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TOPOLOGICAL DEGREE AND PERSISTENT HOMOLOGY FOR A DIVERGENCE-TYPE EQUATION

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Abstract. In this work, we employ topological degree theory to establish the existence of weak solutions for a nonlinear elliptic equation posed on a bounded domain in $\mathbb{R}^N$ with smooth boundary. Then, persistent homology is used to investigate the topological structure of the solution and to monitor the evolution of its critical points, capturing their birth, disappearance, and persistence across multiple scales.

Keywords. fourth-order elliptic; Navier boundary value problem; weak solution; variational method; Persistence homology; Vietoris-Rips complex; Critical points
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1. Introduction

In this paper, we begin by studying the existence of a unique solution for the following stationary degenerate nonlinear elliptic problem:

$$\begin{cases} -\nabla.(a\nabla u) = \lambda|u|^{p-2}u + \mu|u|^{q-2}u(m(x) - |u|^r) + f(x), & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where $p > 2, q > 2, r > 0$ and λ is a negative parameter, while μ is a positive parameter.

In recent years, many researchers have given importance to reaction-diffusion type equations. The general form of such problems is as follows:

$$u_t = \operatorname{div}(D(u)\nabla u) + c(x, u)$$

This equation is important in many areas such as physical, biophysics [9, 13] , plasma physics [1, 16], solid state physics [14], and chemical reaction design [2].

Therefore, in our study, we examine stationary solutions of the equation (1.1), where the diffusion coefficient follows a divergence form and the reaction terms are considered $\lambda|u|^{p-2}u + \mu|u|^{q-2}u(m(x) - |u|^r) + f(x)$

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The problem (1.1) is considered as a generalization of the Fisher equation in the stationary case related to the following Fisher reaction-diffusion equation [10]:

$$u_t = u_{xx} + ru(1-u)$$

.

The type of equation

$$u_t = u_{xx} + ru(1-u)$$

has been developed in various studies. For example, in [12], King-Yeung et all. they proposed the following logistic model.

$$\begin{cases} \frac{\partial u}{\partial t} = d\Delta u + u\left[m(x) - u\right] & (x,t) \in \Omega \times (0,\infty), \\ \frac{\partial u}{\partial \nu} = 0 & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and $\nu(x)$ denotes the unit outward normal vector at $x \in \partial\Omega$. Here, $u(x,t)$ represents the density of the single species at location x and time t. The parameter $d > 0$ is the diffusion rate and $\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ is the usual Laplace operator.

We present the two problems in the stationary case [4, 5], for which the authors have studied the existence of solutions under certain conditions on the parameters.

$$\begin{cases} d\Delta u + u(m-u) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

$$\begin{cases} d\Delta u + r(x)u\left(1 - \dfrac{u}{K(x)}\right) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

Inspired by the above works and by the reaction-diffusion equation [15], in which the authors studied the following logistic model

$$(P)\begin{cases} u_t = \lambda\left(|u_x|^{p-2}u_x\right)_x + |u|^{q-2}u(1-|u|^r), & (x,t) \in (0,1) \times (0,+\infty), \\ u(0,t) = u(1,t) = 0, & t \in (0,+\infty), \\ u(x,0) = u_0(x), & x \in (0,1), \end{cases}$$

where $p > 2$, $q \geq 2$, $r > 0$, and λ is a positive parameter.

And by employing topological degree theory, we aim to establish the existence and uniqueness of a weak solution to the problem (1.1).

The second objective in this paper is to explore the use of *persistent homology* (PH) to analyze the topological features of solutions to partial differential equations (PDEs), with a particular emphasis on the behavior of their critical points.

Topological Data Analysis (TDA) has emerged as a powerful framework for extracting robust qualitative information from complex data using methods from algebraic topology. At the heart of TDA lies persistent homology, which captures the birth and death of topological features—such as connected components, loops,

and voids—across multiple scales. This multiscale perspective allows us to distinguish meaningful structures from topological noise in a coordinate-free and stable manner.

