Gulf Journal of Mathematics
Vol 24, Issue 2 (2026) 1-23
ISSN: 2309-4966
https://doi.org/10.56947/behe4e76

RENORMALIZED SOLUTIONS FOR A TWO-COMPONENT PARABOLIC PROBLEM

OUSSAMA MESSBAHI[1], YOUSSEF HAJJI[2*], MOHAMED BADR BENBOUBKER[3] and HASSANE HJIAJ[4]

Abstract. In this paper, we investigate a class of nonlinear parabolic systems defined on a two-component composite domain with non-homogeneous transmission conditions at the interface. The domain consists of two disjoint open subdomains separated by a Lipschitz interface. Under a homogeneous Dirichlet boundary condition on the exterior boundary and suitable initial data, we establish the existence, uniqueness, and regularity of renormalized solutions for the considered system.

Keywords. Parabolic equations, two-component domain, renormalized solutions, uniqueness of solution.
2020 Mathematics Subject Classification. Primary 35J60; 46E30; Secondary 46E35

1. Introduction

In this article, we study the existence and uniqueness of renormalized solutions for a parabolic problem in a two-component domain and an L^1- data. More precisely, let Ω be a connected bounded open subset of $\mathbb{R}^N$ $(N > 2)$, with $\partial\Omega$ is a boundary of Ω. It's defined as $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$, with the open sets Ω_1 and Ω_2 are two by two disjoint such that $\overline{\Omega_2}$ is included in Ω, $\Omega_1 = \Omega \setminus \overline{\Omega_2}$, and Γ is a Lipschitz boundary of Ω_2. Denote, for $i = 1, 2$

$$Q_T = \Omega \times (0,T), \quad Q_{T,i} = \Omega_i \times (0,T), \quad S_T = \partial\Omega \times (0,T) \quad \text{and} \quad \Sigma_T = \Gamma \times (0,T).$$

Let ν_1 be the unit exterior normal to Ω_1, and we consider a measurable function θ defined on $Q_T \backslash \Sigma_T$, and we denote by $\theta_i = \theta|_{Q_{T,i}}$ the restriction of the measurable function θ to the subdomains $Q_{T,i}$ for $i = 1, 2$.

Date: Received: Jul 31, 2026; Revised: Aug 28, 2026; Accepted: Sep 2, 2026.
* Corresponding author
© The Author(s) 2026. This article is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License. To view a copy of the licence, visit https://creativecommons.org/licenses/by-nc-nd/4.0/.

Our main objective is to study the following parabolic problem, which we formulate as follows

$$\begin{cases} \dfrac{\partial \theta_1}{\partial t} - \operatorname{div}(A(x,t,\theta_1,D\theta_1)) = f(x,t) & \text{in } Q_{T,1}, \\ \dfrac{\partial \theta_2}{\partial t} - \operatorname{div}(A(x,t,\theta_2,D\theta_2)) = f(x,t) & \text{in } Q_{T,2}, \\ \theta_1 = 0 & \text{on } S_T, \\ A(x,t,\theta_1,D\theta_1)\cdot\nu_1 = A(x,t,\theta_2,D\theta_2)\cdot\nu_1 & \text{on } \Sigma_T, \\ A(x,t,\theta_1,D\theta_1)\cdot\nu_1 = -h(x,t)|\theta_1-\theta_2|^{p-2}(\theta_1-\theta_2) & \text{on } \Sigma_T, \\ \theta_1(x,0) = \theta_{1,0} & \text{in } \Omega_1, \\ \theta_2(x,0) = \theta_{2,0} & \text{in } \Omega_2, \end{cases} \tag{1.1}$$

where f and $\theta_{i,0}$ are data belonging to $L^1(\Omega\times(0,T))$ and $L^1(\Omega_i)$, respectively.

We assume that $A : Q_T \times \mathbb{R}\times\mathbb{R}^N \longmapsto \mathbb{R}^N$ is a Carathéodory function, i.e.

(1) $(x,t) \longmapsto A(x,t,s,\xi)$ is a measurable function for every (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$,
(2) $(s,\xi) \longmapsto A(x,t,s,\xi)$ is a continuous function for almost every (x,t) in Q_T,

and satisfying the following conditions:

$$|A(x,t,s,\xi)| \le \beta(K(x,t) + |s|^{p-1} + |\xi|^{p-1}), \tag{1.2}$$

$$(A(x,t,s,\xi) - A(x,t,s,\xi'))\cdot(\xi-\xi') > 0 \quad \text{for any} \quad \xi \neq \xi', \tag{1.3}$$

$$A(x,t,s,\xi)\cdot\xi \ge \alpha|\xi|^p, \tag{1.4}$$

for a.e. $(x,t)\in Q_T$ and all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, where $K(x,t)$ is a positive function belong to $L^{p'}(Q_T)$, with $\beta>0$, and $\alpha>0$. Finally, we assume that

$$f \in L^1(Q_T), \tag{1.5}$$

and the function $h(\cdot,\cdot)$ satisfies

$$h(\cdot,\cdot) \in L^\infty(\Sigma_T) \quad \text{and} \quad 0 < h_0 < h(x,t) \text{ a.e. on } \Sigma_T. \tag{1.6}$$

In the case of one component and $\operatorname{div}(A(x,t,\xi))$ is the operator strictly monotone operator. L. Boccardo and T. Gallouët in [11] was proven the existence of a solution in the distributional sense for (1.1) within the framework of $L^p(0,T;W_0^{1,p}(\Omega))$ for $p > \frac{2N+1}{N+1}$, and by D. Blanchard [9] in the present setting. boundary condition. This extension has been employed in other parts of mathematical research for elliptic and parabolic problem such as [15, 16, 17, 18, 23, 30, 32, 35, 41, 42]. The type of equation under investigation here is much inspired by case elliptic [26], Fulgencio and Guibé have established the existence of renormalized solutions for the quasi-linear elliptic problems where the principal operator involves a matrix field $A(x,u_i)$ for $i=1,2$. Here, u_1 and u_2 are two measurable functions defined on disjoint open subset Ω_1 and Ω_2, separated by an interface Γ. The uniqueness of the solution to this problem was proved in [27]. Moreover, the homogenization was also studied in [20]; we refer the readers to [12, 19, 22, 29, 31, 39, 40] for more details. In [28], which aligns closely with our current work, the authors obtained existence and uniqueness theorems for renormalized solutions to elliptic equations with mixed boundary conditions on perforated domains.

At its core, this models conductive heat flow through a material consisting of two constituent phases. The overall heat conduction in composite materials depends on how conductive each component is, how much of each is present, and how they're mixed together [6, 14, 33]. These investigations employ a single-temperature field approach to model the composite's effective behavior, with researchers focusing on characterizing how interfacial resistance influences effective conductivity. The work encompasses both theoretical analyses [25, 34] and experimental validations [8, 24], examining systems ranging from dilute dispersions [33] and low-concentration mixtures [14] to densely packed composites [6, 7]. Research confirms that interfacial thermal barriers diminish a composite's effective conductivity. Conversely, dual-temperature field approaches have been developed to model porous matrix-liquid configurations [1] and heterogeneous solid systems [13]. While these models don't directly incorporate interfacial resistances, they approximate interphase heat transfer as being linearly dependent on local temperature gradients.

The rest of this paper is organized as follows: In section 2, we introduce the spaces V and $\mathcal{W}$, and we recalls some definitions and technical lemmas. In section 3, we introduce the definition of the renormalized solution for our parabolic equation (1.1), and we show the main result. Sections 4 and Section 5 are dedicated to proving the existence and uniqueness of the solution.

2. Some preliminary results

Let us introduce the spaces

$$V_1 = \{u \in W^{1,p}(\Omega_1) : \ u = 0 \text{ on } \partial\Omega\} \quad \text{with} \quad \|u\|_{V_1} := \|Du\|_{L^p(\Omega)}, \tag{2.1}$$

and

$$V := \{u \equiv (u_1, u_2) : u_1 \in V_1 \text{ and } u_2 \in W^{1,p}(\Omega_2)\}, \tag{2.2}$$

equipped with the norm

$$\|u\|_V^p := \|Du_1\|_{L^p(\Omega_1)}^p + \|Du_2\|_{L^p(\Omega_2)}^p + \|u_1 - u_2\|_{L^p(\Gamma)}^p. \tag{2.3}$$

We defined the setting parabolic space $\mathcal{W}$ by

$$\mathcal{W} := L^p(0, T; V).$$

The space $\mathcal{W}$ endowed by the norm

$$\|u\|_{\mathcal{W}} = \Big(\int_0^T \|u\|_V^p \, dt \Big)^{\frac{1}{p}}.$$

We denote the dual of the space of $\mathcal{W}$ by $\mathcal{W}^\star := L^{p'}(0, T; V^\star)$.

Lemma 2.1. *Let B_0, B and B_1 be a Banach spaces with $B_0 \subset B \subset B_1$. Let us set*

$$Y = \big\{u \ : \ u \in L^{p_0}(0, T; B_0) \quad \text{and} \quad u' \in L^{p_1}(0, T; B_1)\big\}$$

where $p_0 > 1$ and $p_1 > 1$ are reals numbers.
Assuming that the embedding $B_0 \hookrightarrow\hookrightarrow B$ be compact, then

$$Y \hookrightarrow\hookrightarrow L^{p_0}(0, T; B)$$

and this imbedding is compact.

