Gulf Journal of Mathematics
Vol 24, Issue 1 (2026) 1-8

ISSN: 2309-4966
https://doi.org/10.56947/vtekac63

SELF-ADJOINT EXTENSIONS OF SYMMETRIC OPERATORS

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Abstract. We consider self-adjoint extensions of composite symmetric differential operators through the Cayley transform. For fourth- and second-order operators, we compute deficiency indices and analyze spectral properties under different growth conditions. The results indicate when the composite operator's deficiency indices equal the sum of its components, and when it lies in the limit point or limit circle cases. Applications of Levinson's theorem provide explicit spectral characterizations, contributing to a rigorous functional analytic framework for operator extensions.

Keywords. Symmetric Operator Extensions, Differential Operators, Composites, Spectrum
2020 Mathematics Subject Classification. Primary 46L55; Secondary 44B20

1. Introduction and preliminaries

This manuscript formalizes and expands upon concepts outlined in a prior preprint [6]. We consider a $2n$th order formally symmetric differential expression

$$\tau y(x) = w^{-1}(x)\left(\sum_{k=0}^{n}(-1)^k(p_k(x)y^{(k)}(x))^{(k)}\right) \tag{1.1}$$

on $\mathcal{L}^2_w([0,\infty))$. Here p_k are real-valued coefficient functions with $p_n(x) \neq 0$ for all x, and $w(x) > 0$ is a weight. The expression (1.1) generates a symmetric differential operator on a suitable minimal domain. Our aim is to study self-adjoint extensions of composites of such symmetric differential operators with their commutants, to compute deficiency indices in concrete fourth- and second-order cases, and to describe spectral properties under various coefficient growth conditions.

Related work in the Gulf Journal of Mathematics that treats extension theory and spectral stability include [7]. Classical references used throughout include Eastham (1989) and Naimark (1967).

Date: Received: Jul 17, 2026; Revised: Aug 15, 2026; Accepted: Aug 22, 2026.
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1.1. **Notation.** We denote by $\mathcal{H} = \mathcal{L}^2([0,\infty))$ the Hilbert space with the usual inner product. For an unbounded operator T we write $D(T)$ for its domain, $\overline{T}$ for its closure when it exists, and T^* for its Hilbert adjoint. For two possibly unbounded operators T and S we write

$$T \leftrightarrow S$$

to mean that T and S commute on a common dense invariant core $\mathcal{D} \subseteq D(T) \cap D(S)$, i.e. $TSy = STy$ for all $y \in \mathcal{D}$ and $\mathcal{D}$ is invariant under both T and S. For unbounded operators, $TS = ST$ on a core does not automatically imply equality of adjoints; in general one only has $S^*T^* \subseteq (TS)^*$.

1.2. **Self-Adjoint Extensions.** We begin with a lemma clarifying domain issues.

Lemma 1.1. *Let T and S be densely defined symmetric operators on $\mathcal{H}$ and suppose there exists a common dense invariant core $\mathcal{D} \subseteq D(T) \cap D(S)$ such that $T \leftrightarrow S$ on $\mathcal{D}$. Then TS is symmetric on $\mathcal{D}$. Moreover, T^* and S^* commute on the corresponding adjoint domains to the extent permitted by domain inclusions, particularly $S^*T^* \subseteq (TS)^*$.*

Proof. For $x, y \in \mathcal{D}$ we have

$$\langle TSx, y\rangle = \langle Sx, Ty\rangle = \langle x, S^*Ty\rangle,$$

and since T and S map $\mathcal{D}$ into itself and commute on $\mathcal{D}$ the symmetry follows. The adjoint inclusion $S^*T^* \subseteq (TS)^*$ is the standard relation for densely defined operators. Equality would require additional domain conditions, which we do not assume in full generality here. □

The Cayley transform offers a standard approach to characterizing self-adjoint extensions. We recall that for a densely defined closed symmetric operator A the Cayley transform

$$V_A = (A - iI)(A + iI)^{-1}$$

is a partial isometry from $\mathcal{R}(A + iI)$ onto $\mathcal{R}(A - iI)$; self-adjoint extensions of A correspond to unitary extensions of V_A.

