T}_\theta(\xi) \quad \text{and} \quad \left(\mathcal{J}_{\varrho,\gamma}^{\varkappa,\curlywedge} g(\varpi)\right)' \in \mathcal{T}_\theta(\xi),$$

with parameters $\varrho,\gamma,\varkappa,\curlywedge \in \mathbb{C}$, $\Re(\varrho) > 0$, and $g(\varpi)$ as in (1.19).

Remark 2.3. Taking $\eta = 0$ in Definition 2.1 gives $\mathcal{B}_{\varrho,\gamma}^{\varkappa,\curlywedge}(0,\theta,\xi) \equiv \mathcal{M}_{\varrho,\gamma}^{\varkappa,\curlywedge}(\theta,\xi)$. So, $f \in \Sigma$ as in (1.18) belongs to $\mathcal{M}_{\varrho,\gamma}^{\varkappa,\curlywedge}(\theta,\xi)$ if

$$\frac{\zeta(\mathcal{J}_{\varrho,\gamma}^{\varkappa,\curlywedge} f(\zeta))'}{\mathcal{J}_{\varrho,\gamma}^{\varkappa,\curlywedge} f(\zeta)} \in \mathcal{T}_\theta(\xi) \quad \text{and} \quad \frac{\varpi(\mathcal{J}_{\varrho,\gamma}^{\varkappa,\curlywedge} g(\varpi))'}{\mathcal{J}_{\varrho,\gamma}^{\varkappa,\curlywedge} g(\varpi)} \in \mathcal{T}_\theta(\xi),$$

with $\varrho,\gamma,\varkappa,\curlywedge \in \mathbb{C}$, $\Re(\varrho) > 0$, and $g(\varpi)$ as in (1.19).

Theorem 2.4. *Let $f \in \mathcal{B}_{\varrho,\gamma}^{\varkappa,\curlywedge}(\eta,\theta,\xi)$ be of the form (1.18) with $\eta \geq 0$. Then, we have the following bounds on the initial coefficients:*

$$|a_2| \leq \sqrt{\frac{2\theta(1-\xi)}{[\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(2)]^2(\eta^2+\eta-2) + 2\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)(\eta+2)}}. \tag{2.1}$$

$$|a_3| \leq \frac{2\theta(1-\xi)}{[\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(2)]^2(\eta^2+\eta-2) + 2\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)(\eta+2)}. \tag{2.2}$$

For any $\Delta \in \mathbb{R}$, we get

$$|a_3 - \Delta a_2^2| \leq \begin{cases} \dfrac{2\theta(1-\xi)(1-\Delta)}{[\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(2)]^2(\eta^2+\eta-2) + 2\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)(\eta+2)}, & \text{if } \Delta < \Lambda, \\ \dfrac{\theta(1-\xi)}{\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)(\eta+2)}, & \text{if } \Lambda \leq \Delta \leq 2-\Lambda, \\ \dfrac{2\theta(1-\xi)(1-\Delta)}{[\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(2)]^2(\eta^2+\eta-2) + 2\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)(\eta+2)}, & \text{if } \Delta > 2-\Lambda. \end{cases} \tag{2.3}$$

where

$$\Lambda = \frac{2\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)(\eta+2)}{[\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(2)]^2(\eta^2+\eta-2)}. \tag{2.4}$$

We assume

$$2 \leq \theta \leq 4, \quad 0 \leq \xi < 1,$$

and

$$\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(2) = \frac{\Gamma(\gamma)\,\Gamma(\varkappa+\curlywedge)}{2\,\Gamma(\curlywedge)\,\Gamma(\varrho+\gamma)}, \quad \mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3) = \frac{\Gamma(\gamma)\,\Gamma(2\varkappa+\curlywedge)}{6\,\Gamma(\curlywedge)\,\Gamma(2\varrho+\gamma)}. \tag{2.5}$$

