Gulf Journal of Mathematics
Vol 4, Issue 1 (2016) 10-26

ON THE HILBERT FUNCTION OF ZERO-DIMENSIONAL SCHEMES OF THE SMOOTH QUADRIC SURFACE

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Abstract. We study the vanishing of $h^1(\mathcal{I}_Z(a_1,a_2))$ for a zero-dimensional scheme $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$, when $\deg(Z) \le 4a_2+1$ and $a_1 \ge a_2$.

1. Introduction

The postulation of a finite set (or a zero-dimensional scheme) in a projective space $\mathbb{P}^r$ is a classical topic (see the references in [1]). In many cases one needs zero-dimensional schemes ([2], [3], [4]). In the case of Segre-Veronese embeddings one considers zero-dimensional subschemes of a multiprojective space $\mathbb{P}^{n_1} \times \cdots \mathbb{P}^{n_k}$ (they arise in the study of partially symmetric tensors). For the case $k = 2$ we needed some lemmas for $\mathbb{P}^1 \times \mathbb{P}^1$ and we consider a huge generalization of them here (the short Remark 2.1 and Lemmas 2.2 and 2.3 will also written down in a future revision of [2]). In this paper we prove the following results.

Theorem 1.1. *Fix an integer $a \ge 10$. If a is odd, then assume $a \ge 15$. Let $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a zero-dimensional scheme such that $\deg(Z) \le 4a+1$. We have $h^1(\mathcal{I}_Z(a,a)) > 0$ if and only if one of the following cases occurs:*

1. *there are $(o_1,o_2) \in \{(1,0),(0,1),(1,1),(2,1),(1,2)\}$ and $D \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(o_1,o_2)|$ with $\deg(Z \cap D) \ge (o_1+o_2)a+2$;*
2. *Z is the complete intersection of an element of $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,2)|$ and an element of $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,a)|$;*
3. *$\deg(Z) = 4a+1$ and Z is contained in an element of $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,2)|$.*

The main technical skill is to avoid some numerical difficulties with low a and get the full result for higher a. For low a we know the following example (which together with the cases listed in Theorem 1.1 is the only one when $a = 3$).

Example 1.2. Let $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a complete intersection of $A \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(3,2)|$ and $B \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,3)|$. Z is a Cartier divisor of A and $\deg(Z) = 13$. Since $\mathcal{O}_A(3,3) \cong \omega_A(2,3)$ and $h^0(\mathcal{O}_A) = 1$, we have $h^1(A,\mathcal{I}_{Z,A}(3,3)) = 1$. Hence $h^1(\mathcal{I}_Z(3,3)) > 0$. Moreover $h^1(\mathcal{I}_{Z'}(3,3)) = 0$ for all $Z' \nsubseteq Z$.

We prove the following result.

Date: Received: Aug 14, 2015; Accepted: Oct 23, 2015.
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2010 *Mathematics Subject Classification.* Primary 14J99; Secondary 14N05.
Key words and phrases. Hilbert function; postulation; smooth quadric surface; zero-dimensional scheme.

Proposition 1.3. *Fix an integer $a \geq 0$. Let $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a zero-dimensional scheme with $\deg(Z) \leq 4a+1$ and $h^1(\mathcal{I}_Z(a,a)) > 0$. If $a = 0, 1, 2$, then Z is as in Theorem 1.1. If $a = 3$, then either $\deg(Z) = 13$ and Z is as in Example 1.2 or Z is as in Theorem 1.1. If $a = 4$, then there is $F \in |\mathcal{O}(3,3)|$ with $h^1(\mathcal{I}_{F\cap Z}(4,4)) > 0$. If $a \geq 5$, then there is $F \in |\mathcal{O}(4,4)|$ with $h^1(\mathcal{I}_{F\cap Z}(a,a)) > 0$*

For unbalanced line bundles, say $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a_1, a_2)$ with $a_1 > a_2$ we have to modify statements like Theorem 1.1 if we look at schemes $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$ with $\deg(Z) \leq 2a_1 + 2a_2 + 1$ (see Example 3.1).

Proposition 1.4. *Fix integers $a_1 > a_2 \geq 15$. Let $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a zero-dimensional scheme with $\deg(Z) \leq 4a_2 + 1$. We have $h^1(\mathcal{I}_Z(a_1, a_2)) = 0$ if and only if there are $(o_1, o_2) \in \{(1,0), (0,1), (1,1), (2,1), (1,2)\}$ and $D \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(o_1, o_2)|$ such that $\deg(D \cap Z) \geq a_1o_2 + a_2o_1 + 2$.*

The interested reader may easily find several different extension of Proposition 1.3 to the case $a_1 > a_2$.

2. The main lemmas

Set $\mathcal{O} := \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}$. For any zero-dimensional scheme $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$ and for any divisor $D \in |\mathcal{O}(o_1, o_2)|$ let $\mathrm{Res}_D(Z)$ denote the closed subscheme of $\mathbb{P}^1 \times \mathbb{P}^1$ with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal sheaf.

We have $\mathrm{Res}_D(Z) \subseteq Z$ and $\deg(Z) = \deg(\mathrm{Res}_D(Z)) + \deg(D \cap Z)$. For all $(m, n) \in \mathbb{Z}^2$ we have a residual exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Z)}(m - o_1, n - o_2) \to \mathcal{I}_Z(m, n) \to \mathcal{I}_{D\cap Z,Z}(m, n) \to 0$$

Remark 2.1. Fix $T \in |\mathcal{O}(1, 0)|$ and a zero-dimensional scheme $Z \subset T$. Fix $(a_1, a_2) \in \mathbb{N}^2$. We have $h^1(\mathcal{I}_Z(a_1, a_2)) > 0$ if and only if $\deg(Z) \geq a_2 + 2$. We will silently use a similar statement for the other ruling of $\mathbb{P}^1 \times \mathbb{P}^1$ and for a statement similar to Lemma 2.5 with $T \in |\mathcal{O}(1, 2)|$.

Lemma 2.2. *Fix $F \in |\mathcal{O}(1, 1)|$ and integers $(a_1, a_2) \in \mathbb{N}^2$. Let $Z \subset F$ be a zero-dimensional scheme. We have $h^1(\mathcal{I}_Z(a_1, a_2)) > 0$ if and only if either $\deg(Z) \geq a_1 + a_2 + 2$ or there is a proper subcurve G of F, say of type $(o_1, o_2) \in \{(1, 0), (0, 1)\}$, with $\deg(Z \cap G) \geq o_2a_1 + o_1a_2 + 2$.*

Proof. If $Z' \subseteq Z$, then $h^1(\mathcal{I}_{Z'}(a_1, a_2)) \leq h^1(\mathcal{I}_Z(a_1, a_2)$ and $h^1(\mathcal{I}_Z(a_1, a_2)) = h^1(F, \mathcal{I}_{Z,F}(a_1, a_2))$.Hence the " if " part is true.

Now we prove the " only if " part. If F is integral, then the lemma is obvious, because $p_a(F) = 0$ and $\deg(\mathcal{O}_F(a_1, a_2)) = a_1 + a_2$. Assume $F = T + G$ with $T \in |\mathcal{O}(1, 0)|$ and $G \in |\mathcal{O}(0, 1)|$. If $\deg(Z \cap T) \leq a_2 + 1$, then a residual exact sequence gives $h^1(\mathcal{I}_{\mathrm{Res}_T(Z)}(a_1 - 1, a_2)) > 0$. Since $\mathrm{Res}_T(Z) \subset G$, Remark 2.1 gives $\deg(\mathrm{Res}_T(Z)) \geq a_1 + 1$ and hence $\deg(G \cap Z) \geq a_1 + 1$.

Similarly, we get $\deg(\mathrm{Res}_G(Z)) \geq a_2 + 1$.

Hence $\deg(Z) = \deg(\mathrm{Res}_G(Z)) + \deg(Z \cap G) \geq a_1 + a_2 + 2$. □

Lemma 2.3. *Fix integers $a_1 > 0$ and $a_2 > 0$. Let $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a zero-dimensional scheme such that $\deg(Z) \leq a_1 + a_2 + 2$. Then either there is $T \in |\mathcal{O}(1, 0)|$ such that $\deg(T \cap Z) \geq a_2 + 2$ or there is $D \in |\mathcal{O}(0, 1)|$ such that*

$\deg(T \cap Z) \geq a_1 + 2$ *or* $\deg(Z) = a_1 + a_2 + 2$ *and there is* $C \in |\mathcal{O}(1,1)|$ *such that* $Z \subset C$.

Proof. Taking a smaller scheme if necessary, we may assume $h^1(\mathcal{I}_{Z'}(a_1, a_2)) = 0$ for all $Z' \subsetneq Z$. We use induction on the integer $a_1 + a_2$. With no loss of generality we may assume $a_1 \geq a_2$. First assume $a_1 = a_2 = 1$. Since $h^1(\mathcal{I}_Z(1,1)) > 0$, we have $h^0(\mathcal{I}_Z(1,1)) > 0$. Take any $C \in |\mathcal{I}_Z(1,1)|$ and use Lemma 2.2. Now assume $a_1 \geq 2$. Let $F \in |\mathcal{O}(1,0)|$ be an element such that $e := \deg(Z \cap F)$ is maximal. If $e \geq a_2 + 2$, then we take $T = F$. Hence we may assume $e \leq a_1 + 1$, i.e. $h^1(F, \mathcal{I}_{Z \cap F, F}(a_1, a_2)) = 0$. The residual exact sequence of T gives $h^1(\mathcal{I}_{\mathrm{Res}_T(Z)}(a_1 - 1, a_2)) > 0$. Since $e > 0$, we have $\deg(\mathrm{Res}_F(Z)) \leq a_1 + a_2 + 1$. Since $a_1 - 1 > 0$, the inductive assumption gives that either there is $G \in |\mathcal{O}(1,0)|$ with $\deg(G \cap \mathrm{Res}_F(Z)) \geq a_2 + 2$ or there is $A \in |\mathcal{O}(0,1)|$ with $\deg(A \cap \mathrm{Res}_F(Z)) \geq a_1 + 1$ or $\deg(\mathrm{Res}_F(Z)) = a_1 + a_2 + 1$ and there is $C \in |\mathcal{O}(1,1)|$ such that $\deg(C \cap \mathrm{Res}_F(Z)) \geq a_1 + a_2 + 1$. If G exists, then we may take $T = G$ or observe that the maximality property of e would give $e \geq a_2 + 2$.

(a) Assume that C exists. Since $e > 0$ and $\deg(\mathrm{Res}_F(Z)) = a_1 + a_2 + 1$, we get $e = 1$. Since $\deg(\mathrm{Res}_C(Z)) \leq 1$, $a_1 > 0$ and $a_2 > 0$, we have $h^1(\mathcal{I}_{\mathrm{Res}_C(Z)}(a_1 - 1, a_2 - 1)) = 0$. Hence the residual exact sequence of C gives $h^1(C, \mathcal{I}_{Z \cap C, C}(a_1, a_2)) > 0$ and hence $h^1(\mathcal{I}_{Z \cap C}(a_1, a_2)) > 0$. Hence $Z \subset C$.

(b) Assume that A exists. If $\deg(A \cap Z) \geq a_1 + 2$, then we may take $D = A$. Hence we may assume $\deg(A \cap Z) = a_1 + 1$. We have $\deg(\mathrm{Res}_A(Z)) \leq a_2 + 1$. Since $h^1(\mathcal{I}_{A \cap Z}(a_1, a_2)) = 0$, a residual exact sequence gives $h^1(\mathcal{I}_{\mathrm{Res}_A}(a_1, a_2 - 1)) > 0$. Since $a_1 \geq a_2 -$, we get $\deg(\mathrm{Res}_A(Z) = a_2 + 1$ and the existence of $E \in |\mathcal{O}(1,0)|$ with $\mathrm{Res}_A(Z) \subset E$. Since $\deg(\mathrm{Res}_A(Z)) = a_2 + 1$ and $\deg(A \cap Z) = a_1 + 1$, we have $\deg(Z) = a_1 + a_2 + 2$. Since $\mathrm{Res}_A(Z) \subset E$, we have $Z \subset A + E$. Take $C = A + E$. □

Remark 2.4. In the set-up of Lemma 2.3 assume $a_1 \geq a_2$. Lemma 2.3 is trivially true if $a_2 = 0$ and $a_1 \leq 2$, but is is not true if $a_1 \geq 3$, because $h^0(\mathcal{O}(a_1, 0)) = a_1 + 1$ and so $h^1(\mathcal{I}_Z(a_1, a_2)) > 0$ for every $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$ with $\deg(Z) = a_1 + 2$.