Remark 2.2. Let $p > \dfrac{2N}{N+2}$, and we set

$$B_0 = V, \quad B = L^2(\Omega_1) \times L^2(\Omega_2) \text{ or } L^2(\Gamma) \times L^2(\Gamma) \text{ and } B_1 = V^\star,$$

with $p_0 = p$ and $p_1 = p'$ In view of the Lemma 2.1, we obtain

$$\left\{u \ : \ u \in V \quad \text{and} \quad u' \in V^\star\right\} \subseteq Y \hookrightarrow\hookrightarrow L^1(Q_{T,1}) \times L^1(Q_{T,2}), \tag{2.4}$$

and

$$\left\{u \ : \ u \in V \quad \text{and} \quad u' \in V^\star\right\} \subseteq Y \hookrightarrow\hookrightarrow L^1(\Sigma_T) \times L^1(\Sigma_T). \tag{2.5}$$

Moreover, in view of [4], we have

$$\mathcal{W}_p = \left\{u \ : \ u \in \mathcal{W} \quad \text{and} \quad u' \in \mathcal{W}^\star\right\} \subseteq C([0,T]; L^2(\Omega_1) \times L^2(\Omega_2)). \tag{2.6}$$

Definition 2.3. Let $k > 0$, we consider the truncation function $T_k(\cdot) : \mathbb{R} \longmapsto \mathbb{R}$, given by

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \leq k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k, \end{cases}$$

Proposition 2.4. *(see [5] [18]) Let $u := (u_1, u_2) : Q_T \backslash \Sigma_T \longmapsto \mathbb{R}$ be a measurable function such that $T_k(u) \in \mathcal{W}$ for every $k \geq 1$. For $i = 1, 2$,*
i) *there exists a unique measurable function $v_i : Q_{T,i} \to \mathbb{R}^N$ such that for all $k \geq 1$,*

$$DT_k(u_i) = v_i \chi_{\{|u_i|<k\}} \quad a.e. \text{ in } Q_{T,i},$$

where $\chi_{\{|u_i|<k\}}$ denotes the characteristic function of $\{x \in \Omega_i : |u_i(x)| < k\}$. We define v_i as the gradient of u_i and write $v_i = Du_i$.
ii) *if $\sup\limits_{k\geq 1} \dfrac{1}{k}\|T_k(u)\|_{\mathcal{W}}^p < \infty$, then there exists a unique measurable function $w_i : \Sigma_T \longmapsto \mathbb{R}$, for $i = 1, 2$ such that*

$$\gamma_i(T_k(u_i)) = T_k(w_i) \quad a.e. \text{ on } \Sigma_T \quad \text{for all} \quad k \geq 1,$$

where $\gamma_i : W^{1,p}(\Omega_i) \longmapsto L^p(\Gamma)$ is the trace operator, we define the function w_i as the trace of u_i on Σ_T and set

$$\gamma_i(u_i) = w_i.$$

Lemma 2.5. *(see [16]) Let $u = (u_1, u_2)$ be a measurable function satisfying*

$$T_k(u) \in (L^\infty(0, T, L^2(\Omega_1)) \times (L^\infty(0, T, L^2(\Omega_2))) \cap \mathcal{W}, \tag{2.7}$$

for every $k > 0$ and such that

$$\sum_{i=1}^{2} \sup_{t\in(0,T)} \int_{\Omega_i} |T_k(u_i)|^2 \, dx + \sum_{i=1}^{2} \int_0^T \int_{\Omega_i} |DT_k(u_i)|^2 \, dx \, dt \leq Mk, \tag{2.8}$$

where M is constant, then $|u_i| \in L^{\frac{N(p-1)+p}{N},\infty}(Q_{T,i})$, and $|Du_i| \in L^{\frac{N(p-1)+p}{N+1},\infty}(Q_{T,i})$. Moreover

$$\| |u_i| \|_{L^{\frac{N(p-1)+p}{N},\infty}(Q_{T,i})} \leq CM, \tag{2.9}$$

and

$$\| |Du_i| \|_{L^{\frac{N(p-1)+p}{N+1},\infty}(Q_{T,i})} \leq CM + meas(Q_T)^{\frac{N+1}{N+2}}. \tag{2.10}$$

Remark 2.6. Let $p > 2 - \frac{1}{N+1}$. Let u be a measurable function satisfying (2.7) – (2.8). In view of Lemma 2.5, we have

$$u \in L^q(0,T;W^{1,q}_{\Gamma}(\Omega_1)) \times L^q(0,T;W^{1,q}(\Omega_2)) \quad \text{for all} \quad 1 \le q < \frac{N(p-1)+p}{N+1}. \tag{2.11}$$

Moreover

$$\gamma_i(u_i) \in L^q(\Sigma_T). \tag{2.12}$$

Lemma 2.7. *(see. [18]) Let $(u_n := (u_{n,1}, u_{n,2}))_{n\in\mathbb{N}}$ be a sequence in $\mathcal{W}$ such that $u_n \rightharpoonup u$ weakly in $\mathcal{W}$ and $\frac{\partial u_n}{\partial t} \in \mathcal{W}^*$. Assuming that (1.2) – (1.4), and*

$$\begin{aligned} &\sum_{i=1}^{2} \int_{Q_{T,i}} \Big(A(x,t,u_{n,i},Du_{n,i}) - A(x,t,u_{n,i},Du_i) \Big)(Du_{n,i} - Du_i)\, dx\, dt \\ &\longrightarrow 0 \quad as \quad n \to \infty, \end{aligned} \tag{2.13}$$

hold, then $u_n \to u$ strongly in $\mathcal{W}$ for a subsequence.

3. Main results

Firstly, we introduce the definition of renormalized solutions for our parabolic problem. We set $\Omega_1^{(1)} = \Omega_1 \cup \Gamma$.

Definition 3.1. Let $\theta = (\theta_1, \theta_2) : Q_T \setminus \Sigma_T \longmapsto \mathbb{R}$ be a measurable function. θ is called a renormalized solution of problem (1.1) if

$$T_k(\theta) \in \Big(L^\infty(0,T,L^2(\Omega_1)) \times (L^\infty(0,T,L^2(\Omega_2)) \Big) \cap \mathcal{W} \quad \text{for any} \quad k > 0,$$

$$\lim_{h\to\infty} \frac{1}{h} \sum_{i=1}^{2} \int_{\{(x,t)\in Q_{T,i}, \quad |\theta_i|\le h\}} A(x,t,\theta_i,D\theta_i) \cdot D\theta_i \, dx\, dt = 0, \tag{3.1}$$

$$\lim_{h\to\infty} \frac{1}{h} \int_{\Sigma_T} h(x,t)|\theta_1-\theta_2|^{p-2}(\theta_1-\theta_2)(T_h(\theta_1)-T_h(\theta_2))\, d\sigma\, dt = 0, \tag{3.2}$$

and

$$|\theta_1-\theta_2|^{p-2}(\theta_1-\theta_2)(T_k(\theta_1)-T_k(\theta_2)) \in L^1(\Sigma_T) \quad \text{for all} \quad k \ge 1, \tag{3.3}$$

such that θ verifying the following equality

$$\begin{aligned} &-\sum_{i=1}^{2} \int_{Q_{T,i}} S(\theta_i) \frac{\partial \varphi_i}{\partial t}\, dx\, dt + \int_{\Sigma_T} h(x,t)(\theta_1-\theta_2)(\varphi_1 S(\theta_1) - \varphi_2 S(\theta_2))\, d\sigma\, dt \\ &+ \sum_{i=1}^{2} \int_{Q_{T,i}} A(x,t,\theta_i,D\theta_i)(S'(\theta_i)\varphi_i D\theta_i + S(\theta_i) D\varphi_i)\, dx\, dt \\ &= \int_{Q_T} f S(\theta)\varphi\, dx\, dt + \sum_{i=1}^{2} \int_{\Omega_i} S(\theta_{0,i})\varphi_i(x,0)\, dx, \end{aligned} \tag{3.4}$$

for every $\varphi = (\varphi_1, \varphi_1) \in C_0^\infty((\Omega_1^{(1)} \times (0,T[) \times C_0^\infty(\overline{\Omega_2} \times (0,T[)$ and for any smooth function $S(\cdot) \in C_0^2(\mathbb{R})$ (or $W^{2,\infty}(\mathbb{R})$) such that $S'(\cdot)$ has a compact support.

Remark 3.2. Let $S(\cdot) \in C_0^2(\mathbb{R})$ such that $S'(\cdot)$ has a compact support, thanks to (3.3), we have

$$|\theta_1 - \theta_2|^{p-2}(\theta_1 - \theta_2)S(\theta_1)\varphi_1(\cdot) \in L^1(\Sigma_T)$$

and

$$|\theta_1 - \theta_2|^{p-2}(\theta_1 - \theta_2)S(\theta_2)\varphi_2(\cdot) \in L^1(\Sigma_T),$$

for any $\varphi = (\varphi_1, \varphi_2) \in C_0^\infty((\Omega_1^{(1)} \times (0,T)) \times C_0^\infty(\overline{\Omega_2} \times (0,T))$.

We investigate the existence and uniqueness of renormalized solutions to Problem (1.1). The existence part is presented below.

Theorem 3.3. *Let* $p > \dfrac{2N}{N+2}$, *assuming that the conditions* (1.2) − (1.4) *and* (1.6) *hold true, then the problem* (1.1) *has at least one renormalized solution.*

We now state an additional condition under which uniqueness of the renormalized solution to (1.1) is guaranteed.

Theorem 3.4. *Let* $p > 2 - \frac{1}{N+1}$. *Assume that conditions (1.2)-(1.4) hold. Moreover, assume that for any compact set* $K \subset \mathbb{R}$, *there exists a measurable function* $c_K(x,\tau) \in L^{p'}(Q_T)$ *and a constant* $\gamma_K > 0$ *such that*

$$|A(x,\tau,s,\xi) - A(x,\tau,\overline{s},\xi)| \leq (c_K(x,\tau) + \gamma_K|\xi|^{p-1})|s - \overline{s}| \tag{3.5}$$

for almost every $(x,\tau) \in Q_T$, *for every* $\xi \in \mathbb{R}^N$, *and for all* $s, \overline{s} \in K$. *Then the renormalized solution of problem (1.1) is unique.*

4. Existence of renormalized solutions

Now, we will prove Theorem 3.3 by using approximate methods and some a priori estimation techniques. this section is divided into three subsections: The subsection 4.1 introduce the approximate problem. Some a priori estimates and limiting arguments are presented in the subsection 4.2. We conclude the proof of the Theorem 3.3 in the last subsection 4.3.

4.1. **Approximate problems.**

Let us consider a sequence $\{f_n\}_{n=1}^\infty \subset L^1(Q_T) \cap L^\infty(Q_T)$ satisfying:

(i): $f_n \to f$ Strongly in $L^1(Q_T)$.