Proof. Since f belongs to $\mathcal{B}_{\varrho,\gamma}^{\varkappa,\curlywedge}(\eta,\theta,\xi)$, it follows that

$$\frac{\zeta^{1-\eta}\left(\mathcal{J}_{\varrho,\gamma}^{\varkappa,\curlywedge} f(\zeta)\right)'}{[\mathcal{J}_{\varrho,\gamma}^{\varkappa,\curlywedge} f(\zeta)]^{1-\eta}} = u(\zeta) \quad \text{and} \quad \frac{\varpi^{1-\eta}\left(\mathcal{J}_{\varrho,\gamma}^{\varkappa,\curlywedge} g(\varpi)\right)'}{[\mathcal{J}_{\varrho,\gamma}^{\varkappa,\curlywedge} g(\varpi)]^{1-\eta}} = v(\varpi) \tag{2.6}$$

where u and v are analytic in $\mathcal{T}_\theta(\xi)$, with series

$$u(\zeta) = 1 + u_1\zeta + u_2\zeta^2 + \cdots, \quad v(\varpi) = 1 + v_1\varpi + v_2\varpi^2 + \cdots. \tag{2.7}$$

The series expansions give

$$\begin{aligned}\frac{\zeta^{1-\eta}\left(\mathcal{J}_{\varrho,\gamma}^{\varkappa,\curlywedge} f(\zeta)\right)'}{[\mathcal{J}_{\varrho,\gamma}^{\varkappa,\curlywedge} f(\zeta)]^{1-\eta}} &= 1 + \mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(2)(1+\eta)a_2\zeta \\ &\quad + \frac{1}{2}\Big([\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(2)]^2(\eta^2+\eta-2)a_2^2 + 2\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)(\eta+2)a_3\Big)\zeta^2 + \cdots\end{aligned} \tag{2.8}$$

and for the inverse,

$$\begin{aligned}\frac{\varpi^{1-\eta}\left(\mathcal{J}_{\varrho,\gamma}^{\varkappa,\curlywedge} g(\varpi)\right)'}{[\mathcal{J}_{\varrho,\gamma}^{\varkappa,\curlywedge} g(\varpi)]^{1-\eta}} &= 1 - \mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(2)(1+\eta)a_2\varpi \\ &\quad + \frac{1}{2}\Big([\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(2)]^2(\eta^2+\eta-2)a_2^2 + 2\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)(\eta+2)(2a_2^2-a_3)\Big)\varpi^2 + \cdots\end{aligned} \tag{2.9}$$

Comparing coefficients with (2.6) and (2.7), we get

$$\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(2)(1+\eta)a_2 = u_1, \tag{2.10}$$

$$\frac{1}{2}[\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(2)]^2(\eta^2+\eta-2)a_2^2 + \mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)(\eta+2)a_3 = u_2, \tag{2.11}$$

$$-\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(2)(1+\eta)a_2 = v_1, \tag{2.12}$$

$$\frac{1}{2}[\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(2)]^2(\eta^2+\eta-2)a_2^2 + \mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)(\eta+2)(2a_2^2-a_3) = v_2. \tag{2.13}$$

Adding (2.10) and (2.12), we have

$$\Big[[\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(2)]^2(\eta^2+\eta-2) + 2\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)(\eta+2)\Big]a_2^2 = u_2 + v_2, \tag{2.14}$$

or

$$a_2^2 = \frac{u_2+v_2}{[\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(2)]^2(\eta^2+\eta-2) + 2\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)(\eta+2)}. \tag{2.15}$$

Now, by applying Lemma 1.1 to equation (2.15), we obtain the following upper bound for $|a_2|$:

$$|a_2|^2 \le \frac{2\theta(1-\xi)}{[\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(2)]^2(\eta^2+\eta-2) + 2\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)(\eta+2)}. \tag{2.16}$$