Lemma 2.5. *Fix* $F \in |\mathcal{O}(2,1)|$ *and integers* $(a_1, a_2) \in \mathbb{N}^2$. *Let* $Z \subset F$ *be a zero-dimensional scheme. We have* $h^1(\mathcal{I}_Z(a_1, a_2)) > 0$ *if and only if either* $\deg(Z) \geq a_1 + 2a_2 + 2$ *or there is a proper subcurve* G *of* F, *say of type* (o_1, o_2), *such that* $\deg(Z \cap G) \geq o_2 a_1 + o_1 a_2 + 2$.

Proof. The " if " part is obviously true, because for all curves $G \subseteq F$ we have $h^1(\mathcal{O}_G(a_1, a_2)) = 0$. We have $h^1(\mathcal{I}_Z(a_1, a_2)) = h^1(F, \mathcal{I}_{Z,F}(a_1, a_2))$. If F is integral, then the lemma is obvious, because $p_a(F) = 0$ and $\deg(\mathcal{O}_F(a_1, a_2)) = a_1 + 2a_2$. We may assume $\deg(Z) \leq a_1 + 2a_2 + 1$. By Lemma 2.2 the thesis is equivalent to the existence of a curve $G \subsetneq F$ with $h^1(G, \mathcal{I}_{Z \cap Z, G}(a_1, a_2)) > 0$. Assume that no such curve exists. Since F is reducible, we have $F = T + E$ with $T \in |\mathcal{O}(1,0)|$. By assumption $\deg(Z \cap T) \leq a_2 + 1$ and $\deg(Z \cap E) \leq a_1 + a_2 + 1$. A residual exact sequence gives $h^1(\mathcal{I}_{\mathrm{Res}_T(Z)}(a_1 - 1, a_2)) > 0$. Since $\mathrm{Res}_T(Z) \subset E$, Lemma 2.2 gives $\deg(Z \cap E) = a_1 + a_2 + 1$ and $Z \cap F \cap T = \emptyset$. A residual exact sequence gives $h^1(\mathcal{I}_{\mathrm{Res}_F(Z)}(a_1 - 1, a_2 - 1)) > 0$. Since $\mathrm{Res}_F(Z) \subset T$, Remark 2.1 gives

$\deg(\mathrm{Res}_F(Z)) \geq a_2 + 1$. Since $Z \cap F \cap T$, we get $\deg(Z) \geq a_1 + 2a_2 + 2$, a contradiction. □

Lemma 2.6. *Fix $E \in |\mathcal{O}(2,2)|$, integers $a_1 \geq 0$, $a_2 \geq 0$ and a zero-dimensional scheme $Z \subset E$. We have $h^1(\mathcal{I}_Z(a_1,a_2)) > 0$ if and only if either there are $(o_1,o_2) \in \{(1,0),(0,1),(1,1),(2,1),(1,2)\}$ and $D \in |\mathcal{O}(o_1,o_2)|$ with $\deg(D \cap Z) \geq o_1a_2 + o_2a_1 + 2$ or Z is the complete intersection of E and an element of $|\mathcal{O}(a_1,a_2)|$ not containing E or $\deg(Z) \geq 2a_1 + 2a_2 + 1$.*

Proof. The " if " part is true because any $D \in |\mathcal{O}(e_1,e_2)|$ with $e_1 \leq 2$ or $e_2 \leq 2$ has $h^0(\mathcal{O}_D(a_1,a_2)) = e_1a_2 + e_2a_1 + 1 - e$ with $e = 0$ if $(e_1,e_2) \neq (2,2)$ and $e = 1$ if $(e_1,e_2) = (2,2)$. Now assume $h^1(\mathcal{I}_Z(a_1,a_2)) > 0$. Since $\deg(Z) \geq 0$ and $\deg(Z) = 0$ if and only if $Z = \emptyset$, we may assume $(a_1,a_2) \neq (0,0)$. In this case we have $h^1(\mathcal{O}(a_1-2,a_2-2)) = 0$ and hence $h^1(\mathcal{I}_Z(a_1,a_2)) = h^1(E,\mathcal{I}_{Z,E}(a_1,a_2))$. First assume that E is integral. Since $p_a(E) = 1$ and $\mathcal{O}_E(a_1,a_2)$ is a line bundle of degree $2a_1+2a_2$ we get that either $\deg(Z) = 2a_1+2a_2+1$ or Z is the complete intersection of E and an element of $|\mathcal{O}(a_1,a_2)|$ not containing E.

Now assume that E is not irreducible. We may assume $\deg(Z) \leq 2a_1 + 2a_2$ and (replacing Z with a smaller scheme) that $h^1(\mathcal{I}_{Z'}(a_1,a_2)) = 0$ for all $Z' \subsetneq Z$. Therefore $h^1(\mathcal{I}_Z(a_1,a_2)) = 1$, i.e. $h^0(\mathcal{I}_Z(a_1,a_2)) = a_1a_2 + a_1 + a_2 + 2 - \deg(Z)$. If we have a decomposition $E = F + T$ with F and T non-zero, we have $\deg(Z) \leq \deg(Z \cap F) + \deg(Z \cap T)$, but equality is not always true; strict inequality holds e.g. if Z reduced and it contains a point of $(F \cap T)_{\mathrm{red}}$; equality holds if Z is a Cartier divisor of E, but this is one of the key points to be proved if $\deg(Z) = 2a_1 + 2a_2$. We always have $\deg(Z) = \deg(\mathrm{Res}_F(Z)) + \deg(Z \cap F)$ and $\mathrm{Res}_F(Z) \subseteq T$ (here Res_F is computed seeing F as a Cartier divisor of $\mathbb{P}^1 \times \mathbb{P}^1$). If there is a subcurve $F \subsetneq E$ such that $h^1(F, \mathcal{I}_{Z \cap F}(a_1,a_2)) > 0$, then we may apply Remark 2.1 and Lemmas 2.2 and 2.5. Now assume that there is no such F.

Let T be any irreducible component of E_{red}, say $T \in |\mathcal{O}(e_1,e_2)|$. Write $E = F + T$ with $F \in |\mathcal{O}(2-o_1, 2-o_2)|$. We have $\mathrm{Res}_F(Z) \subset T$. By assumption $h^1(F, \mathcal{I}_{Z \cap F, F}(a_1,a_2)) = 0$. A residual exact sequence gives $h^1(T, \mathcal{I}_{\mathrm{Res}_F(Z)}(a_1 + o_1 - 2, a_2 + o_2 - 2)) > 0$. Remark 2.1 and Lemmas 2.2 and 2.5 gives $\deg(\mathrm{Res}_F(Z)) \geq o_2(a_1 + o_1 - 2) + o_1(a_2 + o_2 - 2) + 2$.

(a) Assume $E = A + B$ with $A, B \in |\mathcal{O}(1,1)|$ and A, B irreducible (we allow the case $A = B$). We saw that $\deg(\mathrm{Res}_A(Z)) \geq a_1 + a_2$ and hence $\deg(B \cap Z) \geq a_1 + a_2$. We saw that $\deg(\mathrm{Res}_B(Z)) \geq a_1 + a_2$ and hence $\deg(Z) \geq 2a_1 + 2a_2$. To prove the lemma we may assume that $\deg(Z) = 2a_1 + 2a_2$ and hence that all the inequalities above are equalities. Since $\deg(Z) = 2a_1 + 2a_2$, we get $h^0(E, \mathcal{I}_{Z,E}(a_1,a_2)) = 1$, i.e. there is $D \in |\mathcal{O}(a_1,a_2)|$ such that $D \not\supseteq E$ and $Z \subseteq E \cap D$. Since $\deg(Z) = 2a_1 + 2a_2$ we get that $Z = E \cap D$ (and hence we prove the lemma in this case), unless $|\mathcal{I}_Z(a_1,a_2)|$ has a one-dimensional base locus, G. Since $G \subsetneq E$, then either $G = A$ or $G = B$. With no loss of generality we may assume $G = A$. We get $h^0(\mathcal{I}_Z(a_1,a_2)) = h^0(\mathcal{I}_{\mathrm{Res}_A(Z)}(a_1 - 1, a_2 - 1))$. Recall that $h^0(\mathcal{I}_Z(a_1,a_2)) = a_1a_2 - a_1 - a_2 + 2$. Since $\deg(\mathrm{Res}_A(Z)) = a_1 + a_2$ and $\mathrm{Res}_A(Z) \subset B$, B is the base locus of $\mathcal{I}_{\mathrm{Res}_A(Z)}(a_1 - 1, a_2 - 1)$, i.e. $h^0(\mathcal{I}_{\mathrm{Res}_A(Z)}(a_1 - 1, a_2 - 1)) = h^0(\mathcal{O}(a_1 - 2, a_2 - 2)) = a_1a_2 - a_1 - a_1 + 1$, a contradiction.

(b) Assume $E = A + B$ with A, B integral and $A \notin |\mathcal{O}(1,1)|$. Exchanging if necessary the 2 rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ we may assume $A \in |\mathcal{O}(1,0)|$ and $B \in |\mathcal{O}(1,2)|$. Since $h^1(\mathcal{I}_{B\cap Z}(a_1,a_2)) = 0$, then $\deg(Z \cap B) \le 2a_1 + a_2 + 1$ and $\deg(\mathrm{Res}_B(Z)) \ge a_2$. Since $h^1(\mathcal{I}_{\mathrm{Res}_A(Z)}(a_1-1,a_2)) > 0$ and $\mathrm{Res}_A(Z) \subset B$, Lemma 2.5 gives $\deg(\mathrm{Res}_A(Z)) \ge 2a_1+a_2$. Hence $\deg(Z) \ge 2a_1+2a_2$. To prove the lemma we may assume $\deg(Z) = 2a_1 + 2a_2$ and hence many of the previous inequalities are equalities: we get $\deg(A \cap Z) = a_2$, $\mathrm{Res}_B(Z) = A \cap Z$, $\deg(B \cap Z) = 2a_1 + a_2$ and $\mathrm{Res}_A(Z) = B \cap A$. As in step (a) we get that Z is a complete intersection of E and an element of $|\mathcal{O}(a_1,a_2)|$, unless $\mathcal{I}_Z(a_1,a_2)$ has a one-dimensional base locus G. Since $G \subsetneq E$, either $G = A$ or $G = B$. Since $h^1(\mathcal{I}_Z(a_1,a_2)) = 1$, we have $h^0(\mathcal{I}_Z(a_1,a_2)) = a_1a_2 - a_1 - a_2 + 2$. First assume $G = A$ and so $h^0(\mathcal{I}_Z(a_1,a_2)) = h^0(\mathcal{I}_{\mathrm{Res}_A(Z)}(a_1-1,a_2))$. Since $\deg(\mathrm{Res}_A(Z)) = 2a_1 + a_2$ and $\mathrm{Res}_A(Z) \subseteq B$, B is the base locus of $\mathcal{I}_{\mathrm{Res}_A(Z)}(a_1-1,a_2)$ and so $h^0(\mathcal{I}_{\mathrm{Res}_A(Z)}(a_1-1,a_2)) = h^0(\mathcal{O}(a_1-2,a_1-2)) = a_1a_2 - a_1 - a_2 + 1$, a contradiction. Now assume $G = B$ and so $h^0(\mathcal{I}_Z(a_1,a_2)) = h^0(\mathcal{I}_{\mathrm{Res}_B(Z)}(a_1-1,a_2-2))$. Since $\deg(\mathrm{Res}_B(Z)) = a_2$ and $\mathrm{Res}_B(Z) \subset A$, A is the base locus of $\mathcal{I}_{\mathrm{Res}_B(Z)}(a_1-1,a_2-2))$ and so $h^0(\mathcal{I}_Z(a_1,a_2)) = h^0(\mathcal{O}(a_1-2,a_1-2))$, a contradiction.

(c) Assume that $E = T_1 + T_2 + T_3$ with T_1, T_2, T_3 irreducible, $T_1 \in |\mathcal{O}(1,0)|$, $T_2 \in |\mathcal{O}(0,1)|$ and $T_3 \in |\mathcal{O}(1,1)|$. Since $h^1(\mathcal{I}_{Z\cap(T_2+T_3)}(a_1,a_2)) = 0$, we get $h^1(\mathcal{I}_{\mathrm{Res}_{T_2+T_3}(Z)}(a_1-1,a_2-2)) > 0$ and hence $\deg(\mathrm{Res}_{T_2+T_3}(Z)) \ge a_2$. In the same way we get $\deg(\mathrm{Res}_{T_1+T_3}(Z)) \ge a_1$. Since T_3 is irreducible, we also get $\deg(\mathrm{Res}_{T_1+T_2}(Z)) \ge a_1 + a_2$ and so $\deg(Z \cap T_3) \ge a_1 + a_2$. Assume for the moment $\deg(T_1 \cap Z) > a_2$. Since $\deg(Z) \le 2a_1 + 2a_2$, Lemma 2.2 applied to $\mathrm{Res}_{T_1}(Z)$ and the line bundle $\mathcal{O}(a_1-1,a_2)$ gives that either there is $T \in |\mathcal{O}(1,0)|$ with $\deg(T \cap \mathrm{Res}_{T_1}(Z)) \ge a_2 + 2$ (proving the lemma in this case) or there is $D \in |\mathcal{O}(0,1)|$ with $\deg(D \cap \mathrm{Res}_{T_1}(Z)) \ge a_1 + 1$ (and hence $\deg((T_1 \cup D) \cap Z) \ge a_1 + a_2 + 2$, proving the lemma in this case) or there is $C \in |\mathcal{O}(1,1)|$ with $\deg(C \cap \mathrm{Res}_{T_1}(Z)) \ge a_1 + a_2 + 1$ (and hence $\deg((T_1 \cup C) \cap Z) \ge a_1 + 2a_2 + 2$, proving the lemma in this case).