(ii): $|f_n| \leq |f|$ (for example $f_n = T_n(f)$)

We then formulate the approximate problem

$$\begin{cases} \dfrac{\partial \theta_{n.1}}{\partial t} - \operatorname{div}(A(x,t,T_n(\theta_{n,1}), D\theta_{n,1})) = f_n(x,t) & \text{in } Q_{T,1}, \\ \dfrac{\partial \theta_{n,2}}{\partial t} - \operatorname{div}(A(x,t,T_n(\theta_{n,2}), D\theta_{n,2})) = f_n(x,t) & \text{in } Q_{T,2}, \\ \theta_{n,1} = 0 & \text{on } S_T, \\ A(x,t,T_n(\theta_{n,1}), D\theta_{n,1}) \cdot \nu_1 = A(x,t,T_n(\theta_{n,2}), D\theta_{n,2}) \cdot \nu_1 & \text{on } \Sigma_T, \\ A(x,t,T_n(\theta_{n,1}), D\theta_{n,1}) \cdot \nu_1 = -h(x,t)|\theta_{n,1} - \theta_{n,2}|^{p-2}(\theta_{n,1} - \theta_{n,2}) & \text{on } \Sigma_T \\ \theta_{n,1}(x,0) = \theta_{n,1,0} & \text{in } \Omega_1, \\ \theta_{n,2}(x,0) = \theta_{n,2,0} & \text{in } \Omega_2, \end{cases} \tag{4.1}$$

Thanks to Leray-Lions[36] (see also [18]), there exists at last one weak solution $\theta_n \in \mathcal{W}$ for the problem (4.1), i.e.

$$\begin{aligned} &\int_{Q_{T,1}} \frac{\partial \theta_{n,1}}{\partial t} v_1 \, dx \, dt + \int_{Q_{T,1}} \frac{\partial \theta_{n,2}}{\partial t} v_2 \, dx \, dt + \int_{Q_{T,1}} A(x,t,T_n(\theta_{n,1}), D\theta_{n,1}) \cdot Dv_1 \, dx \, dt \\ &+ \int_{Q_{T,2}} A(x,t,T_n(\theta_{n,2}), D\theta_{n,2}) \cdot Dv_2 \, dx \, dt \\ &+ \int_{\Sigma_T} h(x,t)|\theta_{n,1} - \theta_{n,2}|^{p-2}(\theta_{n,1} - \theta_{n,2})(v_1 - v_2) \, d\sigma \, dt = \int_{Q_T} f_n v \, dx \, dt, \end{aligned} \tag{4.2}$$

for any $v \in \mathcal{W}$.

4.2. **A priori estimates.**

Lemma 4.1. *Let $n \in \mathbb{N}^*$, and assuming that (1.2) – (1.6) hold true. Let $\theta_n = (\theta_{n,1}, \theta_{n,2}) \in \mathcal{W}$ be a weak solution of the approximate problem (4.1). Then*

(i) *There exists a constant $C = C(N,\alpha) > 0$ such that*

$$\sum_{i=1}^{2} \sup_{t\in(0,T)} \int_{\Omega_i} |T_k(\theta_{n,i})|^2 \, dx + \|T_k(\theta_n)\|_{\mathcal{W}}^p \le Ck \qquad \text{for all} \quad k > 0.$$

(ii) *There exists a constant $C = C(N,\alpha) > 0$ such that*

$$meas\{|\theta_n| > k\} \le \frac{C}{k^{p-1}}.$$

(iii) *There exists a constant $C = C(N,\alpha) > 0$ such that for $i = 1, 2$,*

$$meas_{\Sigma_T}\{|\gamma_i(\theta_{n,i})| > k\} \le \frac{C}{k^{p-1}}.$$

Proof. (i) For any integer $k \ge 1$, and $t \in (0,T)$. we select the test function $v = \Big(T_k(\theta_{n,1})\chi_{(0,t)}, T_k(\theta_{n,2})\chi_{(0,t)}\Big) \in \mathcal{W}$ in the variational formulation (4.2), which

yields.

$$\begin{aligned}
&\int_{Q_{t,1}} \frac{\partial \theta_{n,1}}{\partial t} T_k(\theta_{n,1})\,dx\,d\tau + \int_{Q_{t,2}} \frac{\partial \theta_{n,2}}{\partial t} T_k(\theta_{n,2})\,dx\,d\tau \\
&\quad + \sum_{i=1}^{2} \int_{Q_{t,i}} A(x,\tau,T_n(\theta_{n,i}), D\theta_{n,i}) \cdot DT_k(\theta_{n,i})\,dx\,d\tau \\
&\quad + \int_{\Sigma_t} h(x,\tau)|\theta_{n,1}-\theta_{n,2}|^{p-2}(\theta_{n,1}-\theta_{n,2})\Big(T_k(\theta_{n,1})-T_k(\theta_{n,2})\Big)\,d\sigma\,d\tau \\
&= \int_{Q_t} f_n T_k(\theta_n)\,dx\,d\tau.
\end{aligned} \tag{4.3}$$

We note the inequalities $S_k(s) \geq \frac{1}{2}|T_k(s)|^2$, and $|S_k(s)| \leq k|s|$. where $S_k(\cdot)$ is a primitive of $T_k(\cdot)$. Hence

$$\begin{aligned}
\int_{Q_{t,i}} \frac{\partial \theta_{n,i}}{\partial t} T_k(\theta_n)\,dx\,d\tau &= \int_\Omega \Big[S_k(\theta_{n,i})\Big]_0^t\,dx \\
&= \int_{\Omega_i} S_k(\theta_{n,i}(t))\,dx - \int_\Omega S_k(\theta_{0,n,i})\,dx \\
&\geq \frac{1}{2}\sup_{t\in(0,T)} \int_{\Omega_i} |T_k(\theta_{n,i}(t))|^2\,dx - k\|\theta_{0,i}\|_{L^1(\Omega_i)}.
\end{aligned} \tag{4.4}$$

From (4.3) − (4.4), and (1.4), since $|\theta_{n,1}-\theta_{n,2}| \geq |T_k(\theta_{n,1})-T_k(\theta_{n,2})|$ a.e. on Σ_T, condition (1.6), it yields

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{2}\sup_{t\in(0,T)}\int_{\Omega_i}|T_k(\theta_{n,i}(t))|^2\,dx + \alpha\sum_{i=1}^{2}\int_{Q_{t,i}}|DT_k(\theta_{n,i})|^p\,dx\,d\tau \\
&+h_0\int_{\Sigma_t}|T_k(\theta_{n,1})-T_k(\theta_{n,2})|^p\,d\sigma\,d\tau \\
&\leq k\int_{Q_t}|f|\,dx\,d\tau + k\sum_{i=1}^{2}\int_{\Omega_i}|\theta_{0,i}|\,dx.
\end{aligned} \tag{4.5}$$

Consequently, there exists a constant $C = C(N,\alpha) > 0$, that remains independent of both the truncation parameter k and the approximation index n, satisfying

$$\sum_{i=1}^{2}\sup_{t\in(0,T)}\int_{\Omega_i}|T_k(\theta_{n,i})|^2\,dx + \|T_k(\theta_n)\|_{\mathcal{W}}^p \leq Ck \qquad \text{for all} \quad k>0. \tag{4.6}$$

(ii) Thanks to Poincaré's inequality, Proposition 2.4 and (4.6), we have

$$\begin{aligned}
k^p \text{meas}\{|\theta_n| > k\} &= \int_{\{|\theta_{n,1}|>k\}} k^p\,dx\,d\tau + \int_{\{|\theta_{n,2}|>k\}} k^p\,dx\,d\tau \\
&\leq \int_{Q_{T,1}} |T_k(\theta_{n,1})|^p\,dx + \int_{Q_{T,2}} |T_k(\theta_{n,2})|^p\,dx\,d\tau \\
&\leq C_7\|DT_k(\theta_{n,1})\|_{L^p(Q_{T,1})}^p + \|T_k(\theta_{n,2})\|_{L^p(Q_{T,2})}^p \\
&\leq C_8\|T_k(\theta_n)\|_{\mathcal{W}}^p \\
&\leq Ck,
\end{aligned}$$

where $C = C(N, \alpha) > 0$ is a positive constant that does not depend on k and n, we deduce that

$$\text{meas}\{|\theta_n| > k\} \leq \frac{C}{k^{p-1}}. \tag{4.7}$$

(iii) For $\gamma_1(\theta_{n,1})$, by applying the Poincaré's inequality and (4.6), we obtain

$$\begin{aligned} k^p \text{meas}_{\Sigma_T}\{|\gamma_1(\theta_{n,1})| > k\} &= \int_{\{|\gamma_1(\theta_{n,1})|>k\}} k^p \, d\sigma \, d\tau \\ &\leq \int_{\Sigma_T} |\gamma_1(T_k(\theta_{n,1})|^p \, d\sigma \\ &\leq \int_0^T \|\gamma_1(T_k(\theta_{n,1}))\|^p_{L^p(\Gamma)} \, d\tau \\ &\leq C_{10} \int_0^T \|T_k(\theta_{n,1})\|^p_{V_1} \, d\tau \\ &\leq Ck. \end{aligned}$$

where $C = C(N, \alpha) > 0$. Hence

$$\text{meas}_\Gamma\{|\gamma_1(\theta_{n,1})| > k\} \leq \frac{C}{k^{p-1}}. \tag{4.8}$$

For $\gamma_2(\theta_{n,2})$, in view of the trace theorem, Proposition 2.4, and (4.6), we have

$$\begin{aligned} k^p \text{meas}_\Gamma\{|\gamma_2(\theta_{n,2})| > k\} &= \int_{\{|\gamma_2(\theta_{n,2})|>k\}} k^p \, d\sigma \\ &\leq \int_{\Sigma_T} |\gamma_1(T_k(\theta_{n,1})|^p \, d\sigma \\ &\leq \int_0^T \|\gamma_1(T_k(\theta_{n,1}))\|^p_{L^p(\Gamma)} \, d\tau \\ &\leq C_{11} \int_0^T \|T_k(\theta_{n,1})\|^p_{W^{1,p}(\Omega_2)} \, d\tau \\ &\leq Ck. \end{aligned}$$

Consequently

$$\text{meas}_\Gamma\{|\gamma_2(\theta_{n,2})| > k\} \leq \frac{C}{k^{p-1}}. \tag{4.9}$$

□

Lemma 4.2. *([12], Lemma 3) Let $k > 0$ and let $\theta_n = (\theta_{n,1}, \theta_{n,2}) \in \mathcal{W}$ be a solution of the approximate problem (4.1). Then, there exist a mesurable functions θ_i (for $i = 1, 2$) and a subsequence of $(\theta_{n,i})_n$ still denoted by $(\theta_{n,i})_n$, which satisfies the following properties*

(i) $\theta_{n,i} \to \theta_i$ *a.e. in* $Q_{T,i}$ *for* $i = 1, 2$.
(ii) $\gamma_i(\theta_{n,i}) \to \gamma_i(\theta_i)$ *a.e. on* Σ_T *for* $i = 1, 2$.
(iii) $T_k(\theta_n) \rightharpoonup T_k(\theta)$ *weakly in* $\mathcal{W}$.
(iv) $T_k(\theta_{n,i}) \to T_k(\theta_i)$ *strongly in* $L^1(Q_{T,i})$ *and a.e. in* $Q_{T,i}$,
(v) $\gamma_i(T_k(\theta_{n,i})) \to \gamma_i(T_k(\theta_i))$ *strongly in* $L^1(\Sigma_T)$ *and a.e. on* Σ_T.