We now proceed to estimate the upper bound of $|a_3|$. By subtracting the two equations (2.10) and (2.12), we obtain

$$a_3 = \frac{u_2-v_2}{2\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)(\eta+2)} + \frac{u_2+v_2}{[\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(2)]^2(\eta^2+\eta-2) + 2\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)(\eta+2)}. \tag{2.17}$$

By performing some simplifications, we obtain

$$a_3 = \frac{u_2\left([\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(2)]^2(\eta^2+\eta-2)+4\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(3)(\eta+2)\right)-v_2\left([\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(2)]^2(\eta^2+\eta-2)\right)}{2\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(3)(\eta+2)\left[[\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(2)]^2(\eta^2+\eta-2)+2\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(3)(\eta+2)\right]}. \tag{2.18}$$

Next, applying Lemma 1.1 to equation (2.18), we derive the following upper bound for $|a_3|$:

$$|a_3| \leq \frac{2\theta(1-\xi)}{[\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(2)]^2(\eta^2+\eta-2)+2\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(3)(\eta+2)}. \tag{2.19}$$

Let $\Delta \in \mathbb{R}$ be fixed. Combining equations (2.15) and (2.18), we deduce that

$$\begin{aligned} a_3 - \Delta a_2^2 = &\frac{u_2 - v_2}{2\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(3)(\eta+2)} \\ &+ \frac{(1-\Delta)(u_2+v_2)}{[\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(2)]^2(\eta^2+\eta-2)+2\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(3)(\eta+2)}. \end{aligned} \tag{2.20}$$

This expression can be rewritten with a common denominator as

$$\begin{aligned} a_3 - \Delta a_2^2 = &\frac{(u_2 - v_2)\Big([\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(2)]^2(\eta^2+\eta-2)+2\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(3)(\eta+2)\Big)}{2\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(3)(\eta+2)\Big([\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(2)]^2(\eta^2+\eta-2)+2\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(3)(\eta+2)\Big)} \\ &+ \frac{2\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(3)(\eta+2)(1-\Delta)(u_2+v_2)}{2\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(3)(\eta+2)\Big([\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(2)]^2(\eta^2+\eta-2)+2\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(3)(\eta+2)\Big)}. \end{aligned} \tag{2.21}$$

Finally, by applying Lemma 1.1 to inequality (2.21), we obtain the following estimate:

$$\begin{aligned} |a_3 - \Delta a_2^2| \leq &\frac{\theta(1-\xi)\Big([\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(2)]^2(\eta^2+\eta-2)+2\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(3)(\eta+2)(2+\Delta)\Big)}{2\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(3)(\eta+2)\Big([\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(2)]^2(\eta^2+\eta-2)+2\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(3)(\eta+2)\Big)} \\ &+ \frac{\theta(1-\xi)\Big(2\Delta\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(3)(\eta+2)-[\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(2)]^2(\eta^2+\eta-2)\Big)}{2\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(3)(\eta+2)\Big([\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(2)]^2(\eta^2+\eta-2)+2\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(3)(\eta+2)\Big)}. \end{aligned} \tag{2.22}$$

which corresponds to equation (2.3). □

3. Corollaries and Special Cases

Corollary 3.1 (Case $\eta = 0$). *Under the assumptions of Theorem 2.4, for $\eta = 0$, the following bounds hold:*

$$|a_2| \leq \sqrt{\frac{2\theta(1-\xi)}{4\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(3)-2[\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(2)]^2}} \quad \textit{and} \quad |a_3| \leq \frac{2\theta(1-\xi)}{4\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(3)-2[\mathcal{L}^{\varkappa,\gamma}_{\varrho,\curlywedge}(2)]^2}$$

and for any $\Delta \in \mathbb{R}$,

$$|a_3 - \Delta a_2^2| \leq \begin{cases} \dfrac{2\theta(1-\xi)(1-\Delta)}{4\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3) - 2[\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(2)]^2} & \text{if } \Delta < \Lambda_0, \\ \dfrac{\theta(1-\xi)}{2\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)} & \text{if } \Lambda_0 \leq \Delta \leq 2 - \Lambda_0, \\ \dfrac{2\theta(1-\xi)(1-\Delta)}{4\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3) - 2[\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(2)]^2} & \text{if } \Delta > 2 - \Lambda_0, \end{cases}$$

where

$$\Lambda_0 = \frac{-2\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)}{[\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(2)]^2}.$$