Therefore $\deg(T_1 \cap Z) = a_2$ and $\mathrm{Res}_{T_2+T_3}(Z) = T_1 \cap Z$. In the same way we get $\deg(T_2 \cap Z) = a_1$ and $\mathrm{Res}_{T_1+T_3}(Z) = T_2 \cap Z$.

Assume for the moment $\deg(T_3 \cap Z) > a_1 + a_2$. Since $h^1(\mathcal{I}_{\mathrm{Res}_{T_3}(Z)}(a_1-1,a_2-1)) > 0$ and $\deg(Z) \le 2a_1+2a_2$, Lemma 2.2 gives that either there is $T \in |\mathcal{O}(1,0)|$ with $\deg(T \cap \mathrm{Res}_{T_3}(Z)) \ge a_2 + 1$ (and hence $\deg(Z \cap (T + T_3)) \ge a_1 + 2a_2 + 2$, proving the lemma) or there is $D \in |\mathcal{O}(0,1)|$ with $\deg(D \cap \mathrm{Res}_{T_3}(Z)) \ge a_1 + 1$ (and hence $\deg(Z \cap (D + T_3)) \ge 2a_1 + a_2 + 2$, proving the lemma)

Therefore we may assume that $\deg(T_3 \cap Z) = a_1 + a_2$, $T_3 \cap Z = \mathrm{Res}_{T_1+T_2}(Z)$ and $\deg(Z) = 2a_1 + 2a_2$. In this case Z is a complete intersection of E and an element of $|\mathcal{O}(a_1,a_2)|$, unless the one-dimensional part, G, of the base locus of $\mathcal{I}_Z(a_1,a_2)$ is non-empty. We have $G \subsetneq E$.

First assume $G = T_1 \cup T_2$. We get $a_1a_2 - a_1 - a_2 + 2 = h^0(\mathcal{I}_Z(a_1,a_2)) = h^0(\mathcal{I}_{\mathrm{Res}_{T_1+T_2}(Z)}(a_1-1,a_2-1))$. Since $\mathrm{Res}_{T_1+T_2}(Z) = T_3 \cap Z$, $\deg(T_3 \cap Z) = a_1 + a_2$ and T_3 is irreducible, T_3 is in the base locus of $\mathcal{I}_{\mathrm{Res}_{T_1+T_2}(Z)}(a_1-1,a_2-1)$ and so $T_3 \subset G$, i.e. $G = E$, a contradiction.

Now assume $G = T_1$. We get $a_1a_2-a_1-a_2+2 = h^0(\mathcal{I}_Z(a_1,a_2)) = h^0(\mathcal{I}_{\mathrm{Res}_{T_1}(Z)}(a_1-1,a_2))$. Since $\mathrm{Res}_{T_3}(\mathrm{Res}_{T_1}(Z)) = \mathrm{Res}_{T_1+T_3}(Z) = T_2 \cap Z$, we get that T_2 is in the base locus of $\mathcal{I}_{\mathrm{Res}_{T_1}(Z)}(a_1-1,a_2)$ and hence $T_2 \subset G$, a contradiction.

In the same way we conclude if $G = T_2$.

Now assume $G = T_3$. We get $a_1a_2-a_1-a_2+2 = h^0(\mathcal{I}_Z(a_1,a_2)) = h^0(\mathcal{I}_{\mathrm{Res}_{T_3}(Z)}(a_1-1,a_2-1))$. Since $\deg(\mathrm{Res}_{T_2+T_3}(Z)) = a_2$ and we have $\deg(\mathrm{Res}_{T_1+T_3}(Z)) = a_1$, then T_1+T_2 is the base locus of $\mathcal{I}_{\mathrm{Res}_{T_3}(Z)}(a_1-1,a_2-1)$. Hence $h^0(\mathcal{I}_{\mathrm{Res}_{T_3}(Z)}(a_1-1,a_2-1)) = h^0(\mathcal{O}(a_1-2,a_2-2)) = a_1a_2-a_1-a_2+1$, a contradiction.

Now assume $G = T_1+T_3$. We get $a_1a_2-a_1-a_2+2 = h^0(\mathcal{I}_Z(a_1,a_2)) = h^0(\mathcal{I}_{\mathrm{Res}_{T_1+T_3}(Z)}(a_1-2,a_2-1))$. Since $\mathrm{Res}_{T_1+T_3}(Z) = Z \cap T_2$ and $\deg(Z\cap T_2) = a_1$, we get $T_2 \subset G$, a contradiction.

In the same way we conclude if $G = T_2+T_3$.

(d) Assume $E = T_1+T_2+T_3+T_4$ with $T_1, T_2 \in |\mathcal{O}(1,0)|$ and $T_3, T_4 \in |\mathcal{O}(0,1)|$ (we allow the case $T_i = T_j$ for some $i \neq j$). Since $h^1(\mathcal{I}_{T_1\cap Z}(a_1,a_2)) = 0$, then $\deg(Z\cap T_1) \leq a_2+1$. Since $h^1(\mathcal{I}_{\mathrm{Res}_{T_2+T_3+T_4}(Z)}(a_1-1,a_2-2)) > 0$, we have $\deg(\mathrm{Res}_{T_2+T_3+T_4}(Z)) \geq a_2$. Assume for the moment that $a_2 < \deg(\mathrm{Res}_{T_2+T_3+T_4}(Z))$. We get $\deg(T_1\cap Z) \geq a_2+1$ and hencewe also have $\deg(\mathrm{Res}_{T_1}(Z)) \leq 2a_1+a_2-1$. Since $h^1(\mathcal{I}_{\mathrm{Res}_{T_1}(Z)}(a_1-1,a_2)) > 0$, Lemma 2.2 for the line bundle $\mathcal{O}(a_1-1,a_2)$ shows that either there is $T \in |\mathcal{O}(1,0)|$ with $\deg(T\cap\mathrm{Res}_{T_1}(Z)) \geq a_2+2$ (and hence the lemma is true in this case) or there is $D \in |\mathcal{O}(0,1)|$ with $\deg(D\cap\mathrm{Res}_{T_1}(Z)) \geq a_1+1$ (and hence $\deg(D\cup T_1)(Z)) \geq a_1+a_2+2$ and the lemma is true taking $D\cup T_2$) or there is $C \in \mathcal{O}(1,1)|$ with $\deg(C\cap\mathrm{Res}_{T_1}(Z)) \geq a_1-1+a_2+2$ (and hence $\deg(Z\cap(C\cup T_1)) \geq a_1+2a_2+2$, proving the lemma in this case).

Hence $\deg(Z\cap T_1) = a_2$. We also proved that $Z\cap T_1 = \mathrm{Res}_{T_2+T_3+T_4}(Z)$. In the same way we get $\deg(Z\cap T_2) = a_2$, $\deg(Z\cap T_3) = \deg(Z\cap T_4) = a_1$ and $Z\cap T_i = \mathrm{Res}_{E-T_i}(Z)$ for all $i = 1,2,3,4$. As in the previous step we reduce to the case $\deg(Z) = 2a_1+2a_2$, $h^0(\mathcal{I}_Z(a_1,a_2)) = a_1a_2-a_1-a_2$ and that the one-dimensional part G of the base locus of $\mathcal{I}_Z(a_1,a_2)$ is non-empty. We have $G \subsetneq E$.

First assume $G \supseteq T_1$. We get $h^0(\mathcal{I}_Z(a_1,a_2)) = h^0(\mathcal{I}_{\mathrm{Res}_{T_1}(Z)}(a_1-1,a_2))$. Since $Z\cap T_3 = \mathrm{Res}_{T_1+T_2+T_4}(Z)$ and $\deg(Z\cap T_3) = a_1$, T_3 is contained in the base locus of $\mathcal{I}_{\mathrm{Res}_{T_1}(Z)}(a_1-1,a_2)$ and so $T_3 \subset G$. In the same way we get $T_4 \subset G$ and so $G = T_1\cup T_2\cup T_3$. Hence $h^0(\mathcal{I}_Z(a_1,a_2)) = h^0(\mathcal{I}_{\mathrm{Res}_{T_1\cup T_3\cup T_4}(Z)}(a_1-1,a_2-2))$. Since $Z\cap T_2 = \mathrm{Res}_{T_1\cup T_3\cup T_4}(Z)$ and $\deg(Z\cap T_2) = a_2$, we get $T_2 \subset G$, a contradiction.

In the same way we conclude if G contains another component of E. □

Remark 2.7. Take D as in Lemma 2.6 with $D \in |\mathcal{O}(o_1,o_2)|$ and $0 \leq o_i \leq 2$ and $(o_1,o_2) \neq (2,2)$. If $2o_1+2o_2 < o_1a_2+o_2a_2+2$, then Bezout theorem gives that one of the irreducible components of D is an irreducible component of E.

Lemma 2.8. *Fix an integer $a \geq 5$. Let $Z \subset \mathbb{P}^1\times\mathbb{P}^1$ be a zero-dimensional scheme such that $\deg(Z) \leq 2a+6$ and $h^1(\mathcal{I}_Z(a,a)) > 0$. Then there is $C \in |\mathcal{O}(1,1)|$ with $h^1(\mathcal{I}_{C\cap Z}(a,a)) > 0$.*

Proof. Since $3a+2 > \deg(Z)$, by Lemma 2.6 it is sufficient to prove the existence of some $F \in |\mathcal{O}(2,2)|$ such that $h^1(\mathcal{I}_{F\cap Z}(a,a)) > 0$. Look at the set-up of step (a) of the proof of Theorem 1.1 and Proposition 1.3; (in particular we use

E_i, Z_i, e_i, and (3.1)). Since $\deg(Z) - 1 \geq 8\lfloor a/2 \rfloor$, there is a minimal positive integer $f \leq \lfloor a/2 \rfloor$ such that $h^1(E_f, \mathcal{I}_{Z_{f-1}\cap E_f, E_f}(a+2-2f, a+2-2f)) > 0$. By Lemma 2.6 we may assume $f \geq 2$. Since $h^1(\mathcal{I}_{E_f \cap Z_{f-1}}(a-2f+2, a-2f+2)) > 0$, Lemma 2.6 shows that either $\deg(E_f \cap Z_{f-1}) \geq 4(a-2f+2)$ or there is $D \in (|\mathcal{O}(1,0)| \cup |\mathcal{O}(0,1)|)$ with $\deg(D \cap E_f \cap Z_{f-1}) \geq a-2f+4$ or there is $C \in |\mathcal{O}(1,1)|$ with $\deg(C \cap E_f \cap Z_{f-1}) \geq 2a-4f+6$ or there is $C_1 \in (|\mathcal{O}(2,1) \cup |\mathcal{O}(1,2)|)$ with $\deg(C_1 \cap E_f \cap Z_{f-1}) \geq 3a-6f+8$.

(a) Assume $\deg(E_f \cap Z_{f-1}) \geq 4(a-2f+2)$. Since the sequence $\{e_i\}$ is non-increasing we get $2a+6 \geq \deg(Z) \geq 4f(a-2f+2)$. The function $t \mapsto 4t(a+2-2t)$ is increasing if $0 \leq (a+2)/4$ and decreasing for $t \geq (a+2)/4$. Since $2 \leq f \leq a/2$, we get a contradiction.

(b) Assume the existence of $C_1 \in (|\mathcal{O}(2,1)| \cup |\mathcal{O}(1,2)|)$ with $\deg(C_1 \cap E_f \cap Z_{f-1}) \geq 3a-6f+8$. Since $h^0(\mathcal{O}(1,0)) = 2$, the maximality property of e_{f-1} gives $e_{f-1} \geq 3a-6f+9$. Hence $2a+6 \geq \deg(Z) \geq f(3a-6f+9)-1$. The function $t \mapsto 3t(3a+9-6t)-1$ is increasing if $0 \leq t < (3a+9)/12$ and decreasing if $t \geq (3a+9/12$. Since $2 \leq f \leq a/2$, we get $2a+6 \geq 6a-10$, contradicting the assumption $a \geq 5$.