Remark 4.3. In view of Fatou's Lemma, and (4.5), we get

$$\begin{aligned}&\int_{\Sigma_T} h(x,t)|\theta_1-\theta_2|^{p-2}(\theta_1-\theta_2)(T_k(\theta_1)-T_k(\theta_2))\,d\sigma\,dt\\&\leq \liminf_{n\to\infty}\int_{\Sigma_T} h(x,t)|\theta_{n,1}-\theta_{n,2}|^{p-2}(\theta_{n,1}-\theta_{n,2})(T_k(\theta_{n,1})-T_k(\theta_{n,2}))\,d\sigma\,dt\\&\leq C_6 k,\end{aligned}$$

Lemma 4.4. *([12], Lemma 4) Under (1.2) – (1.6), for all $n>0$ let $\theta_n=(\theta_{n,1},\theta_{n,2})\in\mathcal{W}$ be a solution of the approximate problem (4.1), then*

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\int_{\{|\theta_{n,i}|\leq h\}} A(x,t,\theta_{n,i},D\theta_{n,i})\cdot D\theta_{n,i}\,dx\,dt=0\qquad \text{for}\quad i=1,2, \tag{4.10}$$

and

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\int_{\Sigma_T} h(x,t)|\theta_{n,1}-\theta_{n,2}|^{p-2}(\theta_{n,1}-\theta_{n,2})(T_h(\theta_{n,1})-T_h(\theta_{n,2}))\,d\sigma\,dt=0. \tag{4.11}$$

Lemma 4.5. *Under the same assumptions of the Lemma 4.1, we have*

$$T_k(\theta_n)\longrightarrow T_k(\theta)\qquad \text{strongly in}\quad \mathcal{W} \tag{4.12}$$

for a subsequence. Moreover

$$D\theta_{n,i}\longrightarrow D\theta_i\qquad \text{a.e. in}\quad Q_{T,i}\quad \text{for}\quad i=1,2. \tag{4.13}$$

Proof. Let $\mu\geq 0$, we introduce the time mollification u_μ of a function $u\in L^2(0,T;H^1(\Omega))$, by

$$u_\mu(x,t)=\mu\int_{-\infty}^{t}\bar{u}(x,s)\exp(\mu(s-t))\,ds\qquad \text{where}\quad \bar{u}(x,s)=u(x,s)\chi_{(0,T)}(s).$$

For more details on the properties of time mollification, see [2].

For any $h>k\geq 1$, we set $\psi_h(s)=1-\dfrac{|T_{2h}(s)-T_h(s)|}{h}$,

By taking $v=(T_k(\theta_n)-(T_k(\theta))_\mu)\psi_h(\theta_n)$ as a test function in (4.1) yields

$$\begin{aligned}&\sum_{i=1}^{2}\int_0^T\int_{\Omega_i}\frac{\partial\theta_{n,i}}{\partial t}(T_k(\theta_{n,i})-(T_k(\theta_i))_\mu)\psi_h(\theta_{n,i})\,dx\,dt\\&+\sum_{i=1}^{2}\int_{Q_{i,T}} A(x,t,T_n(\theta_{n,i}),D\theta_{n,i})(DT_k(\theta_{n,i})-D(T_k(\theta_i))_\mu)\psi_h(\theta_{n,i})\,dx\,dt\\&+\sum_{i=1}^{2}\int_{Q_{i,T}} A(x,t,T_n(\theta_{n,i}),D\theta_{n,i})\cdot D\theta_{n,i}\psi_h'(\theta_{n,i})(T_k(\theta_{n,i})-(T_k(\theta_i))_\mu)\,dx\,dt\\&+\int_{\Sigma_T} h(x,t)|\theta_{n,1}-\theta_{n,2}|^{p-2}(\theta_{n,1}-\theta_{n,2})(T_k(\theta_{n,1})-(T_k(\theta_1))_\mu)\psi_h(\theta_{n,1})\,d\sigma\,dt\\&-\int_{\Sigma_T} h(x,t)|\theta_{n,1}-\theta_{n,2}|^{p-2}(\theta_{n,1}-\theta_{n,2})(T_k(\theta_{n,2})-(T_k(\theta_2))_\mu)\psi_h(\theta_{n,2})\,d\sigma\,dt\\&=\int_{Q_T} f_n(T_k(\theta_n)-(T_k(\theta))_\mu)\psi_h(\theta_n)\,dx\,dt,\end{aligned} \tag{4.14}$$

In view of Lemma 7.1 (see Appendix in [3]).

$$\sum_{i=1}^{2}\int_0^T\int_{\Omega_i}\frac{\partial\theta_{n,i}}{\partial t}(T_k(\theta_{n,i})-(T_k(\theta_i))_\mu)\psi_h(\theta_{n,i})\,dx\,dt\geq\varepsilon_1(n,\mu,h). \tag{4.15}$$

The defintion of $\psi_h(\cdot)$ allows us to reformulate the second term as

$$\begin{aligned}
&\sum_{i=1}^{2}\int_{Q_{i,T}}A(x,t,T_n(\theta_{n,i}),D\theta_{n,i})(DT_k(\theta_{n,i})-D(T_k(\theta_i))_\mu)\psi_h(\theta_{n,i})\,dx\,dt\\
&\geq\sum_{i=1}^{2}\int_{Q_{i,T}}(A(x,t,T_k(\theta_{n,i}),DT_k(\theta_{n,i}))-A(x,t,T_k(\theta_{n,i}),DT_k(\theta_i)))\cdot(DT_k(\theta_{n,i})\\
&\quad-DT_k(\theta_i))\,dx\,dt+\sum_{i=1}^{2}\int_{Q_{i,T}}A(x,t,T_k(\theta_{n,i}),DT_k(\theta_i))\cdot(DT_k(\theta_{n,i})-DT_k(\theta_i))\,dx\,dt\\
&\quad+\sum_{i=1}^{2}\int_{Q_{i,T}}A(x,t,T_k(\theta_{n,i}),DT_k(\theta_{n,i}))\cdot(DT_k(\theta_i)-D(T_k(\theta_i))_\mu)\,dx\,dt\\
&\quad-\sum_{i=1}^{2}\int_{\{k<|\theta_{n,i}|\leq 2h\}}|A(x,t,T_{2h}(\theta_{n,i}),DT_{2h}(\theta_{n,i}))||D(T_k(\theta_i))_\mu|\,dx\,dt,
\end{aligned} \tag{4.16}$$

Using same arguments in [3], we can show that the last three terms of the left-hand side of (4.16) are tends to 0, as n and h goes to $+\infty$. It follows that

$$\begin{aligned}
&\sum_{i=1}^{2}\int_{Q_{i,T}}A(x,t,T_n(\theta_{n,i}),D\theta_{n,i})(DT_k(\theta_{n,i})-D(T_k(\theta_i))_\mu)\psi_h(\theta_{n,i})\,dx\,dt\\
&\quad\geq\sum_{i=1}^{2}\int_{Q_{i,T}}(A(x,t,T_k(\theta_{n,i}),DT_k(\theta_n))-A(x,t,T_k(\theta_{n,i}),DT_k(\theta_i)))\cdot(DT_k(\theta_{n,i})\\
&\qquad-DT_k(\theta_i))\,dx\,dt+\varepsilon_8(n,h).
\end{aligned} \tag{4.17}$$

Moreover, since $\psi_h'(s)=-\dfrac{1}{h}\chi_{\{h<|s|\leq 2h\}}$, and from (4.10) therefore

$$\begin{aligned}
&\Big|\sum_{i=1}^{2}\int_{Q_{i,T}}A(x,t,T_n(\theta_{n,i}),D\theta_{n,i})\cdot D\theta_{n,i}\psi_h'(\theta_{n,i})(T_k(\theta_{n,i})-(T_k(\theta_i))_\mu)\,dx\,dt\Big|\\
&\leq\frac{2k}{h}\sum_{i=1}^{2}\int_{\{h<|\theta_{n,i}|\leq 2h\}}A(x,t,T_n(\theta_{n,i}),D\theta_{n,i})\cdot D\theta_{n,i}\,dx\,dt\longrightarrow 0\quad\text{as}\quad n,h\longrightarrow\infty.
\end{aligned} \tag{4.18}$$

By the definition of $\psi_h(\cdot)$ we have $\psi_h(\theta_{n,1})=\psi_{2h}(\theta_{n,1})\psi_{2h}(\theta_{n,1})$, we can write the third term on the right-hand side of (4.16), as follows

$$\int_{\Sigma_T}h(x,t)|\theta_{n,1}-\theta_{n,2}|^{p-2}(\theta_{n,1}-\theta_{n,2})(T_k(\theta_{n,1})-(T_k(\theta_1))_\mu)\psi_h(\theta_{n,1})\,d\sigma\,dt=I_1+I_2. \tag{4.19}$$

where

$$\begin{aligned} I_1 &= \int_{\Sigma_T} h(x,t)|\theta_{n,1}-\theta_{n,2}|^{p-2}(\theta_{n.1}-\theta_{n,2})(T_k(\theta_{n,1})-(T_k(\theta_1))_\mu) \\ &\quad \times \psi_{2h}(\theta_{n,1})(\psi_h(\theta_{n,1})-\psi_h(\theta_{n,2}))\,d\sigma\,dt \\ I_2 &= \int_{\Sigma_T} h(x,t)|T_{4h}(\theta_{n,1})-T_{2h}(\theta_{n,2})|^{p-2}(T_{4h}(\theta_{n,1})-T_{2h}(\theta_{n,2})) \\ &\quad \times (T_k(\theta_{n,1})-(T_k(\theta_1))_\mu)\psi_{2h}(\theta_{n,1})\psi_h(\theta_{n,2}))\,d\sigma\,dt. \end{aligned} \tag{4.20}$$