Theorem 1.2. *Let T and S be densely defined symmetric operators with a common invariant core $\mathcal{D}$ on which they commute. Then:*

1. *TS is symmetric on $\mathcal{D}$.*
2. *The closure $\overline{TS}$ (when densely defined) admits symmetric extensions; self-adjoint extensions correspond to unitary extensions of the Cayley transform of $\overline{TS}$.*
3. *If T and S are injective and the necessary domain/range conditions hold so that the Cayley transforms of T and S commute on their natural ranges, then a self-adjoint extension of $\overline{TS}$ can be realized unitarily as the composite of commuting self-adjoint extensions of T and S.*

Proof. (1) This is immediate from Lemma 1.1.

(2) By application of von Neumann's theory: a densely defined closed symmetric operator has self-adjoint extensions parametrized by partial isometries between deficiency spaces; through the Cayley transform, these correspond to unitary extensions of the partial isometry $V_{\overline{TS}}$.
(3) Suppose that we have additional hypotheses which ensure that Cayley transforms V_T and V_S commute on the relevant ranges. Under these hypotheses unitary extensions $\widetilde{V}_T, \widetilde{V}_S$ exist with $\widetilde{V}_T\widetilde{V}_S = \widetilde{V}_S\widetilde{V}_T$, and reversing the Cayley transform yields commuting self-adjoint extensions $\widetilde{T}, \widetilde{S}$ whose product gives a self-adjoint extension of $\overline{TS}$.
We note that commutativity of unbounded operators does not automatically imply commutativity of their Cayley transforms without domain control, hence the explicit domain/range hypotheses. For more details, see Nelson's examples [9]

□

1.3. **Composite Differential Operators.** We now apply the results to concrete differential operators of order four and two. Let

$$\tau_1 y(x) = w^{-1}(x)\left\{\sum_{k=0}^{2}(-1)^k(p_k(x)y^{(k)}(x))^{(k)}\right\},$$

and

$$\tau_2 y(x) = w^{-1}(x)\left\{\sum_{j=0}^{1}(-1)^j(c_j y^{(j)}(x))^{(j)}\right\},$$

be formally symmetric fourth- and second-order expressions, defined on $\mathcal{L}^2([0,\infty))$ with $w(x)$ taken as 1for simplicity. Assume $p_2(x) \neq 0$ and $c_1 \neq 0$ for all $x \geq 0$.

Under the assumptions of Theorem 1.2 we consider the sixth-order composite expression

$$\tau_0 y := \tau_2(\tau_1 y),$$

which after expansion, can be written explicitly as

$$\begin{aligned}\tau_2(\tau_1 y(x)) \;=\; & -[c_1p_2(x)y''(x)]^{(iv)} + \{c_0p_2(x)y''(x) + [c_1p_1(x)y'(x)]'\}'' - \\ & \{c_0p_1(x)y'(x) + [c_1p_0(x)y(x)]'\}' + c_0p_0(x)y(x). \qquad (1.2)\end{aligned}$$

One of the convenient ways of expressing equation 1.2 is by converting it to a first-order system by using quasi-derivatives as described by [10].

1.4. **Quasi-derivatives and first-order system.** We introduce quasi-derivatives of the form:

$$y_0 = y(x), \qquad y_1 = y'(x), \qquad y_2 = y''(x),$$
$$y_3 = (-c_1p_2y''(x))', \qquad y_4 = -y^{(4)}(x) + c_0p_2y''(x) + (c_1p_1y'(x))',$$
$$y_5 = -y^{(5)}(x) + c_0p_1y'(x) + (c_1p_0y(x))',$$