(c) Assume the existence of $C \in |\mathcal{O}(1,1)|$ with $\deg(C \cap E_f \cap Z_{f-1}) \geq 2a-4f+6$. Since $h^0(\mathcal{O}(1,1)) = 4$, the maximality property of e_{f-1} gives $e_{f-1} \geq 2a-4f+9$. Thus $2a+6 \geq \deg(Z) \geq f(2a-4f+9)-3$. Since the concave function $t(2a+9-2t)$ has a maximum at $(2a+9)/2 > a/2$, $2f \leq a$, and $2a+6 < 4a-5$ if $a \geq 6$, we get $a = 5$ and hence $f = 2$. We may assume $h^1(\mathcal{I}_{C\cap Z}(5,5)) = 0$ (otherwise the lemma is proved). Hence $h^1(\mathcal{I}_{\mathrm{Res}_C(Z)}(4,4)) > 0$. Since $f = 2$ and $a = 5$, we have $\deg(\mathrm{Res}_C(Z)) \leq 8$. Lemma 2.2 gives the existence of $C' \in |\mathcal{O}(1,1)|$ with $h^1(\mathcal{I}_{C' \cap \mathrm{Res}_C(Z)}(4,4)) > 0$. Hence $h^1(\mathcal{I}_{(C+C')\cap Z}(5,5)) > 0$. Since $C+C' \in |\mathcal{O}(2,2)|$, Lemma 2.6 concludes the proof in in this case.

(d) Assume the existence of $D \in (|\mathcal{O}(1,0)| \cup |\mathcal{O}(0,1)|)$ with $\deg(D \cap E_f \cap Z_{f-1}) \geq a-2f+4$. Without losing generality we may assume $D \in |\mathcal{O}(1,0)|$. Since $h^0(\mathcal{O}(1,2)) = 6$, the maximality property of the integer e_{f-1} gives $e_{f-1} \leq a-2f+9$. Hence $2a+6 \geq \deg(Z) \geq f(a+9-2f)-5$. The function $t \mapsto t(a+9-2t)-5$ is strictly increasing if $2 \leq t \leq (a+9)/4$ and decreasing if $t \geq (a+9)/4$. Since $2a+6 < 3a+4$ and $2a+6 < 9a/2-5$, we get $f = 2$. Set $W_0 := Z$. Let F_1 be any element of $|\mathcal{O}(2,2)|$ containing D and such that $g_1 := \deg(Z \cap F_1)$ is maximal. We have $g_1 \geq a+5$. Set $W_1 := \mathrm{Res}_{F_1}(Z)$. Define recursively the integer g_i, $F_i \in |\mathcal{O}(2,2)|$ and W_i in the following way. Let F_i be any element of $|\mathcal{O}(2,2)|$ such that $g_i := \deg(W_{i-1} \cap F_i)$ is maximal. Set $W_i := \mathrm{Res}_{F_i}(W_{i-1})$. We have a residual exact sequences (3.1) with W_j instead of Z_j and F_j instead of E_j. We have $g_i \geq g_{i+1}$ if $i \geq 2$. If $g_i \leq 7$, then $W_i = \emptyset$. Since $\deg(Z) - g_1 \leq a+1$, there is a minimal integer $g \leq a/2$ such that $h^1(F_g, \mathcal{I}_{F_g \cap W_{g-1}, F_g}(a-2g+2, a-2g+2)) > 0$. As in step (c) we get $g = 2$ and the existence of $D' \in (|\mathcal{O}(1,0)| \cup |\mathcal{O}(0,1)|)$ with $\deg(D' \cap F_2 \cap W_1) \geq a$. Set $(o_1, o_2) := (2,0)$ if $D' \in |\mathcal{O}(1,0)|$ and $(o_1, o_2) := 0$ if $D' \in |\mathcal{O}(0,1)|$. By Lemma 2.6 we may assume $h^1(\mathcal{I}_{D+D'}(a,a)) = 0$. Thus a residual exact sequence gives $h^1(\mathcal{I}_{\mathrm{Res}_{D+D'}((Z)a-o_1, a-o_2)}) > 0$. Since $\deg(\mathrm{Res}_{D+D'}(Z)) \leq 6$ and $a \geq 5$, Lemma 2.3 gives $a \leq 6$ and that if $a = 6$, then $\deg(Z) = 12$, $D' \in |\mathcal{O}(1,0)|$ and

the existence of $D'' \in |\mathcal{O}(1,0)|$ with $\mathrm{Res}_{D+D'}(Z) \subset D''$; taking D'' instead of D' in this case we get a contradiction.

Now assume $a = 5$. First assume $(o_1, o_2) = (1,1)$. By Lemma 2.3 there is $C' \in |\mathcal{O}(1,1)|$ with $h^1(\mathcal{I}_{C' \cap \mathrm{Res}_{D+D'}(Z)}(4,4)) > 0$. Hence $h^1(\mathcal{I}_{(C+C') \cap Z}(5,5)) > 0$. It is sufficient to apply Lemma 2.6 to the zero-dimensional scheme $(C + C') \cap Z$. Now assume $(o_1, o_2) = (2,0)$. By Lemma 2.3 there is $M \in |\mathcal{O}(0,1)|$ with $\deg(M \cap C' \cap \mathrm{Res}_{D+D'}(Z)) \geq 5$. Since $\deg((M + D + D') \cap Z) \geq 15$, we have $\deg(\mathrm{Res}_{M+D+D'}(Z)) \leq 1$ and hence $h^1(\mathcal{I}_{\mathrm{Res}_{M+D+D'}(Z)}(2,5)) = 0$. The residual sequence of $M + D + D'$ gives $h^1(D + D' + M, \mathcal{I}_{Z \cap (D+D'+M)}(5,5)) > 0$. Let V be the element of $\{D, M, M'\}$ with maximal $\deg(Z \cap V)$. Since $\mathcal{O}(x,y)|J \cong \mathcal{O}_J(0,y)$ for all $J \in |\mathcal{O}(1,0)$ and all $(x,y) \in \mathbb{Z}^2$, two residual sequences give that $\deg(V \cap Z) \geq 7$. □

Lemma 2.9. *Fix an integer $a \geq 5$. Let $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a zero-dimensional scheme such that $\deg(Z) \leq 2a + 3$ and $h^1(\mathcal{I}_Z(a-2, a)) > 0$. If $a = 5, 6$ assume that $\deg(Z) \leq 2a + 1$. Then there is $C \in |\mathcal{O}(1,1)|$ with $h^1(\mathcal{I}_{C \cap Z}(a,a)) > 0$.*

Proof. First assume $a \geq 7$. Fix any $M \in |\mathcal{O}(0,2)|$ such that $\deg(M \cap Z)$ is maximal. We may assume (taking a residual), that $h^1(\mathcal{I}_{M \cap Z}(a-2, a)) = 0$. Hence $h^1(\mathcal{I}_{\mathrm{Res}_M(Z)}(a-2, a-2)) > 0$. Since $\deg(\mathrm{Res}_M(Z)) \leq -1 + 2a + 3 = 2(a-2) + 6$, Lemma 2.8 gives the existence of $C \in |\mathcal{O}(1,1)|$ with $h^1(\mathcal{I}_{C \cap \mathrm{Res}_M(Z)}(a-2, a-2)) > 0$. Take a maximal subcurve A of C such that $h^1(\mathcal{I}_{\mathrm{Res}_M(Z) \cap A}(a-2, a-2)) > 0$ and write $A \in |\mathcal{O}(u_1, u_2)|$ with $(u_1, u_2) \in \{(0,1), (1,0), (1,1)\}$. First assume $(u_1, u_2) = (1,1)$. We have $\deg(A \cap \mathrm{Res}_M(Z)) \geq 2a-2$. We may assume $h^1(\mathcal{I}_{A \cap Z}(a-2, a)) = 0$. Hence $h^1(\mathcal{I}_{\mathrm{Res}_A(Z)}(a-3, a-1)) > 0$. Since $\deg(\mathrm{Res}_A(Z)) \leq 5$ and $a \geq 7$, we got a contradiction. Now assume $(u_1, u_2) = (1,0)$. We have $\deg(A \cap Z) \geq a$ and we may assume that $h^1(\mathcal{I}_{\mathrm{Res}_A(Z)}(a-3, a)) > 0$. Since $\deg(\mathrm{Res}_A(Z)) \leq a + 3$, Lemma 2.3 gives that either there is $D \in |\mathcal{O}(1,0)|$ with $\deg(D \cap \mathrm{Res}_A(Z)) \geq a + 2$ (and in this case the lemma is proved) or there is $D' \in |\mathcal{O}(0,1)|$ with $\deg(D' \cap \mathrm{Res}_A(Z)) \geq a - 1$. Since $D' \in |\mathcal{O}(1,0)|$ the maximality property of M gives $\deg(Z \cap M) \geq 2a - 2$. We also have $\deg(\mathrm{Res}_M(Z)) \geq a - 1$ and hence $3a - 3 \geq 2a + 3$, contradicting the assumption $a \geq 7$.

Now assume $a = 6$ and so $\deg(Z) \leq 13$. Take $F \in |\mathcal{O}(2,2)|$ with maximal $\deg(Z \cap F)$. By Lemma 2.6 we may assume $h^1(\mathcal{I}_{Z \cap F}(4,6)) = 0$ and hence $h^1(\mathcal{I}_{\mathrm{Res}_F(Z)}(2,4)) > 0$. Since $\deg(\mathrm{Res}_F(Z)) \leq 5$, there is $T \in |\mathcal{O}(0,1)|$ such that $\deg(T \cap Z) \geq 4$. Since $h^0(\mathcal{O}(2,1)) = 6$, there is $G \in |\mathcal{O}(2,2)|$ with $G \supset T$ and $\deg(G \cap Z) \geq 9$. Since we may assume that $h^1(\mathcal{I}_{\mathrm{Res}_G(Z)}(2,4)) > 0$ and $\deg(\mathrm{Res}_G(Z)) \leq 4$, then $\deg(\mathrm{Res}_G(Z)) = 4$ and there is $T' \in |\mathcal{O}(0,1)|$ containing $\mathrm{Res}_G(Z)$. We may assume $h^1(\mathcal{I}_{Z \cap (T+T')}(4,6)) = 0$ and so we get $h^1(\mathcal{I}_{\mathrm{Res}_{T+T'}(Z)}(6,4)) > 0$. Since $\deg(\mathrm{Res}_{T+T'}(Z)) \leq 5$, we obtained a contradiction.

Now assume $a = 5$ and so $\deg(Z) \leq 11$. Take $F \in |\mathcal{O}(2,2)|$ with maximal $\deg(Z \cap F)$. By Lemma 2.6 we may assume $h^1(\mathcal{I}_{Z \cap F}(3,5)) = 0$ and hence $h^1(\mathcal{I}_{\mathrm{Res}_F(Z)}(1,3)) > 0$. Since $\deg(\mathrm{Res}_F(Z)) \leq 3$, we get $\deg(Z \cap F) = 8$, $\deg(\mathrm{Res}_F(Z)) = 3$ and $\mathrm{Res}_F(Z)$ is contained in an element $T \in |\mathcal{O}(0,1)|$. Since $h^0(\mathcal{O}(1,2)) = 6$, there is $G \in |\mathcal{O}(2,2)|$ with $G \supset T'$ and $\deg(G \cap Z) \geq 8$. As above we get $\deg(G \cap Z) = 8$, $\deg(\mathrm{Res}_G(Z)) = 3$ and the existence of

$T' \in |\mathcal{O}(0,1)|$. Since $h^1(\mathcal{I}_{Z\cap(T+T')}(3,5)) = 0$, we get $h^1(\mathcal{I}_{\mathrm{Res}_{T+T'}(Z)}(3,3)) > 0$. Since $\deg(\mathrm{Res}_{T+T'}(Z)) \le 5$ we get the existence of $D_1 \in (|\mathcal{O}(1,0)| \cup |\mathcal{O}(0,1)|)$. If $D_1 \in |\mathcal{O}(1,0)|$, then we obtained $U \in |\mathcal{O}(2,2)|$ and hence Lemma 2.6 concludes the proof. If $D_1 \in |\mathcal{O}(0,1)|$, then we get $Z \subset D_1 + T + T'$. Since $h^1(\mathcal{I}_Z(3,5)) > 0$, the divisor V among D_1, T, T' with maximal $\deg(V \cap Z)$ has $\deg(V \cap Z) \ge 5$, because $\mathcal{O}_{D+T+T'}(3,5) \cong \mathcal{O}_{D+T+T'}(3,0)$. □