Corollary 3.2 (Case $\eta = 1$)**.** *Under the assumptions of Theorem 2.4, for* $\eta = 1$, *the following bounds hold:*

$$|a_2| \leq \sqrt{\frac{\theta(1-\xi)}{3\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)}} \quad \text{and} \quad |a_3| \leq \frac{\theta(1-\xi)}{3\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)}$$

and for any $\Delta \in \mathbb{R}$,

$$|a_3 - \Delta a_2^2| \leq \begin{cases} \dfrac{\theta(1-\xi)(1-\Delta)}{3\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)} & \text{if } \Delta < 0, \\ \dfrac{\theta(1-\xi)}{3\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)} & \text{if } 0 \leq \Delta \leq 2, \\ \dfrac{\theta(1-\xi)(1-\Delta)}{3\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)} & \text{if } \Delta > 2, \end{cases}$$

Corollary 3.3 (Case $\theta = 2, \xi = 0$)**.** *Under the assumptions of the previous corollary, for* $\theta = 2$ *and* $\xi = 0$, *the following bounds hold:*

$$|a_2| \leq \sqrt{\frac{2(1-0)}{3\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)}} = \sqrt{\frac{2}{3\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)}} \quad \text{and} \quad |a_3| \leq \frac{2(1-0)}{3\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)} = \frac{2}{3\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)}$$

and for any $\Delta \in \mathbb{R}$,

$$|a_3 - \Delta a_2^2| \leq \begin{cases} \dfrac{2(1-\Delta)}{3\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)} & \text{if } \Delta < 0, \\ \dfrac{2}{3\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)} & \text{if } 0 \leq \Delta \leq 2, \\ \dfrac{2(1-\Delta)}{3\mathcal{L}_{\varrho,\curlywedge}^{\varkappa,\gamma}(3)} & \text{if } \Delta > 2. \end{cases}$$

4. Conclusions

In this paper, we introduced a new subclass of bi-univalent functions using the confluent hypergeometric Mittag–Leffler function. By defining an appropriate linear operator via convolution, we established sharp bounds on the initial coefficients a_2 and a_3 for functions in this class.

Our results include the Fekete–Szegö inequalities, providing estimates for the functional $|a_3 - \Delta a_2^2|$ for any real parameter Δ. These estimates generalize many known results in the literature and unify several coefficient bounds under a single framework.

We also examined special cases for particular values of η, showing how the bounds simplify and become more explicit. These cases highlight the flexibility of the introduced class and its connection to well-known subclasses of bi-univalent functions.

Overall, the obtained results demonstrate that the confluent hypergeometric Mittag–Leffler function is a versatile tool for generating new function classes with controlled growth and rotation. The results open up possibilities for further research in geometric function theory and applications of fractional and special function operators in complex analysis.

Acknowledgement. Acknowledgements could be placed at the end of the text but precede the references.