Where $Y = (y_0, \ldots, y_5)^{\text{tr}}$. Here, tr means transpose. The spectral equation $\tau_0 y = zy$ is equivalent to a first-order system

$$Y'(x,z) = \mathcal{C}(x,z)Y(x,z), \tag{1.3}$$

where $\mathcal{C}(x,z)$ is a 6×6 matrix of the Hamiltonian type

$$\mathcal{C}(x,z) = \begin{pmatrix} A(x) & B(x) \\ C(x,z) & -A(x)^* \end{pmatrix},$$

where A, B and C are 3×3 matrices with non-zero entries given as

$$A_{j,j+1} = 1,\ B_{33} = \frac{1}{-c_1 p_2},\ C_{11} = c_0 p_0 - z,\ C_{22} = c_0 p_1 + c_1 p_0, C_{33} = c_0 p_2 + c_1 p_1.$$

The system (1.3) is just but one way of expressing (1.2) into first-order system. One can equally use the Hamiltonian system with a symplectic matrix $\mathcal{J}$ but for the inner products in this case to make sense in the Hilbert space $\mathcal{L}^2([0,\infty))$, one needs some regularity conditions. For more details, see [2]. The analysis of the first order system (1.3) is based on the Levinson's theorem [3],under suitable growth/decay hypotheses.

1.5. **Growth and decay hypotheses.** We impose the growth/decay conditions used in the analysis:

$$|p_1(x)| \to \infty \quad \text{as } x \to \infty, \qquad p_0(x), p_2(x) = o(p_1(x)), \tag{1.4}$$

and

$$f^{-1}f',\ p_1^{-1}f' \in L^2([0,\infty)), \quad f^{-1}f'',\ p_1^{-1}f'',\ (f^{-1}f')^2 \in L^1([0,\infty)), \tag{1.5}$$

for $f = p_0, p_2$. These conditions are standard in the literature to control the asymptotic behavior of solutions and to apply Levinson's theorem (see [4], [5]).

2. Main results:Spectrum and Deficiency Indices

The first-order system (1.3) can be written in the form

$$Y' = (\Lambda(x,z) + R(x,z))Y,$$

where $\Lambda(x,z)$ is diagonal with entries the eigenvalues $\lambda_k(x,z)$ of $\mathcal{C}(x,z)$ and R is a remainder with integrable off-diagonal entries under the growth and decay conditions (1.4)–(1.5). The Levinson's theorem [3] states that a first-order system (1.3) that can be converted into the form

$$Y'(x,z) = (\Lambda(x,z) + R(x,z))Y(x,z)$$

has solutions of the form

$$y_k(x,z) = (e_k(x,z) + r_k(x,z)) \exp\left(\int_0^x \lambda_k(t,z)dt\right)$$

if $\lambda_k(x,z)$, the eigenvalues of $\mathcal{C}(x,z)$, satisfies the z-uniform dichotomy condition and $R_{ij}(x,z)$ is absolutely integrable. Here, $\Lambda(x,z)$ is a diagonal matix with entries $(\lambda_k(x,z))_{k=1,2,\ldots,2n}$ and $R_{ii}(x,z) = 0$. The dichotomy condition implies that for any $i \neq j$, $\Re(\lambda_i(x,z) - \lambda_j(x,z))$ is of constant sign modulo integrable

terms. Therefore, the applications of Levinson's Theorem and results in Section 1 lead to the following result:

Theorem 2.1. *Let T_1 and T_2 be the minimal symmetric differential operators generated by τ_1 and τ_2 on $\mathcal{L}^2([0,\infty))$, and let T_0 be the minimal operator generated by $\tau_0 = \tau_2 \circ \tau_1$. Assume T_1 and T_2 share a common invariant core $\mathcal{D}$ and that (1.4)–(1.5) hold. Then:*