Lemma 2.10. *Assume $a \ge 5$. Let $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a zero-dimensional scheme such that $\deg(Z) \le 3a+1$ and $h^1(\mathcal{I}_Z(a-1,a)) > 0$. Then either there is $D_1 \in |\mathcal{O}(1,0)|$ with $\deg(D_1 \cap \mathrm{Res}_D(Z)) \ge a+2$ or there is $D_2 \in |\mathcal{O}(0,1)|$ with $\deg(D_2 \cap \mathrm{Res}_D(Z)) \ge a+1$ or there is $C \in |\mathcal{O}(1,1)|$ with $\deg(\mathrm{Res}_D(Z) \cap C) \ge 2a+1$ or there in $C_1 \in |\mathcal{O}(1,2)|$ with $\deg(C_1 \cap \mathrm{Res}_D(Z)) \ge 3a$ or $\deg(Z) = 3a+1$ and $Z \subset C_2$ for some $C_2 \in |\mathcal{O}(2,1)|$.*

Proof. Set $Z_0 := Z$. Let E_1 be any element of $|\mathcal{O}(2,2)|$ such that $e_1 := \deg(Z \cap E_1)$ is maximal. Set $Z_1 := \mathrm{Res}_{E_1}(Z)$. Define recursively the integer e_i, $E_i \in |\mathcal{O}(2,2)|$ and Z_i in the following way. Let E_i be any element of $|\mathcal{O}(2,2)|$ such that $e_i := \deg(Z_{i-1} \cap E_i)$ is maximal. Set $Z_i := \mathrm{Res}_{E_i}(Z_{i-1})$. We have a residual exact sequence

$$0 \to \mathcal{I}_{Z_i}(a-2i-1, a-2i) \to \mathcal{I}_{Z_{i-1}}(a-2i+1, a-2i+2) \to \mathcal{I}_{Z_{i-1}\cap E_i, E_i}(a_1 - 2i + 1, a_2 - 2i + 2) \to 0 \tag{2.1}$$

Since $h^0(\mathcal{O}(2,2)) = 9$, if $e_i \le 7$, then $Z_i = \emptyset$. Since $a \ge 5$, then $8\lfloor (a-1)/2 \rfloor \le 3a$. Since $e_i \ge 8$ if $Z_i \ne \emptyset$, $\deg(Z) - 1 \le 3a \le 8\lfloor (a-1)/2 \rfloor$ and $h^1(\mathcal{I}_A(0,1)) = 0$ if either $A = \emptyset$ or A is a point, there is an integer $i \le \lfloor (a-1)/2 \rfloor$ such that $h^1(\mathcal{I}_{Z_{i-1}\cap E_i}(a-2i+1, a-2i+2)) > 0$ and call f the minimal such an integer. If $f = 1$, then the lemma is true by Lemma 2.6. Therefore we may assume $f \ge 2$. Since $h^1(\mathcal{I}_{E_f \cap Z_{f-1}}(a-2f+1, a-2f+2)) > 0$, Lemma 2.6 shows that either $\deg(E_f \cap Z_{f-1}) \ge 2(2a-4f+3)$ or there is $D \in |\mathcal{O}(1,0)|$ with $\deg(D \cap E_f \cap Z_{f-1}) \ge a+1$ or there is $D_1 \in |\mathcal{O}(0,1)|$ with $\deg(D \cap E_f \cap Z_{f-1}) \ge a-2f+4$ or there is $C \in |\mathcal{O}(1,1)|$ with $\deg(C \cap E_f \cap Z_{f-1}) \ge 2a-4f+5$ or there is $C_1 \in |\mathcal{O}(2,1)$ with $\deg(C_1 \cap E_f \cap Z_{f-1}) \ge 3a-6f+7$ or there is $C_2 \in |\mathcal{O}(1,2)|$ with $\deg(C_2 \cap E_f \cap Z_{f-1}) \ge 3a-6f+6$.

(a) Assume $\deg(E_f \cap Z_{f-1}) \ge 2(2a-4f+3)$. Since $e_i \ge e_f$ for all $i < f$, we get $3a+1 \ge \deg(Z) \ge 2f(2a-4f+3)$. Set $\psi(t) = 2t(2a+3-4t)$. The continuos function $\psi(t)$ is strictly increasing in the interval $(0, (2a+3)/8)$ and symmetric with respect to $(2a+3)/8$. Since $\psi(2) = 8a-10 > 3a+1$ and $\psi(a/2) = 6a > 3a+1$, we get a contradiction.

(b) Assume the existence of $C_2 \in |\mathcal{O}(1,2)|$ with $\deg(C_2 \cap E_f \cap Z_{f-1}) \ge 3a-6f+6$ (an easier proof works if C_1 exists). Since $h^0(\mathcal{O}(0,1)) = 2$, the maximality property of the integer e_{f-1} gives $e_{f-1} \ge 3a-6f+7$. Hence $3a+1 \ge \deg(Z) \ge f(3a+7-6f)-1$. The continuos function $\eta(t) := t(3a+7-6t)-1$ is strictly increasing if $0 < t < (3a+7)/12$ and strictly decreasing if $t > (3a+7)/12$. We have $\eta((a-1)/2) > \eta(2) = 6a-11 > 3a+1$, excluding this case.

(c) Assume the existence of $C \in |\mathcal{O}(1,1)|$ with $\deg(C \cap E_f \cap Z_{f-1}) \ge 2a-4f+5$. Since $Z_{f-1} \ne \emptyset$ and $h^0(\mathcal{O}(1,1)) = 4$, we get $e_{f-1} \ge 3+2a-4f+5$. Hence $3a+1 \ge f(2a-4f+8)-3$. Set $\varphi(t) = 2t(a+4-2t)-3$. The

continuos function $\varphi(t)$ is strictly increasing if $0 < t < (a+4)/4$ and strictly decreasing if $t > (a+4)/4$ and symmetric with respect to $(a+4)/4$. We have $\varphi((a-1)/2) > \varphi(2) = 4a+7 > \deg(Z)$. Hence this case does not occur.

(d) Assume the existence of $D_1 \in |\mathcal{O}(0,1)|$ with $\deg(D_1 \cap E_f \cap Z_{f-1}) \geq a-2f+3$. Since $h^0(\mathcal{O}(2,1)) = 6$, the maximality property of the integer e_{f-1} gives $e_{f-1} \geq a-2f+8$. Hence $3a+1 \geq \deg(Z) \geq f(a-2f+8)-3$. Let $\tau(t) = t(a-2t+8)-3$. Since $f \leq \lfloor (a-1)/2 \rfloor$, $a \geq 5$, $\tau(t)$ is increasing if $t \leq (a+8)/4$, symmetric with respect $(a+8)/4$ and $\gamma(4) = 4a-3 > 3a+1$, $\gamma((a-1)/2) = 5(a-1)-3 > 3a+1$, we get $f \leq 3$. By Remark 2.1 we may assume $h^1(\mathcal{I}_{D_1 \cap Z}(a-1,a)) = 0$ and so $h^1(\mathcal{I}_{\mathrm{Res}_{D_1}(Z)}(a-1,a-1)) > 0$. Since $f \leq 3$, then $\deg(\mathrm{Res}_{D_1}(Z)) \leq 2a+4$. Hence Lemma 2.8 gives the existence of $C \in |\mathcal{O}(1,1)|$ such that $h^1(\mathcal{I}_{C \cap \mathrm{Res}_{D_1}(Z))}(a-1,a-1)) > 0$. By Lemma 2.2 either $\deg(C \cap \mathrm{Res}_{D_1}(Z)) \geq 2a$ (and so $\deg((C+D_1) \cap Z) \geq 3a-3$ and so $h^1(\mathcal{I}_{\mathrm{Res}_{C+D_1}(Z)}(a-2,a-2)) = 0$, proving the lemma) or there is $A \in |\mathcal{O}(1,0)|$ with $\deg(A \cap \mathrm{Res}_{D_1}(Z)) \geq a+1$ or there is $B \in |\mathcal{O}(0,1)|$ with $\deg(B \cap \mathrm{Res}_{D_1}(Z)) \geq a+1$ (proving the lemma in this case). Assume the existence of A. Since $h^0(\mathcal{O}(1,1)) = 4$, we get $e_1 \geq (a+1)+(a-3)+3 = 2a+1$ and so $f = 2$. Hence $e_1 \geq (a+1)+(a-1)+3 = 2a+3$ and so $e_2 \leq a-2$. Hence $h^1(\mathcal{I}_{Z_1}(a-3,a-2)) = 0$, i.e. $f = 1$, a contradiction.

(e) Assume the existence of $D \in |\mathcal{O}(1,0)|$ with $\deg(D \cap E_f \cap Z_{f-1}) \geq a-2f+4$. As in step (d) we get $f \leq 3$. We get $\deg(\mathrm{Res}_D(Z)) \leq 2a+3$; if $f = 2$ (e.g., if $a = 5,6$), then $\deg(\mathrm{Res}_D(Z)) \leq 2a+1$. Hence Lemma 2.9 gives the existence of $C \in |\mathcal{O}(1,1)|$ with $h^1(\mathcal{I}_{C \cap \mathrm{Res}_D(Z))}(a-2,a)) > 0$ and so either $\deg(C \cap \mathrm{Res}_D(Z)) \geq 2a$ (and so $\deg(\mathrm{Res}_{C+D}(Z)) \leq 3$ and so $h^1(\mathcal{I}_{(C+D) \cap Z}(a-1,a)) > 0$, concluding the proof by Lemma 2.5) or there is $A \in |\mathcal{O}(1,0)|$ with $\deg(A \cap \mathrm{Res}_D(Z)) \geq a+1$ (concluding the proof in this case) or there is $B \in |\mathcal{O}(0,1)|$ with $\deg(B \cap \mathrm{Res}_D(Z)) \geq a$. Hence B exists. Since $h^0(\mathcal{O}(1,1)) = 4$, we get $e_1 \geq 3+a+(a-2)$. Hence $f = 2$. Hence $e_1 \geq 3+a+a$. Since $e_2 \leq a-2$, as in step (d) we have $f = 1$, a contradiction. □

Lemma 2.11. *Fix integers $a_1 > 0$, $a_2 \geq 2$ and $E \in |\mathcal{O}(1,3)|$. Let $Z \subset E$ be a zero-dimensional scheme such that $h^1(\mathcal{I}_Z(a_1,a_2)) > 0$. Then either there is $T \in |\mathcal{O}(0,1)|$ with $T \subset E$, and $\deg(T \cap Z) \geq a_1+2$ or there is $b \in \{0,1,2,3\}$ and $C \in |\mathcal{O}(1,b)|$, $C \subseteq E$ and $\deg(C \cap Z) \geq ba_1+a_2+2$.*

Proof. Let $F \subseteq E$ be a minimal subcurve of F such that $h^1(\mathcal{I}_{F \cap Z}(a_1,a_2)) > 0$. Write $F \in |\mathcal{O}(u_1,u_2)|$ for some u_1,u_2. If $u_1 = 1$ and $u_2 \leq 2$, use Lemma 2.5. If $(u_1,u_2) = (0,1)$, then use Remark 2.1. Let α be the maximal integer $\deg(V \cap Z)$ with $V \in \mathcal{O}(0,1)$. If $\alpha \leq a_1+1$, then several residual exact sequences show that if $u_1 = 0$, then $\alpha \geq a_1+2$ and so $u_2 = 1$,. Now assume $(u_1,u_2) = (1,3)$, i.e. $F = E$. If E is irreducible, then the lemma is trivial, because E is smooth and rational and $h^1(\mathcal{O}(a_1-1,a_2-3)) = 0$. Hence we may assume that E is not integral and hence there is $T \in |\mathcal{O}(0,1)|$ with $E = T+G$ with $G \in |\mathcal{O}(1,2)|$. Assume $\deg(T \cap Z) \leq a_1+1$. Since $h^1(T, \mathcal{I}_{Z \cap T}(a_1,a_2)) = 0$ by the minimality property of E, we get $h^1(\mathcal{I}_{\mathrm{Res}_T(Z)}(a_1,a_2-1)) > 0$. Since $h^1(G, \mathcal{I}_{Z \cap G}(a_1,a_2)) = 0$ by the minimality property of E, we get $h^1(\mathcal{I}_{\mathrm{Res}_G(Z)}(a_1-1,a_2-2)) > 0$. Since $\mathrm{Res}_G(Z) \subset T$, we get $\deg(T \cap \mathrm{Res}_G(Z)) \geq a_1+1$ and hence $\deg(T \cap \mathrm{Res}_G(Z)) = \deg(T \cap Z) = a_1+1$.

Lemma 2.5 and Bezout gives that either there is $c \in \{0,1,2\}$ and $U \in |\mathcal{O}(1,c)|$ with $U \subset G$ and $\deg(\mathrm{Res}_T(Z) \cap U) \geq a_2 - 1 + ca_1 + 2$ or there is $T' \in |\mathcal{O}(0,1)|$ with $\deg(T' \cap \mathrm{Res}_T(Z)) \geq a_1 + 2$ (a case excluded by the minimality of E). Set $G' := T + U$. We have $G' \in |\mathcal{O}(1,c+1)|$ and $\deg(Z \cap G') \geq a_1 + 1 + a_2 - 1 + ca_1 + 2$. Take $b = c + 1$. □

3. The end of the proofs

Proof of Theorem 1.1 and Proposition 1.3: The " if " part of Theorem 1.1 is true because any $D \in |\mathcal{O}(o_1, o_2)|$ with $o_1 \leq 2$ or $o_2 \leq 2$ has $h^0(\mathcal{O}_D(a_1, a_2)) = o_1 a_2 + o_2 a_1 + 1 - e$ with $e = 0$ if $(o_1, o_2) \neq (2,2)$ and $e = 1$ if $(o_1, o_2) = (2,2)$.