While PH has been successfully applied in various scientific fields—ranging from material science to biology—its application to the study of PDEs remains relatively underexplored. In our setting, we leverage PH to gain insight into the global structure of a PDE solution by examining the evolution of its critical points (maxima, minima, saddle points) as the level sets of the solution change.

By constructing a filtration induced by the scalar field defined by the solution, we apply persistent homology to track how critical topological features emerge, persist, and eventually vanish. This analysis provides a new lens through which to interpret the qualitative behavior of solutions to PDEs [8], revealing their structural complexity and potential bifurcation patterns in a rigorous topological framework.

2. Variational framework and main results

In this section, we present the essential background needed to prove Theorem 2.1.

Throughout this article Ω is an open, smooth and bounded subset of $\mathbb{R}^N$. We denote the norm of u in $L^p(\Omega)$ by $\|u\|_p = (\int_\Omega |u(x)|^p\, dx)^{\frac{1}{p}}$. The notation $\|.\|$ refers to the corresponding norm, $\rightharpoonup$ denotes weak convergence, and $\to$ denotes strong convergence.

We consider the Sobolev space

$$H^1(\Omega) := \{u \in L^2(\Omega) : u_{x_i} \in L^2(\Omega), 1 \le i \le N\},$$
$$H_0^1(\Omega) := \{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\}.$$

We denote $X = H_0^1(\Omega)$ endowed with the standard inner product:

$$\langle u, v\rangle = \int_\Omega \nabla u.\nabla v dx.$$

and the induced norm:

$$\|u\| = \Big(|\nabla u|^2)\Big)^{1/2}, \quad u \in X,$$

The first main results of this paper are presented in the following theorems:

Theorem 2.1. *Assume that $f \in X^*$. Then for each $\lambda < 0, \mu \ge 0$ and $m < 0$, problem (1.1) has a unique nontrivial weak solution.*

Note that X is a separable and reflexive real Banach space.

By the Sobolev embedding theorem, there is a positive constant C such that:

$$\|u\|_{L^\theta} \le C\|u\|$$

where $\|u\|_{L^\theta}$ denotes the norm of L^θ respectively for $1 \le \theta \le \frac{2N}{N-4}$.

To prove the existence of solutions for problem (1.1), we recall the following result (see [3]).

Theorem 2.2. *Let $(X, \|.\|)$ be a real reflexive Banach space and let $I : X \to X^*$ be an operator satisfying the conditions:*

(1) *I is coercive,*
(2) *I is hemicontinuous,*
(3) *I is monotone on the space X,*

Then the operator equation: $I(u) = f$ has at least one solution $u \in X$ for every $f \in X^*$. If, moreover, I is strictly monotone then the equation $I(u) = f$ has precisely one solution $u \in X$ for every $f \in X^*$.

3. PROOF OF MAIN RESULTS

In order to prove the existence and uniqueness of a solution to problem (1.1), we first recall some definitions and preliminary results that will be used in the analysis.

Definition 3.1. We say that a function $u \in X$ is a weak solution of problem (1.1) if:
$\int_\Omega a\nabla u.\nabla v dx = \lambda \int_\Omega |u|^{p-2}uv\, dx + \mu \int_\Omega |u|^{q-2}uv(m(x) - |u|^r)\, dx + \int_\Omega f(x)v\, dx$
for all $v \in X$

Definition 3.2. We say that $u \in X$ is a weak solution of the operator equation

$$B(u) = f,$$

if for all $f \in X^*$, it holds that

$$\langle B(u), v\rangle = \langle f, v\rangle.$$

for all $v \in X$
We put:

$$B(u) = -\nabla.(a\nabla u) - \lambda|u|^{p-2}u - \mu|u|^{q-2}u(m(x) - |u|^r)$$

Then

$$(B(u), v) = \int_\Omega a\nabla u.\nabla v dx - \lambda \int_\Omega |u|^{p-2}uv\, dx - \mu \int_\Omega |u|^{q-2}uv(m(x) - |u|^r)\, dx$$

Remark 3.3. In accordance with the theory of monotone operators, and based on Definitions (3.1) and (3.2), showing that $u \in X$ is a weak solution to problem (1.1) is equivalent to finding a solution to the operator equation

$$B(u) = f.$$

In the remainder of this section, we prove **Theorem** (2.1). We always assume the assumptions stated above.