1. *$T_1 \leftrightarrow T_2$ on $\mathcal{D}$ and $T_0 = T_1T_2$ is symmetric on $\mathcal{D}$.*
2. *If T_1 and T_2 are injective and the Cayley transforms commute on the relevant ranges, T_0 admits self-adjoint extensions expressible as composites of self-adjoint extensions of T_1 and T_2.*
3. *Assume that $\frac{p_{k-1}}{p_k} > 0$, $k = 1, 2$ and $\frac{c_0}{c_1} > 0$, then $defT_0 = (3, 3)$. Moreover, if $|p_1(x)|^{-\frac{1}{2}}$ is integrable, then $defT_0 = (6, 6)$ if $\frac{p_{k-1}}{p_k} < 0$ and $\frac{c_0}{c_1} < 0$. The spectrum of selfadjoint operator extension of T_0 is discrete.*
4. *Suppose that the signs of $p_0(x)$, $p_1(x)$, c_0 and c_1 are the same while that of $p_2(x)$ is different, then $defT_0 = defT_1 + defT_2$.*
5. *Absolutely continuous spectrum of the selfadjoint operator extension of T_0 of various multiplicities can only be obtained if $|p_1(x)|^{-\frac{1}{2}}$ is not integrable and at least one of the rational functions $\frac{p_{k-1}}{p_k}$ and $\frac{c_0}{c_1}$ is negative.*

Proof. The proof will be divided into sections.

1. follows from the operator-theoretic discussion in section 1.
2. follows from the domain hypotheses in section 1
3. Assume that $\frac{p_{k-1}}{p_k}$ and $\frac{c_0}{c_1}$, $k = 1, 2$ are positive or negative definite for all $x \in [0,\infty)$, then by application of z-uniform Levinson's theorem [3], we need to perform asymptotic integration on the first order system (1.3) to obtain solutions. This requires the computation of the eigenvalues $\lambda_n(x,z)$, $n = 1,\ldots,6$. In a summarised version, if the first order system (1.3) can be converted into Levinson's form $Y'(x,z) = [\Lambda(x,z) + R(x,z)]Y(x,z)$ where $\Lambda(x,z)$ is a diagonal matix with entries $(\lambda_k(x,z))_{k=1,2,\ldots,2n}$ and $R(x,z)$ is such that $\lambda_n(x,z)$ satisfies the z-uniform dichotomy condition $(\Re(\lambda_s(x,z) - \lambda_m(x,z))$ is constant sign modulo integrable terms and $R_{ii}(x,z) = 0$, $R_{ij}(x,z) \in \mathcal{L}^1$, then the solutions of (1.3) have asymptotic form of

$$y_n(x,z) = (e_n(x,z) + o(1)) \exp(\int_0^x \lambda_n(t,z)dt). \tag{2.1}$$

In order to compute the eigenvalues, we determine $\mathcal{P}(x,\lambda,z) = \det(\mathcal{C}(x,z) - \lambda \cdot I_6)$ that results into

$$\mathcal{P}(x,\lambda,z) = -c_1p_2\lambda^6 + (c_0p_2 + c_1p_1)\lambda^4 - (c_0p_1 + c_1p_0)\lambda^2 + c_0p_0 - z. \tag{2.2}$$

Assuming that (1.4) holds and approximating the λ-roots, only the leading terms, by application of Eastham's techniques [5], we obtain three clusters of roots of equal magnitudes as follows

$$\lambda_{1\pm} \approx \pm(\frac{p_0}{p_1})^{\frac{1}{2}} + O(p_1^{-\frac{3}{2}}), \quad \lambda_{2\pm} \approx \pm(\frac{c_0}{c_1})^{\frac{1}{2}} + O(p_1^{-1}),$$

$$\lambda_{3\pm} \approx \pm(\frac{p_1}{p_2})^{\frac{1}{2}} + O(p_1^{-\frac{1}{2}}).$$