Due to (4.11), and the Lipschitz continuity of $\psi_h(\cdot)$ we obtain

$$\begin{aligned} |I_1 \leq & \frac{2k}{h}\int_{\Sigma_T} h(x,t)|\theta_{n,1}-\theta_{n,2}|^{p-1}|T_h(\theta_{n,1})-T_h(\theta_{n,2})|\,d\sigma\,dt \\ & \longrightarrow 0 \quad \text{as} \quad n,h\longrightarrow\infty. \end{aligned} \tag{4.21}$$

Since, for $i=1,2$, $T_k(\theta_{n,1})\to T_k(\theta_1)$ and $(T_k(\theta_i))_\mu\to T_k(\theta_1)$ strongly in $L^1(\Sigma_T)$ as $n\to\infty$ and $\mu\to\infty$ respectively, then

$$|I_2|\leq (6h)^{p-1}\int_{\Sigma_T} h(x,t)|T_k(\theta_{n,1})-(T_k(\theta_1))_\mu)|\,d\sigma\,dt\longrightarrow 0 \quad \text{as} \quad n,\mu\to\infty. \tag{4.22}$$

Hence, combining (4.19) – (4.22) we get

$$\begin{aligned} &\int_{\Sigma_T} h(x,t)|\theta_{n,1}-\theta_{n,2}|^{p-2}(\theta_{n,1}-\theta_{n,2})(T_k(\theta_{n,1})-(T_k(\theta_1))_\mu)\psi_h(\theta_{n,1})\,d\sigma\,dt \\ &\quad \leq \varepsilon(n,\mu,h). \end{aligned} \tag{4.23}$$

Similary, we get

$$\begin{aligned} &\int_{\Sigma_T} h(x,t)|\theta_{n,1}-\theta_{n,2}|^{p-2}(\theta_{n,1}-\theta_{n,2})(T_k(\theta_{n,2})-(T_k(\theta_2))_\mu)\psi_h(\theta_{n,2})\,d\sigma\,dt \\ &\quad \leq \varepsilon(n,\mu,h). \end{aligned} \tag{4.24}$$

Moreover, it easy to show that

$$\int_{Q_T} f_n(T_k(\theta_n)-(T_k(\theta))_\mu)\psi_h(\theta_n)\,dx\,dt\longrightarrow 0 \quad \text{as} \quad n,\mu,h\longrightarrow\infty. \tag{4.25}$$

By combining (4.14), (4.15), (4.17) – (4.18) and (4.23) – (4.25) we deduce that

$$\begin{aligned} \sum_{i=1}^{2}\int_{Q_{i,T}} &(A(x,t,T_k(\theta_{n,i}),DT_k(\theta_{n,i}))-A(x,t,T_k(\theta_{n,i}),DT_k(\theta_i))) \\ &\times(DT_k(\theta_{n,i})-DT_k(\theta_i))\,dx\,dt\leq\varepsilon(n,\mu,h). \end{aligned} \tag{4.26}$$

Therefore, by letting n and h tend to infinity in the previous inequality, and using Lemma 2.7, we deduce that (4.12) and (4.13). □

Remark 4.6. By Fatou's Lemma and (4.11), we obtain

$$\lim_{h\to\infty}\frac{1}{h}\int_{\Sigma_T} h(x,t)|\theta_1-\theta_2|^{p-2}(\theta_1-\theta_2)(T_h(\theta_1)-T_h(\theta_2))\,d\sigma\,dt=0. \tag{4.27}$$

Remark 4.7. For $i = 1, 2$, the mesurable functions θ_i satisfies 3.1 i.e.

$$\lim_{h\to\infty} \frac{1}{h} \int_{\{|\theta_i|\le h\}} A(x,t,\theta_i, D\theta_i)\cdot D\theta_i \, dx = 0$$

Indeed, for $i = 1, 2$, we have $A(x,t,T_n(\theta_{n,i}), D\theta_{n,i})\cdot D\theta_{n,i} \to A(x,t,\theta_i,D\theta_i)\cdot D\theta_i$ a.e. in $Q_{T,i}$. Thanks to Fatou's Lemma and (4.10), we conclude that

$$\begin{aligned}
&\lim_{h\to\infty} \frac{1}{h} \int_{\{|\theta_i|\le h\}} A(x,t,\theta_i, D\theta_i)\cdot D\theta_i \, dx\, dt \\
&\le \lim_{h\to\infty} \liminf_{n\to\infty} \frac{1}{h} \int_{\{|\theta_{n,i}|\le h\}} A(x,t,T_n(\theta_{n,i}), D\theta_{n,i})\cdot D\theta_{n,i} \, dx\, dt \\
&\le \lim_{h\to\infty} \limsup_{n\to\infty} \frac{1}{h} \int_{\{|\theta_{n,i}|\le h\}} A(x,t,T_n(\theta_{n,i}), D\theta_{n,i})\cdot D\theta_{n,i} \, dx = 0.
\end{aligned}$$

4.3. **Proof of Theorem 3.3.**
By combining the observations from Remark 4.3, Remark 4.6 and Remark 4.7, we deduce that for $i = 1, 2$, θ_i satisfies (3.1) − (3.3). Moreover, a straightforward verification shows that $T_k(\theta) := (T_k(\theta_1), T_k(\theta_2)) \in \Big(L^\infty(0,T,L^2(\Omega_1)) \times (L^\infty(0,T,L^2(\Omega_2))\Big) \cap \mathcal{W}$,
Now we will prove (3.4): let $S \in C_0^2(\mathbb{R})$ with $\mathrm{supp}(S'(\cdot)) \subset [-M, M]$ for some $M > 0$, and let $\varphi \in C_0^\infty((\Omega_1^{(1)} \times (0,T[) \times C_0^\infty(\overline{\Omega_2} \times (0,T[)$. By taking $v = S'(\theta_n)\varphi \in \mathcal{W}$ as a test function in the approximate problem (4.2), we get

$$K_{1,1}^n + K_{1,2}^n + K_{2,1}^n + K_{2,2}^n + K_{3,1}^n + K_{3,2}^n + K_{4,1}^n + K_{4,2}^n + K_{5,1}^n + K_{5,2}^n = K_6^n. \tag{4.28}$$

$$\begin{aligned}
K_{1,i}^n &= \int_0^T \int_{\Omega_i} \frac{\partial S(\theta_{n,i})}{\partial t} \varphi_i \, dx\, dt \\
K_{2,i}^n &= \int_0^T \int_{\Omega_i} A(x,t,T_n(\theta_{n,i}), D\theta_{n,i})\cdot D\theta_{n,i} S''(\theta_{n,i})\varphi_i \, dx\, dt \\
K_{3,i}^n &= \int_0^T \int_{\Omega_i} A(x,t,T_n(\theta_{n,i}), D\theta_{n,i}) S'(\theta_{n,i}) D\varphi_i \, dx\, dt \\
K_{4,i}^n &= \int_0^T \int_{\Gamma} h(x,t)|\theta_{n.1} - \theta_{n,2}|^{p-2}(\theta_{n.1} - \theta_{n,2}) S(\theta_{n,i})\varphi_i \, d\sigma\, dt \\
K_5^n &= \int_0^T \int_{\Omega} f_n S(\theta_n)\varphi \, dx\, dt.
\end{aligned}$$

Since $S(\cdot)$ is bounded and continuous, with $S(\theta_{n,i}) \to S(\theta_i)$ a.e. in $Q_{T,i}$ and weak$-*$ in $L^\infty(Q_{T,i})$. Therefore

$$\begin{aligned}
\lim_{n\to\infty} K_{1,i}^n &= -\lim_{n\to\infty} \int_0^T \int_{\Omega_i} S(\theta_{n,i}) \frac{\partial \varphi_i}{\partial t} \, dx\, dt - \lim_{n\to\infty} \int_{\Omega_i} S(\theta_{n,0,i})\varphi(x,0)\, dx \\
&= -\int_0^T \int_{\Omega_i} S(\theta_i) \frac{\partial \varphi_i}{\partial t} \, dx\, dt - \int_{\Omega_i} S(\theta_{0,i})\varphi(x,0)\, dx.
\end{aligned} \tag{4.29}$$

For the term $K^n_{2,i}$ we can whritten

$$A(x,t,T_n(\theta_{n,i}),D\theta_{n,i})D\theta_{n,i}S''(\theta_i) = A(x,t,T_M(\theta_{n,i}),DT_M(\theta_{n,i}))DT_M(\theta_{n,i})S''(\theta_{n,i}), \tag{4.30}$$

we have $A(x,t,T_M(\theta_{n,i}),DT_M(\theta_{n,i})) \to A(x,t,T_M(\theta_i),DT_M(\theta_i))$ a.e. in Q_T, furthermore by using the assumption (4.1) we have $(A(x,t,T_M(\theta_{n,i}),DT_M(\theta_{n,i})))_n$ is bounded in $(L^{p'}(Q_{T,i}))^N$, we arrive that and

$$A(x,t,T_M(\theta_{n,i}),DT_M(\theta_{n,i}) \quad \overset{\rightharpoonup}{n\to\infty} \quad A(x,t,T_M(\theta_i),DT_M(\theta_i)) \tag{4.31}$$

weakly in $(L^{p'}(Q_{T,i}))^N$, forthurmore, and since

$$DT_M(\theta_{n,i})S''(\theta_{n,i})\varphi_i \longrightarrow DT_M(\theta_i)S''(\theta_i)\varphi_i \quad \text{ strongly in } \ (L^p(Q_{T,i}))^N,$$

therefore

$$\begin{aligned}\lim_{n\to\infty} K^n_{2,i} &= \int_0^T\int_{\Omega_i} A(x,t,T_M(\theta_i),DT_M(\theta_i))\cdot DT_M(\theta_i)S''(\theta_i)\varphi_i\,dx\,dt \\ &= \int_0^T\int_{\Omega_i} A(x,t,\theta_i,D\theta_i)\cdot D\theta_i S''(\theta_i)\varphi_i\,dx\,dt.\end{aligned} \tag{4.32}$$