References

1. D.Aharonov & S.Friedland, (1975). On Functions of Bounded Boundary Rotation. Annales Academiae Scientiarum Fennicae. Series A I. Mathematica, 585. https://doi.org/10.5186/aasfm.1975.585
2. A. Alatawi, M. Darus, & B. Alamri, (2023). Applications of Gegenbauer Polynomials for Subfamilies of Bi-Univalent Functions Involving a Borel Distribution-Type Mittag-Leffler Function. Symmetry, 15, 785. https://doi.org/10.3390/sym15040785
3. A. Alotaibi, M. Arif, M. A. Alghamdi,& S. Hussain, (2020). Starlikeness Associated with Cosine Hyperbolic Function. Mathematics, 8, 1118. https://doi.org/10.3390/math8071118
4. W. G. Atshan, M. El-Ityan,& A. S. Tayyah, (2025). Fekete–Szegö Inequalities for Bi-Univalent Functions Involving Cauchy and Bernoulli Polynomials. Gulf Journal of Mathematics, 20(2), 317–329. https://doi.org/10.56947/gjom.v20i2.3194
5. O. Bazighifan, N. Alshammari, K. S. Al-Ghafri,& L. F. Iambor, (2024). Differential Equations of Fourth-Order with p-Laplacian-Like Operator: Oscillation Theorems. Mathematics, 12, 3558. https://doi.org/10.3390/math12223558
6. D. A. Brannan,& J. G. Clunie, (Eds.). (1980). Aspects of Contemporary Complex Analysis. Academic Press, London–New York. https://nla.gov.au/nla.cat-vn1431077
7. D. A. Brannan, J. Clunie,& W. E. Kirwan, (1970). Coefficient Estimates for a Class of Star-Like Functions. Canadian Journal of Mathematics, 22, 476–485.
8. R. G. Buschman, (1964). Convolution Equations with Generalized Laguerre Polynomial Kernels. SIAM Review, 6, 166–167. https://doi.org/10.1137/1006035
9. M. Çağlar, M. El-Ityan,S. J. Abd,& A. S. Tayyah, (2026). Third Hankel Determinant for a Class of Bi-Univalent Functions Defined in Balloon-Shaped Domain. Complex Analysis and Synergies, 12, 18. https://doi.org/10.1007/s40627-025-00194-w
10. A. M. Darweesh, A. S. Tayyah, S. H. Hadi,& A. A. Lupaş, (2026). Sandwich Results for Holomorphic Functions Related to an Integral Operator. Fractal and Fractional, 10, 171. https://doi.org/10.3390/fractalfract10030171
11. M. El-Ityan, T. Al-Hawary, B. A. Frasin, & I. Aldawish (2025). Third Hankel Determinant for a Subclass of Bi-Univalent Functions Related to Balloon-Shaped Domain. AIMS Mathematics, 10(11), 25452–25468. https://doi.org/10.3934/math.20251127
12. F. Ghanim, H. F. Al-Janaby,& O. Bazighifan, (2021). Some New Extensions on Fractional Differential and Integral Properties for Mittag–Leffler Confluent Hypergeometric Function. Fractal and Fractional, 5, 143. https://doi.org/10.3390/fractalfract5040143