Here, because $|p_1(x)| \nearrow \infty$ as $x \to \infty$, the magnitude of the roots obeys the relation $\mid \lambda_{1\pm} \mid \ll \mid \lambda_{2\pm} \mid \ll \mid \lambda_{3\pm} \mid$. The leading terms for all the three pairs of the roots are pure real if $\frac{p_{k-1}}{p_k} > 0,\ k = 1, 2$ and $\frac{c_0}{c_1} > 0$. By results of Behncke [1], the $z-$ uniform dichotomy condition is satisfied since each pair of roots will lead to one square integrable and one non-square integrable solution irrespective of the uniform dichotomy condition. But if any of the rational functions $\frac{p_{k-1}}{p_k} < 0,\ k = 1, 2$ and $\frac{c_0}{c_1} < 0$ for all $x \in [0, \infty)$, then the associated root is pure imaginary and hence the z-uniform dichotomy condition is proved off the real axis. Suppose that $z = z_0 + i\eta$, z_0, η real numbers, $\eta > 0$ and small, then we have for the $\lambda_{1\pm}$ if $\frac{p_0}{p_1} < 0$, that

$$\lambda_{1\pm}(x, z) \approx \pm i \mid \frac{p_0}{p_1} \mid^{\frac{1}{2}} \left\{1 + \frac{z}{2c_0p_0} + \text{lower order terms}\right\}.$$

The second term within the brackets on the right hand side has the influence of Imz which results into a real number approximately $\mp\frac{\eta}{2|c_0p_1^{\frac{1}{2}}p_0^{\frac{1}{2}}|}$. Therefore, off the real axis we have

$$\text{Re}\lambda_{1\pm}(x, z) \approx \mp\frac{\eta}{2|c_0p_1^{\frac{1}{2}}p_0^{\frac{1}{2}}|}.$$

A similar argument can be applied to determine explicitly the correction terms or the real parts of λ-roots off the real axis. Approximately, for $\lambda_{1\pm}$, $\lambda_{2\pm}$ and $\lambda_{3\pm}$ the real parts as a results of correction terms are $O(p_1^{-\frac{1}{2}})$, $O(p_1^{-1})$ and $O(p_1^{-\frac{3}{2}})$ respectively. Off real axis, the roots $\lambda_{n\pm}$ satisfy the z-uniform dichotomy condition since $\Re(\lambda_{n\pm})$ has different signs. For any $\lambda_{s\pm}$ and $\lambda_{m\pm}$, $s \neq m$ the z-uniform dichotomy condition is satisfied since the correction terms are of different sizes as explained above. The system can then be diagonalised twice to achieve the Levinson's form using the eigenvectors u_n as the columns of the diagonalising matrix. Here, $\mathcal{C}(x, z)u_n = \lambda_n(x, z)u_n$. Once the Levinson's form has been obtained, application of Levinson's theorem will result into solutions of the form (2.1). The square integrability of the solutions depends on the sign of the real part of the eigenvalues $\lambda_n(x, z)$. If $\frac{p_{k-1}}{p_k} > 0,\ k = 1, 2$ and $\frac{c_0}{c_1} > 0$, then each pair of eigenvalues $\lambda_{1\pm}(x, z), \lambda_{2\pm}(x, z)$ and $\lambda_{3\pm}(x, z)$ will have one root with positive real part and the other with negative real part. Thus, each pair will lead to one square integrable solution and one non-square integrable solution. Hence, def$T_0 = (3, 3)$ and T_0 is said to be in limit point case. The spectrum of self-adjoint extensions of T_0 is continuous, typically covering $[0, \infty)$. Assume that $\frac{p_{k-1}}{p_k} < 0,\ k = 1, 2$, $\frac{c_0}{c_1} < 0$, and suppose that $|p_1(x)|^{-\frac{1}{2}}$ is integrable, then all the eigensolutions $y_n(x, z)$ will be square integrable both in the upper and lower half planes and hence

def$T_0 = (6, 6)$. This is limit circle case and the spectrum of self-adjoint extensions of T_0 is purely discrete.