For the term $K^n_{3,i}$ using similar arguments on $K^n_{2,i}$ we conclude that

$$\begin{aligned}\lim_{n\to\infty} K^n_{3,i} &= \int_0^T\int_{\Omega_i} A(x,t,T_M(\theta_i),DT_M(\theta_i))S'(\theta_i)D\varphi_i\,dx\,dt \\ &= \int_0^T\int_{\Omega_i} A(x,t,\theta_i,D\theta_i)S'(\theta_i)D\varphi_i\,dx\,dt.\end{aligned} \tag{4.33}$$

We now focus on the term $K^n_{4,1}$. Let $h > M$, by the definition of $\psi_h(\cdot)$ and since $S'(\cdot) \subset [-M,M]$, we have $S'(\theta_{n,1}) = S'(\theta_{n,1})\psi_h(\theta_{n,1}) = S'(\theta_{n,1})\psi_h(\theta_{n,1})\psi_{2h}(\theta_{n,1})$ a.e. in Σ_T. It follows that

$$\begin{aligned}&\int_0^T\int_\Gamma h(x,t)|\theta_{n.1}-\theta_{n,2}|^{p-2}(\theta_{n.1}-\theta_{n,2})S'(\theta_{n,1})\varphi_1\,d\sigma\,dt \\ &= \int_0^T\int_\Gamma h(x,t)|\theta_{n.1}-\theta_{n,2}|^{p-2}(\theta_{n.1}-\theta_{n,2})S'(\theta_{n,1})\psi_{2h}(\theta_{n,1})\psi_h(\theta_{n,1})\varphi_1\,d\sigma\,dt \\ &= \int_0^T\int_\Gamma h(x,t)|\theta_{n.1}-\theta_{n,2}|^{p-2}(\theta_{n.1}-\theta_{n,2})\psi_{2h}(\theta_{n,1})S'(\theta_{n,1})(\psi_h(\theta_{n,1})-\psi_h(\theta_{n,2}))\varphi_1\,d\sigma\,dt \\ &+ \int_0^T\int_\Gamma h(x,t)|T_{4h}(\theta_{n.1})-T_{2h}(\theta_{n,2})|^{p-2}(T_{4h}(\theta_{n.1})-T_{2h}(\theta_{n,2}))\psi_{2h}(\theta_{n,1})\psi_h(\theta_{n,2})S'(\theta_{n,1})\varphi_1 d\sigma\,dt. \\ &= J_1+J_2\end{aligned} \tag{4.34}$$

Thanks to the Lipschitz continuity of $\psi_h(\cdot)$ and due to (4.11) we have

$$\begin{aligned}|J_1| &\le \frac{\|S(\cdot)\|_{L^\infty(\mathbb{R})}\|\varphi_1\|_{L^\infty(\Sigma_T)}}{h}\int_{\Sigma_T} h(x,t)|\theta_{n.1}-\theta_{n,2}|^{p-2}|T_h(\theta_{n,1})-T_h(\theta_{n,2})|\,d\sigma\,dt \\ &\longrightarrow 0 \quad \text{as} \quad n,h\to\infty.\end{aligned} \tag{4.35}$$

We look to the last term of (4.34), thanks to Lebesgue dominated convergence theorem and by using Lemma 4.2(v), we have

$$\int_{\Sigma_T} h(x,t)|T_{4h}(\theta_{n,1}) - T_{2h}(\theta_{n,2})|^{p-2}(T_{4h}(\theta_{n.1}) - T_{2h}(\theta_{n,2}))\psi_{2h}(\theta_{n,1})\psi_h(\theta_{n,2})S(\theta_{n,1})\varphi_1 \, d\sigma \, dt$$
$$\longrightarrow \int_{\Sigma_T} h(x,t)|T_{4h}(\theta_1) - T_{2h}(\theta_2)|^{p-2}(T_{4h}(\theta_1) - T_{2h}(\theta_2))\psi_{2h}(\theta_1)S(\theta_1)\psi_h(\theta_2)\varphi_1 \, d\sigma \, dt \quad \text{as} \quad n \to \infty.$$

as n tends to infinity. According to Remark 3.2, we have $h(x,t)|\theta_1 - \theta_2|^{p-2}(\theta_1 - \theta_2)S(\theta_1) \in L^1(\Sigma_T)$. Thus

$$\begin{aligned}
&\int_0^T \int_\Gamma h(x,t)|T_{4h}(\theta_1) - T_{2h}(\theta_2)|^{p-2}(T_{4h}(\theta_1) - T_{2h}(\theta_2))\psi_{2h}(\theta_1)S(\theta_1)\psi_h(\theta_2)\varphi_1 \, d\sigma \, dt \\
&= \int_0^T \int_\Gamma h(x,t)|\theta_1 - \theta_2|^{p-2}(\theta_1 - \theta_2)\psi_{2h}(\theta_1)S(\theta_1)\psi_h(\theta_2)\varphi_1 \, d\sigma \, dt \\
&\longrightarrow \int_0^T \int_\Gamma h(x,t)|\theta_1 - \theta_2|^{p-2}(\theta_1 - \theta_2)S(\theta_1)\varphi_1 \, d\sigma \, dt \quad \text{as} \quad h \to \infty.
\end{aligned} \tag{4.36}$$

By combining (4.34) $-$ (4.36), we obtain

$$\lim_{h\to\infty} \limsup_{n\to\infty} K_{4,1}^n = \int_0^T \int_\Gamma h(x,t)|\theta_1 - \theta_2|^{p-2}(\theta_1 - \theta_2)S(\theta_1)\varphi_1 \, d\sigma \, dt. \tag{4.37}$$

In the same way, we get

$$\lim_{h\to\infty} \limsup_{n\to\infty} K_{4,2}^n = \int_0^T \int_\Gamma h(x,t)|\theta_1 - \theta_2|^{p-2}(\theta_1 - \theta_2)S(\theta_2)\varphi_2 \, d\sigma \, dt. \tag{4.38}$$

Concerning now the term K_5^n we get

$$\int_0^T \int_{\Omega/\Gamma} f_n S(\theta_n)\varphi \, dx \, dt \longrightarrow \int_0^T \int_{\Omega/\Gamma} f S(\theta)\varphi \, dx \, dt \quad \text{as} \quad n \to \infty. \tag{4.39}$$

By passing to the limit as n tends to infinity in (4.28), and using the results established in (4.29), and (4.32) $-$ 4.33) and (4.37) $-$ (4.39), we deduce that

$$\begin{aligned}
&-\sum_{i=1}^2 \int_0^T \int_{\Omega_i} S(\theta_i)\frac{\partial \varphi_i}{\partial t} \, dx \, dt + \sum_{i=1}^2 \int_0^T \int_{\Omega_i} A(x,t,\theta_i, D\theta_i) \cdot (S'(\theta_i)\varphi_i D\theta_i \\
&+ S(\theta_i)D\varphi_i) \, dx \, dt + \int_0^T \int_\Gamma h(x,t)|\theta_1 - \theta_2|^{p-2}(\theta_1 - \theta_2)(\varphi_1 S(\theta_1) - \varphi_2 S(\theta_2)) \, d\sigma \, dt \\
&= \int_0^T \int_\Omega f S(\theta)\varphi \, dx \, dt + \sum_{i=1}^2 \int_{\Omega_i} S(\theta_{0,i})\varphi_i(x,0) \, dx,
\end{aligned}$$

which complete the proof of the Theorem 3.3.

5. Uniqueness

In this section, we prove Theorem 3.4. Inspired by the methods in [10] and [17], we rely on a crucial lemma

Lemma 5.1. *(see. [10]) Under the assumption (1.2) − (1.4), and (3.5), any renormalized solution θ of the problem (1.1) satisfies the following estimate for any $h \geq 1$ and $0 < \mu < 1$*

$$\frac{1}{\mu}\int_{\{h-\mu<|\theta_i|<h+\mu\}} A(x,t,\theta_i,D\theta_i)\cdot D\theta_i\,dx\,dt \leq \varepsilon(\mu,h), \tag{5.1}$$

where $\lim\limits_{h\to\infty}\limsup\limits_{\mu\to 0}\varepsilon(\mu,h)=0.$

5.0.1. *Proof of theorem (3.4).*
Let $f \in L^1(Q_T)$ and $\theta_{0,i} \in L^1(\Omega_i)$. assuming that θ^1 and θ^2 are two renormalized solutions for the parabolic problem (1.1).
Let $h > 0$ and $0 < \mu < 1$, we define the function $T_{h,\mu}(r)$ as follows :

$$\begin{cases} T_{h,\mu}(0)=0, & \\ (T_{h,\mu})'(r)=1 & \text{for } |r|<h, \\ (T_{h,\mu})'(r)=\dfrac{1}{\mu}(h+\mu-|r|) & \text{for } h\leq|r|\leq h+\mu, \\ (T_{h,\mu})'(r)=0 & \text{for } |r|>h+\mu. \end{cases} \tag{5.2}$$

Firstly, we have $T_{h,\mu}(\cdot) \in W^{2,\infty}(\mathbb{R})$, with $\operatorname{supp}(T'_{h,\mu}(s)) \subset [-h-1, h+1]$, and since $\varphi^1 = \dfrac{1}{k}T_k(T_{h,\mu}(\theta^1)-T_{h,\mu}(\theta^2))\chi_{(0,t)} \in \mathcal{W}\cap(L^\infty(Q_{T,1})\times L^\infty(Q_{T,2}))$, by taking φ^1 as a test function for the problem (1.1) with θ^1 as a renormalized solution and $S = T_{h,\mu}$, and $\varphi^2 = \dfrac{1}{k}T_k(T_{h,\mu}(\theta^2)-T_{h,\mu}(\theta^1))\chi_{(0,t)}$ as a test function for the problem (1.1) with θ^2 as a renormalized solution and $S = T_{h,\mu}$, we set

$$S_k(r)=\int_0^r T_k(s)\,ds=\begin{cases}\frac{r^2}{2} & \text{if } |r|\leq k,\\ k|r|-\frac{k^2}{2} & \text{if } |r|>k.\end{cases}$$