Recall the following well known inequalities , see for example [6]

Lemma 3.4. *For any vectors $X, Y \in \mathbb{R}^n$, the following inequalities hold:*

$$C_p\langle |X|^{p-2}X - |Y|^{p-2}Y, X - Y\rangle \geq \begin{cases} |X - Y|^p & \text{if } p \geq 2 \\ \frac{|X-Y|^2}{(|X|+|Y|)^{2-p}} & \text{if } 1 \leq p \leq 2, \end{cases} \tag{3.1}$$

in which $\langle \cdot, \cdot \rangle$ denotes the usual dot product in $\mathbb{R}^n$, and C_p is a constant depending on p.

Lemma 3.5. *For any vectors $X, Y \in \mathbb{R}^n$ and for $p > 1$ the following inequalities hold:*

$$\langle |X|^{p-2}X - |Y|^{p-2}Y, X - Y\rangle \leq C_p|X - Y|(|X| + |Y|)^{p-2} \tag{3.2}$$

in which $\langle \cdot, \cdot \rangle$ denotes the usual dot product in $\mathbb{R}^n$, and C_p is a constant depending on p.

To prove our result (2.1), we will show that the operator B satisfies the three conditions of the theorem (2.2).

First, let us show that $B : X \to X^*$ is coercive.

Since $\mu > 0$ and $m(x) < 0$, it follows that

$$-\mu \int_\Omega |u|^q \big(m(x) - |u|^r\big)\, dx > 0.$$

Assume that $\|u\| > 1$. Using compact embedding, we obtain :

$$\begin{aligned}(B(u), u) &= \int_\Omega a|\nabla|^2 dx - \lambda \int_\Omega |u|^p\, dx - \mu \int_\Omega |u|^q (m(x) - |u|^r)\, dx \\ &\geq a\|u\|^2 - \lambda c_1 \|u\|^p\end{aligned}$$

Hence

$$\frac{(B(u), u)}{\|u\|} \geq a\|u\| - \lambda c_1 \|u\|^{p-1}$$

By assumptions $\lambda < 0, p > 2$ it follows that B is coercive.

Next, let us show that $B : X \to X^*$ is hemicontinuous, i.e., for all $u, w, v \in X$, the mapping

$$t \mapsto \langle B(u + tw), v\rangle$$

is continuous on $\mathbb{R}$.

We compute:

$$\langle B(u + tw), v\rangle = \int_\Omega a(x)\nabla(u + tw) \cdot \nabla v\, dx - \lambda \int_\Omega |u + tw|^{p-2}(u + tw)v\, dx$$

$$-\mu \int_\Omega |u + tw|^{q-2}(u + tw)(m(x) - |u + tw|^r)v\, dx.$$

The first term is affine in t and thus continuous.

To handle the nonlinear terms, consider two real numbers $t_n \to t$. We study the difference

$$\langle B(u + t_n w) - B(u + tw), v\rangle.$$

For the second term, define:

$$g_n = |u + t_n w|^{p-2}(u + t_n w), \quad g = |u + tw|^{p-2}(u + tw).$$

Using the inequality (3.2):

$$\langle g_n - g, w\rangle \leq C_p \int_\Omega |t_n - t| \cdot |w|^2 (|u + t_n w| + |u + tw|)^{p-2} dx,$$

and assuming $p > 2$, the Sobolev embeddings ensure that $u + t_n w \in L^p(\Omega)$. The right-hand side tends to zero as $t_n \to t$ due to the Dominated Convergence Theorem, since $|u + t_n w| \to |u + tw|$ a.e. and is bounded in L^p.