Now we check the " only if " part. Assume $h^1(\mathcal{I}_Z(a,a)) > 0$. Since the cases $a = 0, 1$ are trivial and the case $a = 2$ follows from Lemma 2.6, we assume $a \geq 3$. Set $Z_0 := Z$. Let E_1 be any element of $|\mathcal{O}(2,2)|$ such that $e_1 := \deg(Z \cap E_1)$ is maximal. Set $Z_1 := \mathrm{Res}_{E_1}(Z)$. Define recursively the integer e_i, $E_i \in |\mathcal{O}(2,2)|$ and Z_i in the following way. Let E_i be any element of $|\mathcal{O}(2,2)|$ such that $e_i := \deg(Z_{i-1} \cap E_i)$ is maximal. Set $Z_i := \mathrm{Res}_{E_i}(Z_{i-1})$. We have a residual exact sequence

$$0 \to \mathcal{I}_{Z_i}(a-2i, a-2i) \to \mathcal{I}_{Z_{i-1}}(a-2i+2, a-2i+2) \to \mathcal{I}_{Z_{i-1} \cap E_i, E_i}(a_1 - 2i + 2, a_2 - 2i + 2) \to 0 \tag{3.1}$$

Since $h^0(\mathcal{O}(2,2)) = 9$, if $e_i \leq 7$, then $Z_i = \emptyset$.

(a) Assume that a is even; since will use some parts of this proof also for odd a we sometimes makes upper or lower bounds which allow a odd. Since $e_i \geq 8$ if $Z_i \neq \emptyset$, $\deg(Z) \leq 4a + 1$ and $h^1(\mathcal{I}_A) = 0$ if either $A = \emptyset$ or A is a point, there is an integer $i \leq a/2$ such that $h^1(\mathcal{I}_{Z_{i-1} \cap E_i}(a - 2i + 2, a - 2i + 2)) > 0$ and call f the minimal such an integer. By Lemma 2.6 if $f = 1$, then the lemma is true. Therefore we may assume $f \geq 2$. Since $h^1(\mathcal{I}_{E_f \cap Z_{f-1}}(a - 2f + 2, a - 2f + 2)) > 0$, Lemma 2.6 shows that either $\deg(E_f \cap Z_{f-1}) \geq 4(a - 2f + 2)$ or there is $D \in (|\mathcal{O}(1,0)| \cup |\mathcal{O}(0,1)|)$ with $\deg(D \cap E_f \cap Z_{f-1}) \geq a - 2f + 4$ or there is $C \in |\mathcal{O}(1,1)|$ with $\deg(C \cap E_f \cap Z_{f-1}) \geq 2a - 4f + 6$ or there is $C_1 \in (|\mathcal{O}(2,1) \cup |\mathcal{O}(1,2)|)$ with $\deg C_1 \cap E_f \cap Z_{f-1}) \geq 3a - 6f + 8$.

(a1) Assume $\deg(E_f \cap Z_{f-1}) \geq 4(a - 2f + 2)$. Since $e_i \geq e_f$ for all $i < f$, we get $4a + 1 \geq \deg(Z) \geq 4f(a - 2f + 2)$. Set $\psi(t) = 4t(a + 2 - 2t)$. The continuos function $\psi(t)$ is strictly increasing in the interval $(0, (a+2)/4)$ and strictly decreasing if $t \geq (a+2)/4$. Since $\psi(2) = \psi(a/2) = 8a > 4a + 1$, we get a contradiction.

(a2) Assume the existence of $C_1 \in (|\mathcal{O}(2,1)| \cup |\mathcal{O}(1,2)|)$ with $\deg(C_1 \cap E_f \cap Z_{f-1}) \geq 3a - 6f + 8$. Since $h^0(\mathcal{O}(0,1)) = h^0(\mathcal{O}(0,1)) = 2$, the maximality property of the integer e_{f-1} gives $e_{f-1} \geq 3a - 6f + 9$. Hence $4a + 1 \geq \deg(Z) \geq 3f(a + 3 - 2f) - 1$. The continuos function $\eta(t) := 3t(a + 3 - 2t) - 1$ is strictly increasing if $0 < t < (a+3)/4$ and strictly decreasing if $t > (a+3)/4$. We have $\eta(3) = 9a - 28 > 4a + 1$ if $a \geq 6$. We have $\eta(2) = 6a - 7 > 4a + 1$ if $a \geq 10$. We have $\eta(a/2) = 9a/2 - 1 > 4a + 1$ (if $a \geq 6$). Hence either $a = 4$ or $f = 2$ and $a = 6, 8$. For the case $a = 4$ see step (d). Assume $5 \leq a \leq 8$ and $f = 2$ and

hence $e_2 \geq 3a-4$, $e_1 \geq 3a-3$. Thus $4a+1 \geq \deg(Z) \geq e_1+e_2 \geq 6a-7$, a contradiction.

(a3) Assume the existence of $C \in |\mathcal{O}(1,1)|$ with $\deg(C \cap E_f \cap Z_{f-1}) \geq 2a-4f+6$. Since $Z_{f-1} \neq \emptyset$ and $h^0(\mathcal{O}(1,1)) = 4$, we get $e_{f-1} \geq 3+2a-4f+6$. Hence $4a+1 \geq 3(f-1)+f(2a-4f+6)$. Set $\varphi(t) = 2t(a+3-2t)-3$. The continuos function $\varphi(t)$ is strictly increasing if $0 < t < (a+3)/4$ and strictly decreasing if $t > (a+3)/4$. We have $\varphi(2) = 4a+7 > \deg(Z)$. We have $\varphi(a/2-2) = 7(a-4)-3 > 4a+1$ if $3a > 28$, i.e. if $a \geq 10$. We have $\varphi(a/2) = 3a-3$. Since $2f \leq a$, then $\deg(C \cap Z_{e-1} \cap E_f) \geq 6$. Since $h^0(\mathcal{O}(1,1)) = 4$, the maximality of the integer e_{f-1} gives $e_{f-1} \geq 9$. Hence $9(f-1)+6 \leq 4a+1$ and so $f \leq (4a+4)/9$. We have $\varphi((4a+4)/9) = 8(a+1)(a+21)/81-3$. We have $8(a+1)(a+21)/81-3 > 4a+1$ if $2(a+21) > 81$, i.e. if $a \geq 10$. Assume $a = 6,7,8,9$. First assume $f = 2$. We get $e_2 \geq 2+2(a-2) = 2a-2$ and $e_1 \geq 3+e_2 \geq 2a+1$. Hence $\deg(Z) \geq e_1+e_2 \geq 4a+4$, a contradiction. If $f \geq 4$ (i.e. $f = 4$ and $a = 8,9$), then $e_4 \geq 2a-10$, $e_1 \geq e_2 \geq e_3 \geq 2a-7$ and hence $4a+1 \geq \deg(Z) \geq e_1+e_2+e_3+e_4 \geq 8a-21$, a contradiction. Now assume $f = 3$. We get $e_3 \geq 2a-6$ and $e_1 \geq e_2 \geq 2a-3$. If $a \geq 7$ we get $e_1+e_2+e_3 \geq 6a-12 > 4a+1$, a contradiction. For the case $a = 6$, we covered all cases with $\deg(Z) \leq 23 = 4a-1$. If $a = 6$ and $\deg(Z) \geq 4a$ we take $F_1 \in |\mathcal{O}(2,2)|$ with $F_1 \supset C$ and $g_1 := \deg(F_1 \cap C)$ maximal. As in step (a4) below we get some integers g_i, $i \geq 2$ and $g \leq 3$; since $g \leq 3$, we get $e_1 \geq 1+(2a-6)+(a-2) = 11$; since $e_2 \geq 9$ and $e_1 \geq 6$, we get a contradiction.

(a4) Assume the existence of $D \in (|\mathcal{O}(1,0)| \cup |\mathcal{O}(0,1)|)$ with $\deg(C \cap E_f \cap Z_{f-1}) \geq a-2f+4$. With no loss of generality we may assume $D \in |\mathcal{O}(1,0)|$. Since $h^0(\mathcal{O}(1,2)) = 6$, the maximality property of the integer e_{f-1} gives $e_{f-1} \leq a-2f+9$. Hence $4a+1 \geq \deg(Z) \geq f(a+9-2f)-5$. We get that either $f \leq 4$ or $a = 10$ and $f = 5$ (remember that $a \geq 2f$). Set $W_0 := Z$. Let F_1 be any element of $|\mathcal{O}(2,2)|$ containing D and such that $g_1 := \deg(Z \cap F_1)$ is maximal. We have $g_1 \geq a-2f+9$. Set $W_1 := \mathrm{Res}_{F_1}(Z)$. Define recursively the integer g_i, $F_i \in |\mathcal{O}(2,2)|$ and W_i in the following way. Let F_i be any element of $|\mathcal{O}(2,2)|$ such that $g_i := \deg(W_{i-1} \cap F_i)$ is maximal. Set $W_i := \mathrm{Res}_{F_i}(W_{i-1})$. We have a residual exact sequences (3.1) with W_j instead of Z_j and F_j instead of E_j. We have $g_i \geq g_{i+1}$ if $i \geq 2$. If $g_i \leq 7$, then $W_i = \emptyset$. Since $\deg(Z)-g_1 \leq 3a+2f-9$, there is a minimal integer $g \leq a/2$ such that $h^1(F_g, \mathcal{I}_{F_g \cap W_{g-1}, F_g}(a-2g+2, a-2g+2)) > 0$. By Lemma 2.6 we may assume $g \geq 2$. By Lemma 2.6 either $\deg(F_g \cap W_{g-1}) \geq 4(a-2g+2)$ or there is $D' \in (|\mathcal{O}(1,0)| \cup |\mathcal{O}(0,1)|)$ with $\deg(D' \cap F_g \cap W_{f-1}) \geq a-2g+4$ or there is $C' \in |\mathcal{O}(1,1)|$ with $\deg(C' \cap E_f \cap Z_{f-1}) \geq 2a-4g+6$ or there is $C'' \in (|\mathcal{O}(2,1)| \cup |\mathcal{O}(1,2)|)$ with $\deg(C'' \cap E_f \cap Z_{f-1}) \geq 3a-6g+8$. All cases with C' or C'' or with $\deg(F_g \cap W_{g-1}) \geq 4(a-2g+2)$ are done for $a \geq 10$ as in step (a3). Hence we may Since $h^0(\mathcal{O}(1,2)) = h^0(\mathcal{O}(2,1)) = 6$, we have $g_i \geq a-2g+9$ if $2 \leq i < g$. We also have $g \leq 4$, unless $a = 10$, in which also $g = 5$ could be allowed. Assume $a = 10$ and $f = 5$. We get $e_1 = e_2 = e_3 = e_4 = 9$ and $e_5 = 8$; since $h^0(\mathcal{O}(2,3)(-D-D')) \geq 3$, we have $e_1 \geq 2+4+4$, a contradiction. Hence $f \leq 4$ and in the same way we see that $g \leq 4$. Since $h^0(\mathcal{O}(2,3)(-D-D')) \geq 3$, $f \leq 4$ and $g \leq 4$, we get $e_1 \geq 2+(a-2)+(a-2)$. Hence $4a+1 \geq \deg(Z) \geq 2a-2+(f-1)(a-2f+4)+5(f-2)$; if $a \geq 4$ we exclude the case $f = 4$. Hence $f \leq 3$ and we also exclude in the same way the case $g = 4$.