Then, adding the results, we obtain

$$\begin{aligned}&\frac{1}{k}\sum_{i=1}^{2}\int_{\Omega_i} S_k(T_{h,\mu}(\theta_i^1(t))-T_{h,\mu}(\theta_i^2(t)))\,dx \\ &+A_{1,k}^{h,\mu}+A_{1,k}^{h,\mu}+\tilde{A}_{1,k}^{h,\mu}+\tilde{A}_{1,k}^{h,\mu}+G_{1,k}^{h,\mu}-G_{2,k}^{h,\mu}=F_k^{h,\mu}.\end{aligned} \tag{5.3}$$

where

$$A_{i,k}^{h,\mu} = \frac{1}{k}\int_{Qt,i}\Big(T'_{h,\mu}(\theta_i^1)A(x,\tau,\theta_i^1,D\theta_i^1) - T'_{h,\mu}(\theta_i^2)A(x,\tau,\theta_i^2,D\theta_i^2)\Big) \times DT_k(T_{h,\mu}(\theta_i^1) - T_{h,\mu}(\theta_i^2))\,dx\,d\tau$$

$$\tilde{A}_{i,k}^{h,\mu} = \frac{1}{k}\int_{Qt,i}\Big(T''_{h,\mu}(\theta_i^1)A(x,\tau,\theta_i^1,D\theta_i^1)\cdot D\theta_i^1 - T''_{h,\mu}(\theta_i^2)A(x,\tau,\theta_i^2,D\theta_i^2)\cdot D\theta_i^2\Big) \times T_k(T_{h,\mu}(\theta_i^1) - T_{h,\mu}(\theta_i^2))\,dx\,d\tau$$

$$G_{i,k}^{h,\mu} = \frac{1}{k}\int_{\Sigma_t} h(x,\tau)\Big(|\theta_1^1-\theta_2^1|^{p-2}(\theta_1^1-\theta_2^1)T'_{h,\mu}(\theta_i^1) - |\theta_1^2-\theta_2^2|^{p-2}(\theta_1^2-\theta_2^2)T'_{h,\mu}(\theta_i^2)\Big) \times T_k(T_{h,\mu}(\theta_i^1) - T_{h,\mu}(\theta_i^2))\,d\sigma\,d\tau$$

$$F_k^{h,\mu} = \frac{1}{k}\int_0^t\int_\Omega f(T'_{h,\mu}(\theta^1) - T'_{h,\mu}(\theta^2)))T_k(T_{h,\mu}(\theta^1) - T_{h,\mu}(\theta^2))\,dx\,d\tau.$$

By definition of $S_k(\cdot)$, we obtain

$$\limsup_{k\to 0}\lim_{\mu\to 0}\frac{1}{k}\int_{\Omega_i} S_k(T_{h,\mu}(\theta_i^1(t)) - T_{h,\mu}(\theta_i^2(t)))\,dx = \int_{\Omega_i}|T_k(\theta_i^1(t)-\theta_i^2(t)|\,dx. \tag{5.4}$$

For the term $A_{i,k}^{h,\mu}$, we have $T'_{h,\mu}(\theta_i^1)A(x,\tau,\theta_i^1,D\theta_i^1) \to A(x,\tau,T_h(\theta_i^1),DT_h(\theta_i^1))$ a.e. in $Q_{T,i}$ and since

$$|T'_{h,\mu}(\theta_i^1)A(x,\tau,\theta_i^1,D\theta_i^1)| \le 2|A(x,\tau,T_{h+1}(s),DT_{h+1}(\theta_i^1))| \in L^{p'}(Q_{T,i}). \tag{5.5}$$

In view of Lebesgue dominated convergence theorem we have

$$T'_{h,\mu}(\theta_i^1)A(x,\tau,\theta_i^1,D\theta_i^1)\overrightarrow{\mu\to 0}A(x,\tau,T_h(\theta_i^1),DT_h(\theta_i^1)) \quad \text{strongly in} \quad (L^{p'}(Q_{t,i}))^N. \tag{5.6}$$

Similarly, we have

$$T'_{h,\mu}(\theta_i^2)A(x,\tau,\theta_i^2,D\theta_i^2)\overrightarrow{\mu\to 0}A(x,\tau,T_h(\theta_i^2),DT_h(\theta_i^2)) \quad \text{strongly in} \quad (L^{p'}(Q_{t,i}))^N. \tag{5.7}$$

Moreover, since $T_{h,\mu}(\theta_i^1) \overset{\rightarrow}{\mu \to 0} T_h(\theta_i^1)$ strongly in $\mathcal{W}$, we conclude that

$$\begin{aligned}
A_{i,k}^h &= \lim_{\mu\to 0} A_{i,k}^{h,\mu} \\
&= \lim_{\mu\to 0} \frac{1}{k} \int_{Q_{t,i}} \Big((T_{h,\mu})'(\theta_i^1) A(x,\tau,\theta_i^1, D\theta_i^1) - (T_{h,\mu})'(\theta_i^2) A(x,\tau,\theta_i^2, D\theta_i^2) \Big) \\
&\qquad \times DT_k(T_{h,\mu}(\theta_i^1) - T_{h,\mu}(\theta_i^2))\, dx\, d\tau \\
&= \frac{1}{k} \int_{Q_{t,i}} \Big((A(x,\tau,T_h(\theta_i^1), DT_h(\theta_i^1)) - A(x,\tau,T_h(\theta_i^2), DT_h(\theta_i^2)) \Big) \\
&\qquad \times DT_k(T_h(\theta_i^1) - T_h(\theta_i^2))\, dx\, d\tau \\
&= \frac{1}{k} \int_{Q_{t,i}} \Big((A(x,\tau,T_h(\theta_i^1, DT_h(\theta_i^1)) - A(x,\tau,T_h(\theta_i^1), DT_h(\theta_i^2)) \Big) \\
&\qquad \times DT_k(T_h(\theta_i^1) - T_h(\theta_i^2))\, dx\, d\tau \\
&\quad + \frac{1}{k} \int_{Q_{t,i}} \Big((A(x,\tau,T_h(\theta_i^1), DT_h(\theta_i^2)) - A(x,\tau,T_h(\theta_i^2), DT_h(\theta_i^2)) \Big) \\
&\qquad \times DT_k(T_h(\theta_i^1) - T_h(\theta_i^2))\, dx\, d\tau \\
&= A_{i,1,k}^h + A_{i,2,k}^h.
\end{aligned} \tag{5.8}$$

For the first term on the right-hand side of (5.8), in view of (1.4) we have

$$\begin{aligned}
A_{i,1,k}^h \;=\; & \frac{1}{k} \int_{\{|\theta_i^1-\theta_i^2|\le k, |\theta_i^1|\le h, |\theta_i^2|\le h\}} \Big((A(x,\tau,\theta_i^1, D\theta_i^1) - A(x,\tau,\theta_i^1, D\theta_i^2) \Big) \\
& \times (D\theta_i^1 - D\theta_i^2)\, dx\, d\tau \\
& + \frac{1}{k} \int_{\{|\theta_i^1 - T_h(\theta_i^2)|\le k, |\theta_i^1|\le h, |\theta_i^2|> h\}} A(x,\tau,\theta_i^1, D\theta_i^1)\cdot D\theta_i^1\, dx\, d\tau \\
& + \frac{1}{k} \int_{\{|T_h(\theta_i^1) - \theta_i^2|\le k, |\theta_i^1|> h, |\theta_i^2|\le h\}} A(x,\tau,\theta_i^1, D\theta_i^2)\cdot D\theta_i^2\, dx\, d\tau \ge 0.
\end{aligned} \tag{5.9}$$

Concerning the second term on the right-hand side of (5.8), We have

$$(c_k(x,\tau) + \gamma_K |DT_h(\theta_i^2)|^{p-1}) |DT_h(\theta_i^1) - DT_h(\theta_i^2))| \in L^1(Q_{T,i}),$$

By using the assumption (3.5) and by using Lebesgue dominated convergence, we obtain

$$\begin{aligned}
&\limsup_{k\to 0} |A_{i,2,k}^h| \\
&\le \limsup_{k\to 0} \frac{1}{k} \int_0^t \int_{\Omega_i} |A(x,\tau,T_h(\theta_i^1), DT_h(\theta_i^2)) - A(x,\tau,T_h(\theta_i^2), DT_h(\theta_i^2))| \\
&\qquad \times |DT_k(T_h(\theta_i^1) - T_h(\theta_i^2))|\, dx\, d\tau \\
&\le \limsup_{k\to 0} \int_{\{0<|T_h(\theta_i^1) - T_h(\theta_i^2)|\le k\}} (c_k(x,\tau) + \gamma_K |DT_h(\theta_i^2)|^{p-1}) |DT_h(\theta_i^1) - DT_h(\theta_i^2))|\, dx\, d\tau = 0.
\end{aligned} \tag{5.10}$$

By combining (5.8) and (5.9) – (5.10) we conclude that

$$\limsup_{k\to 0} A_{i,k}^h \ge 0 \tag{5.11}$$

Now, we treated the convergence of the behavior $\tilde{A}_{i,k}^{h,\mu}$, in view of Lemma 5.1 we have

$$\begin{aligned}|\tilde{A}_{i,k}^{h,\mu}| &\leq \frac{1}{\mu}\Big(\int_0^t \int_{\{h\leq|\theta_i^1|<h+\mu\}} A(x,\tau,\theta_i^1,D\theta_i^1)\cdot D\theta_i^1\,dx\,d\tau \\ &\quad + \int_0^t \int_{\{h\leq|\theta_i^2|<h+\mu\}} A(x,\tau,\theta_i^2,D\theta_i^2)\cdot D\theta_i^2\,dx\,d\tau\Big) \\ &\leq \varepsilon(\mu,h).\end{aligned} \tag{5.12}$$