(4) We solve the equations $\tau_1 y(x) = zy(x)$ and $\tau_2 y(x) = zy(x)$, then when the associated characteristic polynomials to the first order systems of these equations are equated to zero we obtain respectively $p_2\lambda^4 - p_1\lambda^2 + p_0 - z = 0$ and $-c_1\lambda^2 + c_0 - z = 0$. The zeros of the first polynomial with $p_1(x)$ dominant can be approximated from

$$\lambda_1^2 \approx \frac{p_1}{p_2} + \frac{p_0 - z}{p_1} + O(p_1^{-3}), \quad \lambda_2^2 \approx -\frac{p_0 - z}{p_1} + O(p_1^{-3}).$$

The second polynomial has roots given by $\lambda = (\frac{c_0 - z}{c_1})^{\frac{1}{2}}$. Now suppose that $p_2 > 0$, and all the other coefficients p_1, p_0, c_0 and c_1 are negative, then all the roots in the first polynomial are pure imaginary and if $\mid p_1 \mid^{-\frac{1}{2}}$ is integrable then def$T_1 = (4, 4)$ and if $\mid p_1 \mid^{-\frac{1}{2}}$ is not integrable then def$T_1 = (2, 2)$. In both cases the second polynomial will lead to eigenvalues with one root having negative real part and the other positive real part. Thus def$T_2 = (1, 1)$. These implies that defT_1 + def$T_2 = (5, 5)$ if $\mid p_1 \mid^{-\frac{1}{2}}$ is integrable and defT_1 + def$T_2 = (3, 3)$ if $\mid p_1 \mid^{-\frac{1}{2}}$ is not integrable. By the results in (iii) above, the assumed conditions in the coefficients will also lead to def$T_0 = (5, 5)$ when $\mid p_1 \mid^{-\frac{1}{2}}$ is integrable and def$T_0 = (3, 3)$ when $\mid p_1 \mid^{-\frac{1}{2}}$ is not integrable. A similar analysis confirms the relationship between the deficiency indices of T_0, T_1 and T_2 when $p_2 < 0$, and all the other coefficients p_1, p_0, c_0 and c_1 are positive.

(5) Suppose that $|p_1(x)|^{-\frac{1}{2}}$ is not integrable, then if $\frac{c_0}{c_1} < 0$, then one of the eigensolutions associated with $\lambda_{2\pm}$ will lose its square integrability as $\eta \to 0$ thereby contributing to absolutely continuous spectrum. Similarly, if any of the rational functions $\frac{p_{k-1}}{p_k} < 0$, then one of the associated eigensolution will lose its square integrability as $\eta \to 0$. This will also lead to absolutely continuous spectrum. Thus the absolutely continuous spectrum of the selfadjoint operator extension of T_0 is contained in $[0, \infty)$ if $p_1 > 0$, and it is the entire real line if $p_1 < 0$. Spectral multiplicities of the absolutely continuous spectrum of the selfadjoint operator extension of T_0 are as given in the table below:

Sign of $\frac{p_0}{p_1}$	Sign of $\frac{c_0}{c_1}$	Sign of $\frac{p_1}{p_2}$	Spectral multiplicity
Positive	Positive	Positive	0
Positive	Positive	Negative	1
Positive	Negative	Positive	1
Negative	Positive	Positive	1
Negative	Negative	Positive	2
Negative	Positive	Negative	2
Positive	Negative	Negative	2
Negative	Negative	Negative	3

□

Remark 2.2. The requirement of a common invariant core is essential to avoid pitfalls associated with commutativity of unbounded operators. Where possible, one may replace the core hypothesis by resolvent commutativity or spectral measure commutativity; each notion has different consequences for extension theory.

Remark 2.3. The Cayley transform method used here is constructive but requires care: commutativity of operators on a core does not automatically imply commutativity of their Cayley transforms without verifying domain/range compatibility. Nelson's examples illustrate such subtleties.

Acknowledgement. The authors thank the anonymous referees for constructive comments that helped improve the clarity and rigor of the manuscript.

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