Taking into account the condition on $m(x)$, the same argument applies for the third term involving q and the additional weight $m(x) - |u + t_n w|^r$,

Hence,

$$\langle B(u + t_n w), v\rangle \to \langle B(u + tw), v\rangle,$$

showing that the map is continuous in t, i.e., B is hemicontinuous.

□

The operator

$$B(u) = -\nabla \cdot (a(x)\nabla u) - \lambda|u|^{p-2}u - \mu|u|^{q-2}u(m(x) - |u|^r)$$

is strictly monotone on X, i.e., for all $u, v \in X$, $u \neq v$,

$$\langle B(u) - B(v), u - v\rangle > 0.$$

We begin by considering the difference $B(u) - B(v)$:

$$B(u) - B(v) = -\nabla \cdot (a(x)\nabla u) + \nabla \cdot (a(x)\nabla v) - \lambda(|u|^{p-2}u - |v|^{p-2}v)$$
$$-\mu(|u|^{q-2}u(m(x) - |u|^r) - |v|^{q-2}v(m(x) - |v|^r)).$$

Next, we apply the inequality (3.4) :
for the terms:

$$C_2\langle \nabla u - \nabla v, u - v\rangle \geq \|u - v\|^2.$$

$$C_p\langle |u|^{p-2}u - |v|^{p-2}v, u - v\rangle \geq \|u - v\|^p$$

$$C_{q+r}\langle |u|^{q+r-2}u - |v|^{q+r-2}v, u - v\rangle \geq \|u - v\|^{q+r}.$$

Under the assumptions on a, m, λ and μ, we conclude that

$$\langle B(u) - B(v), u - v\rangle > 0 \quad \text{for} \quad u \neq v,$$

which proves that B is strictly monotone.

□

Thus, B is coercive, hemicontinuous and strictly monotone. From the Theorem (2.2), we prove Theorem (2.1).

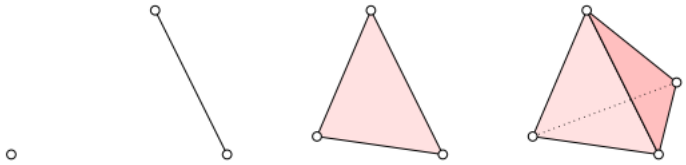
4. Numerical Scheme and Topological Characterization

We numerically investigate the steady-state behavior of a nonlinear elliptic partial differential equation, the problem is discretized using a finite difference method. Finally, the topological features of the resulting solution, particularly the distribution and persistence of critical structures, are analyzed using persistent homology.

4.1. **Topological Tools Overview.**

We briefly review the core concepts in topological data analysis that are employed to interpret the structural properties of the numerical solutions. For further reading, consult [11].

4.1.1. *Simplicial Complexes.* A *simplicial complex* is a finite set of simplices—points, edges, triangles, and higher-dimensional analogues—glued together in a way that captures the shape of a topological space. This representation is well-suited for computations on discretized data.



4.1.2. *Filtrations.* A *filtration* is an increasing sequence of nested simplicial complexes:

$$\emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n,$$

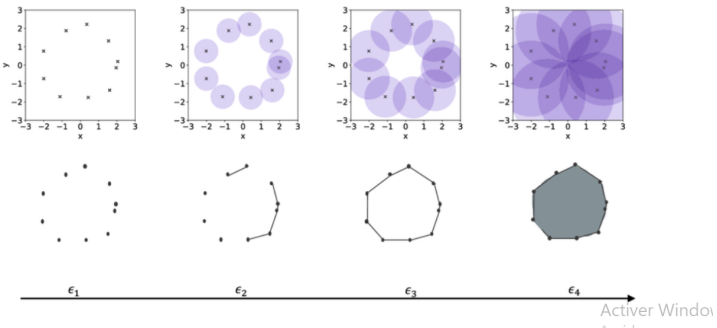
which allows the analysis of topological features (such as holes and components) across multiple spatial scales. This is fundamental for constructing persistence diagrams.