(a4.1) Assume $f = 3$. First assume $g = 2$. We have $e_3 \geq a-2$ and $e_2 \geq a+3$. Taking an element of $|\mathcal{O}(2,2)|$ containing $D+D'$ we get $e_1 \geq 2a+\varepsilon$ with $\varepsilon = 0$ if $D' \in |\mathcal{O}(1,0)|$ and $\varepsilon = 1$ if $D' \in |\mathcal{O}(0,1)|$. Thus $D' \in |\mathcal{O}(1,0)|$, $\deg(Z) = 4a+1$ and the previous inequalities are equalities. Take $G \in |\mathcal{O}(2,2)|$ with $G \supset D \cup D'$ and maximal $\deg(G \cap Z)$. By Lemma 2.6 we may assume $h^1(\mathcal{I}_{\mathrm{Res}_G(Z)}(a-2,a-2)) > 0$ with $\deg(\mathrm{Res}_G(Z)) \leq 2a+1$. By Lemma 2.3 we get that either there is $D_1 \in (|\mathcal{O}(1,0)| \cup |\mathcal{O}(0,1)|)$ with $\deg(D_1 \cap \mathrm{Res}_G(Z)) \geq a$ or there is $C_1 \in |\mathcal{O}(0,1)|$ with $\deg(C_1 \cap (\mathrm{Res}_G(Z)) \geq 2a-2$. C_1 cannot exist, because taking a curve containing $D' \cup C_1$ it would give $e_1 \geq 1+a+2a-2$. There is no $D_1 \in |\mathcal{O}(0,1)|$, because (since $h^0(\mathcal{O}(0,1)) = 2$) there is $G' \in |\mathcal{O}(2,2)|$ with $G' \supset D \cup D' \cup D_1$ and $\deg(G' \cap Z) \geq 1+(a-2)+a+a$ and hence $e_1 \geq 3a-1$. Hence there is $D_1 \in |\mathcal{O}(1,0)|$. We may assume $\deg(Z \cap T) \leq a+1$ if $T = D, D', D_1$. Hence the residual exact sequence of $D+D'+D_1$ gives $h^1(\mathcal{I}_{\mathrm{Res}_{D+D'+D_1}(Z)}(a-3,a)) > 0$. Since $\deg(\mathrm{Res}_{D+D'+D_1}(Z)) \leq a+3$, at least for $a \geq 6$ Lemma 2.2 gives either the existence of $D_2 \in |\mathcal{O}(0,1)|$ with $\deg(D_2 \cap \mathrm{Res}_{D+D'+D_1}(Z)) \geq a-1$ or the existence of $D_3 \in |\mathcal{O}(1,0)|$ with $\deg(D_3$
$capRes_{D+D'+D_1}) \geq a+1$. If D_3 exists, we get a contradiction taking $D_3+D'+D_1$. Assume the existence of D_2. Since we may assume that $\deg(Z \cap D_2) \leq a+1$, we have $h^1(\mathcal{I}_{\mathrm{Res}_{D+D'+D_1+D_2}(Z)}(a-4,a)) > 0$. We have $\deg(\mathrm{Res}_{D+D'+D_1+D_2}(Z)) \leq 3$ and hence we get a contradiction (at least if $a \geq 6$).

Now assume $g = 3$. Take $G_1 \in |\mathcal{O}(2,2)|$ with $G_1 \supset D+D'$ and maximal $\gamma_1 := \deg(G_1 \cap Z)$. We have $\gamma_1 \geq 2a-2+\varepsilon$ with $\varepsilon = 0$ if $D' \in |\mathcal{O}(1,0)|$ and $\varepsilon = 1$ if $D' \in |\mathcal{O}(0,1)|$. Set $W_1' := \mathrm{Res}_{G_1}(Z)$. By Lemma 2.6 we may assume $h^1(\mathcal{I}_{W_1'}(a-2,a-2)) > 0$. As in the first part of step (a4) we get zero-dimensional schemes W_i', divisors $G_i \in |\mathcal{O}(2,2)|$, integers γ_i, $i \geq 2$, and a minimal integer γ with $h^1(G_\gamma, \mathcal{I}_{W'_{\gamma-1} \cap G_\gamma}(a+2-2\gamma, a+2-2\gamma)) > 0$. We first get $\gamma \leq 3$ and then the existence $A \in (|\mathcal{O}(1,0)| \cup |\mathcal{O}(0,1)|)$ with $\deg(A \cap W'_{\gamma-1}) \geq a+4-2\gamma$. Take $(o_1,o_2) \in \mathbb{N}^2$ such that $A+D+D' \in |\mathcal{O}(o_1,o_2)|$.

First assume $(o_1,o_2) = (3,0)$. Since (as in the case $u_1 = 0$ of the proof of Lemma 2.11) we may assume $h^1(\mathcal{I}_{\mathrm{Res}_{A+D+D'}(Z)}(a-3,a)) > 0$, the inequality $\deg(\mathrm{Res}_{A+D+D'}(Z)) \leq a+5$ and Lemma 2.3 give that either there is $B \in |\mathcal{O}(1,0)|$ with $\deg(B \cap \mathrm{Res}_{A+D+D'}(Z)) \geq a+1$ or there is $B' \in |\mathcal{O}(0,1)|$ with $\deg(B' \cap \mathrm{Res}_{A+D+D'}(Z)) \geq a-2$. Assume the existence of B. Since $h^0(\mathcal{O}(0,2)) = 3$, we get $g_1 \geq 2a+1$; since $e_3 \geq a-2$ and $e_2 \geq a+3$, we obtained a contradiction. Assume the existence of B'. If $h^1(\mathcal{I}_{Z \cap (A+D+D'+B')}(a,a)) > 0$, then we use Lemma 2.11. Hence we may assume $h^1(\mathcal{I}_{\mathrm{Res}_{A+D+D'+B'}(Z)}(a-3,a-1)) > 0$. Since $\deg(\mathrm{Res}_{A+D+D'+B'}(Z)) \leq 9$, we get $a \leq 10$ and that if $a \geq 8$, there is $T \in |\mathcal{O}(0,1)|$ with $\deg(T \cap \mathrm{Res}_{A+D+D'+B'}(Z)) \geq a-1$. If T exists, then $A+D+B'+T \in |\mathcal{O}(2,2)|$ and $\deg(Z \cap (A+D+B'+T) \geq 4a-8$; hence $e_2+e_3 \leq 9$, contradicting the inequalities $e_2 \geq a+3$ and $e_3 \geq a-2$. If $6 \leq a \leq 9$ and T does not exists, then we get $T' \in |\mathcal{O}(1,0)|$ with $\deg(T' \cap \mathrm{Res}_{A+D+D'+B'}(Z)) \geq a+1$; using an element of $|\mathcal{O}(2,2)|$ containing $D+T'$ and that $h^0(\mathcal{O}(1,1)) = 4$, we get $e_1 \geq 2a+2$, contradicting the inequalities $e_2 \geq a+3$, $e_3 \geq a-2$.

Now assume $(o_1,o_2) \neq (3,0)$, i.e. $(o_1,o_2) \in \{(2,1),(1,2)\}$. By Lemma 2.5 we may assume $h^1(\mathcal{I}_{Z \cap (A+D+D')}(a,a)) = 0$ and so $h^1(\mathcal{I}_{\mathrm{Res}_{A+D+D'}(Z)}(a-o_1,a-o_2)) > 0$.

We have $\deg(\mathrm{Res}_{A+D+D'}(Z)) \le a+7$. If $a > 12$, Lemma 2.3 gives the existence of either $M \in |\mathcal{O}(1,0)|$ with $\deg(M \cap \mathrm{Res}_{A+D+D'}(Z)) \ge a - o_2 + 2$ or $M' \in |\mathcal{O}(0,1)|$ with $\deg(M' \cap \mathrm{Res}_{A+D+D'}(Z)) \ge a + o_1 + 2$. Call $M'' \in \{M, M'\}$ the element which exists. If $M'' + A + D + D' \in |\mathcal{O}(2,2)|$, then we conclude, because $\deg(\mathrm{Res}_{M''A+D+D'}(Z)) \le 7 < 2a+1 \le e_2 + e_3$. If $M'' + A + D + D' \in (|\mathcal{O}(3,1)| \cup |\mathcal{O}(1,3)|)$, then we use Lemma 2.11. If $6 \le a \le 12$ we must allow the existence of $M'' \in |\mathcal{O}(1,1)|$ with $\deg(M'' \cap \mathrm{Res}_{A+D+D'}(Z)) \ge 2 + a - o_1 + a - o_2 = 2a - 1$, but in this case (since $h^0(\mathcal{O}(1,0)) = 2$) we would get $e_1 \ge 1 + (2a-1) + a - 2$, contradicting the inequalities $e_2 \ge a+3$, $e_3 \ge a-2$.

(a4.2) Assume $f = 2$. The residual sequence of D gives $h^1(\mathcal{I}_{\mathrm{Res}_D(Z)}(a-1,a)) > 0$. Since $\deg(\mathrm{Res}_D(Z)) \le 3a+1$, Lemma 2.10 gives that either there is $D_1 \in |\mathcal{O}(1,0)|$ with $\deg(D_1 \cap \mathrm{Res}_D(Z)) \ge a+2$ (proving the theorem in this case) or there is $D_2 \in |\mathcal{O}(0,1)|$ with $\deg(D_2 \cap \mathrm{Res}_D(Z)) \ge a+1$ or there is $C \in |\mathcal{O}(1,1)|$ with $\deg(\mathrm{Res}_D(Z) \cap C) \ge 2a+1$ or there is $C_1 \in |\mathcal{O}(1,2)|$ with $\deg(C_1 \cap \mathrm{Res}_D(Z)) \ge 3a$ or $\deg(Z) = 3a+1$ and $Z \subset C_2$ for some $C_2 \in |\mathcal{O}(2,1)|$. If C exists, then since $h^0(\mathcal{O}(0,1)) = 2$ we get $e_1 \ge 3a+2$ and hence $e_2 \le a-1$, a contradiction. If C_1 exists, then $e_1 \ge 4a$, a contradiction. If C_2 exists, then $e_1 \ge 3a+2$, because $h^0(\mathcal{O}(0,1)) = 2$, concluding this case. Now assume the existence of D_2. By Lemma 2.5 we may assume $h^1(\mathcal{I}_{Z \cap (D+D_2)}(a,a)) = 0$. Hence $h^1(\mathcal{I}_{\mathrm{Res}_{D+D_2}(Z)}(a-1,a-1)) > 0$. Since $\deg(\mathrm{Res}_{D+D_2}(Z)) \le 2a$, Lemma 2.3 shows that either $\deg(Z) = 4a+1$ and $Z \subset C' + D + D_2$ for some $C' \in |\mathcal{O}(1,1)|$ or there is $D_3 \in (|\mathcal{O}(1,0)| \cup |\mathcal{O}(0,1)|$ with $\deg(D_3 \cap (\mathrm{Res}_{D+D_2}(Z)) \ge a+1$. If C' exists, then Lemma 2.6 proves the theorem. Now assume the existence of D_3. Since $h^0(\mathcal{O}(1,0)) = 2$, we get that $e_1 \ge 1 + \deg(Z \cap (D_3 + D + D_2)) \ge 3a+3$, a contradiction.

(b) Assume a odd. The proof of step (a) work if $\deg(Z) \le 8(a-1)/2 + 2$ and $a \ge 11$, because $h^1(\mathcal{I}_A(1,1)) = 0$ for every zero-dimensional scheme $A \subset \mathbb{P}^1 \times \mathbb{P}^1$ with $\deg(A) \le 2$, but not if $\deg(Z) \ge 4a-1$. If $4a-1 \le \deg(Z) \le 4a+1$ we use the following trick. We first run the program with E_i, e_i, Z_i, F_i. We say that it fails if $h^1(E_i, \mathcal{I}_{E_i \cap Z_{i-1}, E_i}(a+2-2i, a+2-2i)) = 0$ for $i = 1, \dots, \lfloor a/2 \rfloor$. Fix any subscheme $A \subset Z$ with $\deg(A) = 15$. Since $h^0(\mathcal{O}(3,3)) = 16$, there is $F \in |\mathcal{O}(3,3)|$ with $A \subset F$. Set $Z_0 := \mathrm{Res}_F(Z)$. We have $\deg(Z) \le \deg(Z) - 15 \le 4(a-3) + 1$.

(b1) Assume $h^1(F, \mathcal{I}_{F \cap Z}(a,a)) > 0$. Since A is arbitrary, we get the existence of an integer $w \le 16$ such that $h^1(\mathcal{I}_B(a,a)) > 0$ for each zero-dimensional scheme $B \subset Z$ with $\deg(Z) \le w$. If $a \ge 17$, this is absurd. If $a \ge 8$, it implies the existence of $D_1 \in (|\mathcal{O}(1,0)| \cup \mathcal{O}(0,1)|$ such that $\deg(D_1 \cap Z) \ge a+2$. Since $h^0(\mathcal{O}(1,2)) = h^0(\mathcal{O}(2,1)) = 6$, we get $e_1 \ge a+7$. Hence in this case there is an integer $i \le (a-1)/2$ such that $h^1(E_i, \mathcal{I}_{E_i \cap Z_{i-1}, E_i}(a-2i+2, a-2i+2)) > 0$. If $5 \le a \le 7$ and D_1 does not exists, then there is $C \in |\mathcal{O}(o_1, o_2)|$ with $2 \le o_1 + o_2 \le 3$ and $\deg(C \cap B) \ge (o_1 + o_2)a + 2$, and hence $e_1 \ge 2a+5$.