13. R. Hilfer, (Ed.). (2000). Applications of Fractional Calculus in Physics. World Scientific, Singapore. https://doi.org/10.1142/3779
14. P. R. Khalendekar, (1965). On a Convolution Transform Involving Generalized Laguerre Polynomial as Its Kernel. Journal de Mathématiques Pures et Appliquées, 44, 195–197.
15. M. Kurulay & M. Bayram, (2012). Some Properties of the Mittag-Leffler Functions and Their Relation with the Wright Functions. Advances in Difference Equations, 2012, 181. https://doi.org/10.1186/1687-1847-2012-181
16. M. Lewin, (1967). On a Coefficient Problem for Bi-Univalent Functions. Proceedings of the American Mathematical Society, 18, 63–68. https://doi.org/10.1090/S0002-9939-1967-0206255-1
17. Y. Li, K. Vijaya, G. Murugusundaramoorthy,& H.Tang, (2020). On New Subclasses of Bi-Starlike Functions with Bounded Boundary Rotation. AIMS Mathematics, 5, 3346–3356. https://doi.org/10.3934/math.2020215
18. A. A. Lupaş, A. S. Tayyah, & J. Sokoł, (2025). Sharp Bounds on Hankel Determinants for Starlike Functions Defined by Symmetry with Respect to Symmetric Domains. Symmetry, 17, 1244. https://doi.org/10.3390/sym17081244
19. W. Ma & D. Minda, (1994). A Unified Treatment of Some Special Cases of Univalent Functions. In Proceedings of the Conference on Complex Analysis, Nankai Institute of Mathematics, Tianjin, China. International Press, Cambridge, MA, USA, 157–169.
20. S. F. Maktoof, S. J. Abd,& A. S. Tayyah, (2026). On Certain Bi-Univalent Functions Defined via the Struve Function and Symmetric Point Structure. GJOM, 23(1). https://doi.org/10.56947/jzep9t69
21. A. D. MacDonald, (1949). Properties of the Confluent Hypergeometric Function. Journal of Mathematics and Physics. https://doi.org/10.1002/sapm1949281183
22. R. Mendiratta, S. Nagpal,& V. Ravichandran, (2015). On a Subclass of Strongly Starlike Functions Associated with Exponential Function. Bulletin of the Malaysian Mathematical Sciences Society, 38(1), 365–386. https://doi.org/10.1007/s40840-014-0026-8
23. E. Netanyahu, (1969). The Minimal Distance of the Image Boundary from the Origin and the Second Coefficient of a Univalent Function in $|z| < 1$. Archive for Rational Mechanics and Analysis, 32, 100–112.
24. B. Pinchuk, (1971). Functions of Bounded Boundary Rotation. Israel Journal of Mathematics, 10, 6–16. https://doi.org/10.1007/BF02771515
25. T. R. Prabhakar, (1971). A Singular Integral Equation with a Generalized Mittag–Leffler Function in the Kernel. Yokohama Mathematical Journal, 19, 7–15.
26. T. R. Prabhakar, (1969). Two Singular Integral Equations Involving Confluent Hypergeometric Functions. Proceedings of the Cambridge Philosophical Society, 66, 71–89. https://doi.org/10.1017/S0305004100044728
27. R. K. Raina & J. Sokoł, (2015). On Coefficient Estimates for a Class of Starlike Functions. Hacettepe Journal of Mathematics and Statistics, 44(6), 1427–1433. https://doi.org/10.15672/HJMS.2015449676
28. M. S. Robertson, (1936). On the Theory of Univalent Functions. Annals of Mathematics, 37, 374–408. https://doi.org/10.2307/1968451
29. K. Sharma, N. K. Jain,& V.Ravichandran, (2016). Starlike Functions Associated with a Cardioid. Afrika Matematika, 27(5–6), 923–939. https://doi.org/10.1007/s13370-015-0387-7
30. T. S. Taha, (1981). Topics in Univalent Function Theory. Ph.D. Thesis, University of London.
31. A. S. Tayyah & W. G. Atshan, (2025). Starlikeness and Bi-Starlikeness Associated with a New Carathéodory Function. Journal of Mathematical Sciences, 290, 232–256. https://doi.org/10.1007/s10958-025-07604-8
32. A. S. Tayyah,S. Yalçın,& H. Bayram, (2025). Sharp Coefficient Bounds for a Class of Analytic Functions Related to Exponential Function. Mathematics, 13, 3878. https://doi.org/10.3390/math13233878

33. A. K. Wanas, M. El-Ityan, A. S. Tayyah, & A. Cătaş, (2025). Certain Subclasses of Bi-Univalent Functions Involving Caputo Fractional Derivatives with Bounded Boundary Rotation. Mathematics, 13, 3563. https://doi.org/10.3390/math13213563

[1] Department of Mathematics, College of Education for Pure Science, Al-Muthanna University, Iraq.

[2] Department of Accounting, College of Administration and Economics, Al-Farahidi University, Iraq.
Email address: dr.ahmedtalib@mu.edu.iq; ahmed.abd1@uoalfarahidi.edu.iq

[3] Department of Mathematics and Computer Applications, College of Science, Al-Muthanna University, Al-Muthanna, Iraq.

[4] Department of Petroleum Engineering, College of Engineering, Al-Ayen University, Thi-Qar, Iraq.
Email address: sci.rafid@mu.edu.iq