4.1.3. *Homology and Betti Numbers.* To each simplicial complex K, we associate homology groups $H_k(K)$ whose ranks, called Betti numbers, quantify the number of k-dimensional features:

- $\beta_0 = \text{rank}(H_0)$: number of connected components,
- $\beta_1 = \text{rank}(H_1)$: number of 1-dimensional holes (loops),
- $\beta_2 = \text{rank}(H_2)$: number of 2-dimensional cavities, etc.

4.2. **Vietoris–Rips Complex and Persistent Homology.** To extract topological information from the computed solution $u(x, y)$, we construct a Vietoris–Rips complex (see [7]) based on the discretized scalar field.

Definition: Given a point cloud $X \subset \mathbb{R}^2$ and a radius parameter $r > 0$, the open Vietoris–Rips complex $VR_r(X)$ is the simplicial complex whose simplices are all subsets of X with diameter strictly less than r.



This construction enables the generation of barcodes and persistence diagrams, which reveal the robust topological structures present in the solution landscape of the nonlinear PDE.

4.3. **Mathematical Model.** We consider the following nonlinear elliptic problem:

$$\begin{cases} -\nabla(a\nabla u) = \lambda \mid u \mid^{p-2} u + \mu \mid u \mid^{q-2} u(m(x) - \mid u \mid^{r}) & \text{in } \Omega = (0,1)^2, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Where the parameters are fixed as follows:

$$\begin{aligned} a &= 1, & \lambda &= -2, \\ \mu &= 3, & p &= q = 4, \\ r &= 2, & M(x,y) &= -\left(1 + (x-0.5)^2 + (y-0.5)^2\right). \end{aligned}$$

The equation is solved using an iterative fixed-point scheme with relaxation, based on the discretized Laplacian operator.

4.4. **Topological Analysis Based on Persistence.** The numerical solution of the nonlinear elliptic equation was analyzed using persistent homology. The resulting persistence diagram and barcode, computed from the Vietoris–Rips complex associated with critical points of the gradient field, reveal distinct topological features in homological dimensions 0 and 1.