(b2) Assume $h^1(F, \mathcal{I}_{F \cap Z}(a,a)) = 0$. We have $\deg(Z_0) \le \deg(Z) - 15 \le 4a + 1 - 15 = 4(a-3) - 2$. The residual exact sequence of F gives $h^1(\mathcal{I}_{Z_0}(a-3, a-3)) > 0$. We repeat the proof of part (a) with Z_0 and the integer $a-3$ instead of a. Use a $\tilde{}$ for the integers e_i, f and g that we get with this program for $\mathcal{O}(a-3, a-3)$ for the new scheme $Z_0 := \mathrm{Res}_E(Z)$ and e_i, f, g without $\tilde{}$ for the program for Z

and $\mathcal{O}(a,a)$. For $\mathcal{O}(a-3,a-3)$ we cannot exclude the cases $\widetilde{f}=1$ and $\widetilde{g}=1$, because we apply Lemma 2.6 to $\mathcal{O}(a-3,a-3)|$. Assume for instance $\widetilde{g}=1$ and the existence of $D' \in |\mathcal{O}(1,0)|$ such that $\deg(D' \cap Z_0) \geq a-1$. Since $h^0(\mathcal{O}(1,2))=6$, we get $e_1 \geq a+4$. Hence $e_{\lfloor a/2 \rfloor}=0$, contradicting the assumption that the program with E_i, Z_i, e_i does not stop, i.e. there is no integer $i \leq \lfloor a/2 \rfloor$ such that $h^1(\mathcal{I}_{Z_{i-1}\cap E_i}(a-2i+2,a-2i+2))>0$. The case $a-3=10$ and either $\widetilde{f}=5$ or $\widetilde{g}=5$ is excluded, because $\deg(\mathrm{Res}_E(Z)) \leq 4(a-3)-2$. Assume $2 \leq \widetilde{f} \leq 4$ and $2 \leq \widetilde{g} \leq 4$. We get that either $e_{\widetilde{f}} \geq 2(a-3+2-2\widetilde{f})+2$ or $\widetilde{e}_{\widetilde{f}} \geq a-3+4-2\widetilde{f} \geq a-7$ and the existence of $D \in |\mathcal{O}(1,0)| \cup |\mathcal{O}(0,1)|$. Since $h^0(\mathcal{O}(2,1))=6$, in both cases we get $e_1 \geq a-2$. Since $4a+1-(a-2)-8(a-3)/2 = 15-a$, we get $\deg(Z_{(a-1)/2}) \leq \max\{0, 15-a\}$. If $a \geq 13$, then $h^1(\mathcal{I}_{Z_{(a-1)/2}}(1,1))=0$, contradicting the assumption that the program with E_i, Z_i, e_i does not stop.

(c) Assume $a=3$.

(c1) Assume $\deg(Z)=13$, First assume $Z \subset F$ for some $F \in |\mathcal{O}(3,2)|$. Since $h^1(F, \mathcal{I}_{Z,F}(3,3))>0$ and $\mathcal{O}_F(3,3) \cong \omega_X(2,3)$. Duality gives $h^0(F, \mathcal{I}_{Z,F}(2,3))=1$. Take $F_1 \in |\mathcal{O}(2,3)|$ with $Z \subset F_1$. Z is as in Example 1.2 with $F=A$ and $F_1=B$, unless F and F_1 have a common component. Let $G \subset F \cap F_1$ be the one-dimensional part of $F \cap F_1$ with, say, $G \in |\mathcal{O}(o_1,o_2)|$, $o_1 \leq 2$ and $o_2 \leq 2$. We have $\deg(\mathrm{Res}_G(Z)) \leq (F-G)\cdot F_1 - G) = (3-o_1)(3-o_2)+(2-o_2)(2-o_1)$. If Z is not as in Example 1.2 then there is an upper bound for $\deg(G \cap Z)$ depending on (o_1,o_2). If $(o_1,o_2) \in \{(1,0),(0,1)\}$, then $13 \geq 4+8$, a contradiction. If $(o_1,o_2)=(1,1)$, then $13 \geq 7+5$, a contradiction. If $(o_1,o_2) \in \{(2,1),(1,2)\}$, then $13 \geq 9+3$, a contradiction. Now assume $o_1=o_2=2$. In this case we get $\deg(G \cap Z)=12$. If $Z \cap G$ is not as in Lemma 2.6, then $h^1(G, \mathcal{I}_{Z\cap G,G}(3,3)=0$. Since $\deg(\mathrm{Res}_G(Z))=1$, then $h^1(\mathcal{I}_{\mathrm{Res}_G(Z)}(1,1))=0$. A residual exact sequence gives a contradiction.

(c2) Assume $\deg(Z)=12$. Since $h^1(\mathcal{I}_Z(3,3))>0$, then $h^1(\mathcal{I}_Z(2,3))>0$ and $h^1(\mathcal{I}_Z(3,2))>0$. Thus there are $F \in |\mathcal{O}(3,2)|$ and $F_1 \in |\mathcal{O}(2,3)|$ with $Z \subset F \cap F_1$. If $F \cap F_1$ contains no curve, then $h^1(\mathcal{I}_Z(3,3))=0$ by the last sentence of Example 1.2. Hence we may assume that $F \cap F_1$ has a one-dimensional component and call G the union of the one-dimensional components of $F \cap F_1$ with $G \in |\mathcal{O}(o_1,o_2)|$ with, say, $o_1 \geq o_2$. Since $o_1 \leq 2$, by Lemma 2.6 we may assume $h^1(G, \mathcal{I}_{G\cap Z,G}(3,3))=0$ and hence $h^1(\mathcal{I}_{\mathrm{Res}_G(Z)}(3-o_1,3-o_2))$. As in step (c1) we get that in each case $\mathrm{Res}_G(Z)$ is the complete intersection of a form $F-G$ of bidegree $(3-o_1,2-o_2)|$ and a form F_1-G of bidegree $(2-o_1,2-o_2)$. The exact sequence

$$0 \to \mathcal{O}(-2+o_1,-2+o_2) \to \mathcal{O}(0,1) \oplus \mathcal{O}(1,0) \to \mathcal{I}_Z(3-o_1,2-o_2) \to 0$$

gives $h^1(\mathcal{I}_{\mathrm{Res}_G(Z)}(3-o_1,3-o_2))=0$, a contradiction.

(c3) Assume $\deg(Z) \leq 11$. Take $Z_1 \supset Z$ with $\deg(Z_1)=12$. We have $h^1(\mathcal{I}_{Z_1}(3,3))>0$. Use the proof of step (c1) (which didn'used that $h^1(\mathcal{I}_{Z'}(3,3))=0$ for all $Z' \subsetneq Z$).

(d) Assume $a=4$. We have $\deg(Z) \leq 17$. Fix any $G \in |\mathcal{O}(3,3)|$ such that $\deg(G \cap Z) \geq 15$. Since $\deg(\mathrm{Res}_G(Z)) \leq 2$, we have $h^1(\mathcal{I}_{\mathrm{Res}_G(Z)}(1,1))=0$. Hence $h^1(\mathcal{I}_{G\cap Z}(4,4))>0$.

(e) If $a \geq 5$, in the set-up of Proposition 1.3 all cases with either $f = 2$ or $g = 2$ land into a case in which that proposition is true. Hence we always have $f \geq 3$ and $g \geq 3$. Take $F \in |\mathcal{O}(4,4)|$ with $\deg(F \cap Z)$ maximal. Assume $a = 5$. We have $\deg(Z) \leq 21$ and hence $Z \subset F$. Now assume $a = 6$. Since $\deg(Z) \leq 25$, we have $\deg(\mathrm{Res}_F(Z)) \leq 1$ and hence $h^1(\mathcal{I}_{\mathrm{Res}_F(Z)}(2,2)) = 0$. Now assume $a = 7$. Since $\deg(\mathrm{Res}_F(Z)) \leq 5$, if $h^1(\mathcal{I}_{\mathrm{Res}_F(Z)}(3,3)) > 0$, then $\deg(Z) = 29$, $\deg(\mathrm{Res}_F(Z)) = 5$ and there is $D \in (|\mathcal{O}(1,0)| \cup |\mathcal{O}(0,1)|)$ with $\mathrm{Res}_F(Z) \subset D$. Since $h^0(\mathcal{O}(3,4)) = 20$, we may find $G \in |\mathcal{O}(4,4)|$ with $G \supset D$ and $\deg(G \cap Z) \geq 24$. Since we may assume that $h^1(\mathcal{I}_{\mathrm{Res}_G(Z)}(3,3)) > 0$, then $\deg(\mathrm{Res}_G(Z)) = 5$ and there is $D_1 \in (|\mathcal{O}(1,0)| \cup |\mathcal{O}(0,1)|)$ with $\mathrm{Res}_G(Z) \subset D_1$. We repeat two other times and we get divisors D_2, D_3 with $D_i \in (|\mathcal{O}(1,0)| \cup |\mathcal{O}(0,1)|)$ and $\deg(Z \cap (D + D_1 + D_2 + D_3)) \geq 20$. Write $D + D_1 + D_2 + D_3 \in |\mathcal{O}(u_1, u_2)|$ with $u_1 + u_2 = 4$. If $(u_1, u_2) \notin \{(4,0), (0,4)\}$, then we get $\deg(F \cap Z) \geq 25$, a contradiction. Now assume $(u_1, u_2) = (4,0)$ (the case $(u_1, u_2) = (0,4)$ is similar). We have $\deg(\mathrm{Res}_{D+D_1+D_2+D_3}(Z)) \leq 5$ and $h^1(\mathcal{I}_{\mathrm{Res}_{D+D_1+D_2+D_3}(Z)}(3,7)) = 0$. Hence there is $J \in |\mathcal{O}(0,1)|$ such that $\deg(J \cap \mathrm{Res}_{D+D_1+D_2+D_3}(Z)) \geq 5$. Since $J + D + D_1 + D_2 + D_3 \in |\mathcal{O}(4,1)|$, we get $\deg(F \cap Z) > 24$, a contradiction. For $a \geq 8$ we only use that $f \geq 3$ and $g \geq 3$. □

Proof of Proposition 1.4: Set $Z_0 := Z$. We run E_i, Z_i, e_i as in step (a) of the proof just given for $i = 1, \dots, \lfloor a_2/2 \rfloor$ with the exact sequences

$$0 \to \mathcal{I}_{Z_i}(a_1 - 2i, a_2 - 2i) \to \mathcal{I}_{Z_{i-1}}(a_1 + 2 - 2i, a_2 + 2 - 2i) \to$$
$$\mathcal{I}_{Z_{i-1} \cap E_i, E_i}(a_1 + 2 - 2i, a_2 + 2 - 2i) \to 0$$

(a) Assume that $h^1(E_i, \mathcal{I}_{E_i \cap Z_i}(a_1 + 2 - 2i, a_2 + 2 - 2i)) > 0$ for some integer i with $1 \leq i \leq \lfloor a_2/2 \rfloor$ and call f the minimal such an integer. If $f = 1$, then we use Lemma 2.6. Assume $f \geq 2$. We apply Lemma 2.6 and get that either $e_f \geq 2(a_1 + a_2 + 4 - 2i)$ or there is $(o_1, o_2) \in \{(1,0), (0,1), (1,1), (2,1), (1,2)\}$ and $D \in |\mathcal{O}(o_1, o_2)|$ with $\deg(Z_{f-1} \cap E_f \cap D) \geq o_1 a_2 + o_2 a_1 + 2$. Since $a_1 > a_2$ and $\deg(Z) \leq 4a_2 + 1$, in most cases we only need to test divisors $D, D', D_1 \in |\mathcal{O}(1,0)|$ and use the case $u_1 = 0$ at the beginning of the proof of Lemma 2.11.

(b) Assume $h^1(E_i, \mathcal{I}_{E_i \cap Z_i}(a_1 + 2 - 2i, a_2 + 2 - 2i)) = 0$ for all $i = 1, \dots, \lfloor a_2/2 \rfloor$. In this case we have a_2 odd and $\deg(Z) \geq 3 + 8(a_2 - 1)/2$, because we have $h^1(\mathcal{I}_{Z_{(a_2-1)/2-1}}(a_1 - a_2 + 1, 1)) > 0$. We do as in step (b) of the proof of the theorem. □

The following example shows that we cannot get bounds of type $\alpha a_1 + \beta a_2$ with $\alpha > 1$ and α, β not depending on a_1, a_2, and $a_1 \gg a_2$.

Example 3.1. Fix an integer $c \geq 3$ and $C \in |\mathcal{O}(c,1)|$ with C smooth. Fix integers $a_1 \geq a_2 \geq 0$. Let $Z \subset C$ be a zero-dimensional scheme. Since $h^1(\mathcal{O}(a_1 - c, a_2)) = 0$, then $h^1(\mathcal{I}_Z(a_1, a_2)) = h^1(C, \mathcal{I}_{Z,C}(a_1, a_2))$. Hence $h^1(\mathcal{I}_Z(a_1, a_2)) = 0$ if and only if $\deg(Z) \leq ca_2 + a_1 + 1$.

Acknowledgement. The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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