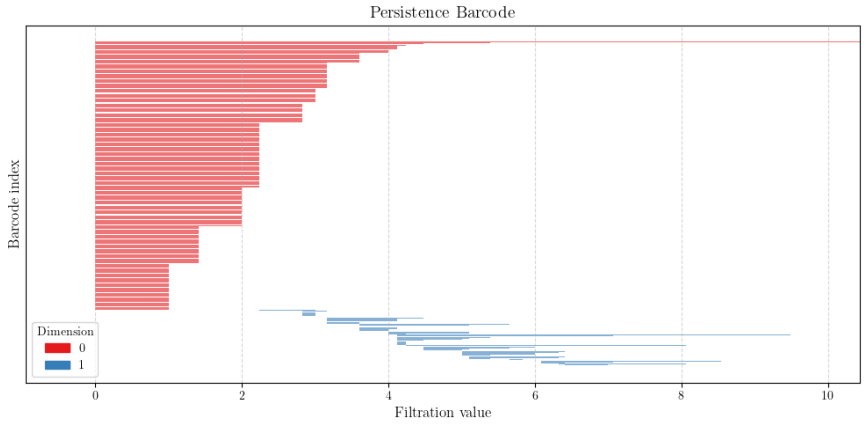


FIGURE 1. Persistence diagram

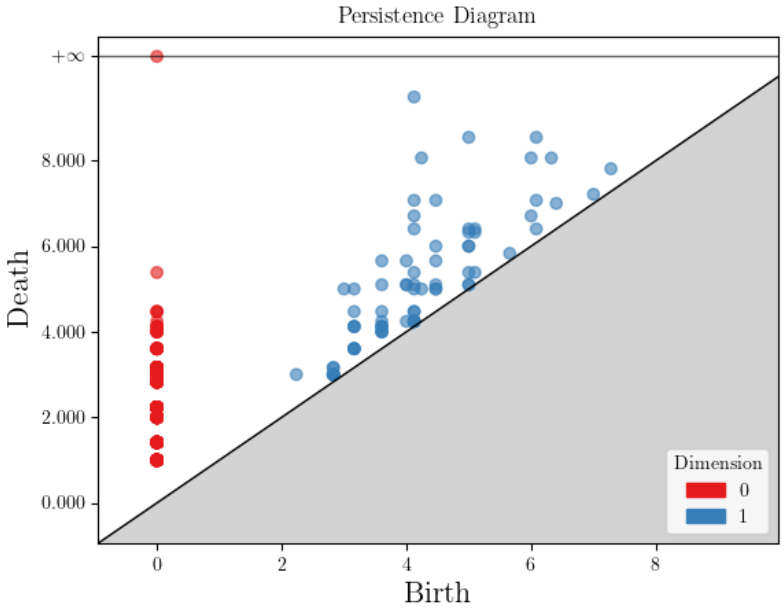


FIGURE 2. Persistence barcode

Homology in Dimension 0 – H_0: Connected Components. The persistence diagram for H_0 reveals a large number of connected components appearing at very low filtration values, which quickly disappear. These short-lived bars represent topological noise or local fluctuations in the solution, commonly observed in scattered point clouds where connections gradually emerge as the filtration radius increases. In contrast, a single bar persists indefinitely, corresponding to the global connected component of the domain $\Omega = [0,1]^2$. This indicates that the scalar field $u(x,y)$ has a highly fragmented local structure at small scales, but maintains a dominant global connectivity—typical of fields defined over connected domains.

Homology in Dimension 1 – H_1: Cycles (Loops). The persistence diagram for H_1 reveals several nontrivial 1-cycles (shown in blue), emerging at filtration values approximately between 3.5 and 8, with significant lifespans. These persistent bars indicate the presence of stable topological structures—loops that remain across a wide range of scales. Such features typically correspond to closed contours or annular regions in the scalar field $u(x,y)$, such as ridges, valleys, or potential wells. Their relatively long persistence reflects the geometric and topological complexity of the stationary solution, likely shaped by the nonlinear source terms in the governing equation. This suggests that the field exhibits organized spatial patterns, with topologically significant zones like closed ridges or annular depressions.

Physical Interpretation

- If the PDE models diffusion, equilibrium dynamics, or a steady-state configuration:

– Persistent H_1 components may represent zones of energetic stability or geometric nodes.
– The initial proliferation of low-persistence H_0 components reflects strong nonlinearity in the equation and the presence of local variations in the field.

Summary

Dimension	Meaning	Observation	Interpretation
H_0	Connected components	One dominant component + noise	Connected domain with local fluctuations
H_1	Topological loops	Persistent loops between 4 and 8	Stable zones, structured ridges or wells

5. Conclusion

This work establishes the existence and uniqueness of a weak solution to a stationary degenerate nonlinear elliptic problem in divergence form. Furthermore, persistent homology is employed to characterize the approximate solution. The persistent homology of the numerical solution reveals both local fluctuations and global organization of the scalar field $u(x, y)$. The dominant connected component in H_0 confirms the physical coherence of a globally connected domain, while the persistent loops in H_1 suggest the presence of physically relevant structures such as closed ridges, valleys, or potential wells. These features reflect the spatial organization induced by the nonlinearities in the PDE. This topological analysis thus provides a meaningful interpretation of the solution's geometry, offering a robust tool to detect and quantify physically significant patterns beyond traditional methods.

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