For the term $G_{i,k}^{h,\mu}$, we have

$$\begin{aligned}G_{i,k}^{h,\mu} &= \frac{1}{k}\int_{\Sigma_t} h(x,\tau)|\theta_1^1-\theta_2^1|^{p-2}(\theta_1^1-\theta_2^1)\chi_{\{|\theta_i^1|\leq h\}}T_k(T_h(\theta_i^1)-T_h(\theta_i^2))\,d\sigma\,d\tau \\ &\quad - \frac{1}{k}\int_{\Sigma_t} h(x,\tau)|\theta_1^2-\theta_2^2|^{p-2}(\theta_1^2-\theta_2^2)\chi_{\{|\theta_i^2|\leq h\}}T_k(T_h(\theta_i^1)-T_h(\theta_i^2))\,d\sigma\,d\tau,\end{aligned} \tag{5.13}$$

since $p>2-\frac{1}{N+1}$ and thanks to remark 2.6 we have $h(x,\tau)|\theta_1^i-\theta_2^i|^{p-2}(\theta_1^i-\theta_2^i)$ belongs to $L^1(\Sigma_T)$, and since $\dfrac{T_k(\theta_i^1-\theta_i^2)}{k}\to \operatorname{sign}(\theta_i^1-\theta_i^2)$ a.e in Σ_T then

$$\begin{aligned}&\lim_{h\to\infty}\limsup_{k\to 0}\lim_{\mu\to 0}\frac{1}{k}\int_{\Sigma_t} h(x,\tau)|\theta_1^1-\theta_2^1|^{p-2}(\theta_1^1-\theta_2^1)\chi_{\{|\theta_i^1|\leq h\}}T_k(T_h(\theta_i^1)-T_h(\theta_i^2))\,d\sigma\,d\tau \\ &= \lim_{h\to\infty}\limsup_{k\to 0}\frac{1}{k}\int_{\{|\theta_i^1|\leq h\}\cap\{|\theta_i^2|\leq h\}} h(x,\tau)|\theta_1^1-\theta_2^1|^{p-2}(\theta_1^1-\theta_2^1)T_k(\theta_i^1-\theta_i^2)\,d\sigma\,d\tau \\ &+ \lim_{h\to\infty}\limsup_{k\to 0}\frac{1}{k}\int_{\{|\theta_i^1|\leq h\}\cap\{|\theta_i^2|> h\}} h(x,\tau)|\theta_1^1-\theta_2^1|^{p-2}(\theta_1-\theta_2)T_k(T_h(\theta_i^1)-T_h(\theta_i^2))\,dx\,d\tau \\ &= \lim_{h\to\infty}\int_{\{|\theta_i^1|\leq h\}\cap\{|\theta_i^2|\leq h\}} h(x,\tau)|\theta_1^1-\theta_2^1|^{p-2}(\theta_1^1-\theta_2^1)\operatorname{sign}(\theta_i^1-\theta_i^2)\,d\sigma\,d\tau \\ &+ \lim_{h\to\infty}\int_{\{|\theta_i^1|\leq h\}\cap\{|\theta_i^2|> h\}} h(x,\tau)|\theta_1^1-\theta_2^1|^{p-2}(\theta_1-\theta_2)\ \operatorname{sign}(T_h(\theta_i^1)-T_h(\theta_i^2))\,dx\,d\tau \\ &= \int_{\Sigma_t} h(x,\tau)|\theta_1^1-\theta_2^1|^{p-2}(\theta_1^1-\theta_2^1)\operatorname{sign}(\theta_i^1-\theta_i^2)\,d\sigma\,d\tau\end{aligned} \tag{5.14}$$

Similary we get

$$\begin{aligned}&\lim_{h\to\infty}\limsup_{k\to 0}\lim_{\mu\to 0}\frac{1}{k}\int_{\Sigma_t} h(x,\tau)|\theta_1^2-\theta_2^2|^{p-2}(\theta_1^2-\theta_2^2)\chi_{\{|\theta_i^2|\leq h\}}T_k(T_h(\theta_i^1)-T_h(\theta_i^2))\,d\sigma\,d\tau \\ &= \int_{\Sigma_t} h(x,\tau)|\theta_1^2-\theta_2^2|^{p-2}(\theta_1^2-\theta_2^2)\operatorname{sign}(\theta_i^1-\theta_i^2)\,d\sigma\,d\tau\end{aligned} \tag{5.15}$$

Combining(5.13) – (5.15) we obtain

$$\lim_{h\to\infty}\limsup_{k\to 0}\lim_{\mu\to 0} G_{1,k}^{h,\mu} = \int_{\Sigma_t}\Big(h(x,\tau)|\theta_1^1-\theta_2^1|^{p-2}(\theta_1^1-\theta_2^1) - h(x,\tau)|\theta_1^2-\theta_2^2|^{p-2}(\theta_1^2-\theta_2^2)\Big)\ \mathrm{sign}(\theta_i^1-\theta_i^2)\,dx\,d\tau. \tag{5.16}$$

Hence

$$\begin{aligned}\lim_{h\to\infty}\limsup_{k\to 0}\lim_{\mu\to 0} G_{1,k}^{h,\mu} - G_{2,k}^{h,\mu} &= \int_{\Sigma_t}\Big(h(x,\tau)|\theta_1^1-\theta_2^1|^{p-2}(\theta_1^1-\theta_2^1) - h(x,\tau)|\theta_1^1-\theta_2^1|^{p-2}(\theta_1^2-\theta_2^2)\Big)\\ &\quad\times(\ \mathrm{sign}(\theta_1^1-\theta_1^2) - \ \mathrm{sign}(\theta_2^1-\theta_2^2))\,dx\,d\tau \geq 0.\end{aligned} \tag{5.17}$$

For the last behavior $F_{h,k}^{\mu}$, we have $f\in L^1(Q_T)$, then by Lebesgue dominated convergence yields that

$$\begin{aligned}\lim_{\mu\to 0}|F_k^{h,\mu}| &\leq \int_{Q_T}|f(x,\tau)|\ |\chi_{\{|\theta^1|\leq h\}}(\theta_i^1)-\chi_{\{|\theta^2|\leq h\}}(\theta_i^2)|\frac{|T_k(T_h(\theta_i^1)-T_h(\theta_i^2))|}{k}\,dx\,d\tau\\ &\leq 2\int_{\{|u|>h\}\cup\{|v|>h\}}|f(x,\tau)|\,dx\,d\tau\longrightarrow 0 \qquad \text{as}\quad h\to\infty.\end{aligned} \tag{5.18}$$

By combining (5.3), (5.4), (5.11) – (5.12), and (5.17) – (5.18) we deduce that

$$\int_{\Omega_i}|T_k(\theta_i^1(t)-\theta_i^2(t)|\,dx\,dx\leq 0. \tag{5.19}$$

Then $\theta_i^1=\theta_i^2$ a.e. in $Q_{T,i}$.

Conflict of interest.
All authors declare that they have no conflicts of interest.

References

[1] Aifantis, E. C., & Beskos, D. E. (1980). Heat extraction from hot dry rock. *Mechanics Research Communications*, *7*(3), 165–170. https://doi.org/10.1016/0093-6413(80)90007-5

[2] Aharouch, L., Azroul, E., & Rhoudaf, M. (2008). Strongly nonlinear variational parabolic problems in weighted sobolev spaces. *Australian Journal of Mathematical Analysis and Applications*, *5*(2), Article 13, 1–25.

[3] Benboubker, M. B., Hajji, Y., Hjiaj, H., & Messbahi, O. (2026). Study of some nonlinear elliptic problem in a two-component domain with Hardy potential and L^1-data. *Asia Pacific Journal of Mathematics*, *13*, 119. https://doi.org/10.28924/APJM/13-119.

[4] Bendahmane, M., Wittbold, P., & Zimmermann, A. (2010). Renormalized solutions for a nonlinear parabolic equation with variable exponents and L^1-data. *Journal of Differential Equations*, *249*(6), 1483–1515. https://doi.org/10.1016/j.jde.2010.05.011.

[5] Bénilan, Ph., Boccardo, L., Gallouët, T., Gariepy, R., Pierre, M., & Vazquez, J. L. (1995). An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, *22*(2), 241–273.

[6] Benveniste, Y. (1987). Effective thermal conductivity of composites with a thermal contact resistance between the constituents: Non dilute case. *Journal of Applied Physics*, *61*(8), 2840–2843.https://doi.org/10.1063/1.337877

[7] Benveniste, Y., Chen, T., & Dvorak, G. J. (1990). The effective thermal conductivity of composites reinforced by coated cylindrically orthotropic fibers. *Journal of Applied Physics*, *67*(6), 2878–2884.https://doi.org/10.1063/1.345463

[8] Bhatt, H., Donaldson, K. Y., & Hasselman, D. P. H. (1990). Role of the interfacial thermal barrier in the effective diffusivity/conductivity of SiC-fiber reinforced reaction-bonded silicon nitride. *Journal of the American Ceramic Society*, *73*(2), 312–316.https://doi.org/10.1111/j.1151-2916.1990.tb06511.x

[9] Blanchard, D. (1993). Truncations and monotonicity methods for parabolic equations. *Nonlinear Analysis: Theory, Methods & Applications*, *21*(10), 725–743.https://doi.org/10.1016/0362-546X(93)90120-H.

[10] Blanchard, D., Murat, F., & Redwane, H. (2001). Existence and uniqueness of a renormalized solution for a fairly general class of nonlinear parabolic problems. *Journal of Differential Equations*, *177*(2), 331–374.https://doi.org/10.1006/jdeq.2000.4013

[11] Boccardo, L., & Gallouët, T. (1989). Nonlinear elliptic and parabolic equations involving measure data. *Journal of Functional Analysis*, *87*(1), 149–169.https://doi.org/10.1016/0022-1236(89)90005-0..

[12] Bouzelmate, A., Hajji, Y., Hjiaj, H., & Inssaf, R. (2026). Renormalized solutions for a quasilinear elliptic problem in a two-component domain under Fourier boundary conditions. *Rendiconti del Circolo Matematico di Palermo Series 2*, *75*(3), Article 77.https://doi.org/10.1007/s12215-026-01394-x.

[13] Chen, P. J., & Gurtin, M. E. (1968). On a theory of heat conduction involving two temperatures. *Zeitschrift für angewandte Mathematik und Physik ZAMP*, *19*(4), 614–627.https://doi.org/10.1007/BF01594969.

[14] Chiew, Y. C., & Glandt, E. D. (1987). Effective conductivity of dispersions: The effect of resistance at the particle surfaces. *Chemical Engineering Science*, *42*(11), 2677–2685.https://doi.org/10.1016/0009-2509(87)87018-5

[15] Chrif, M., Hajji, Y., & Hjiaj, H. (2025). Entropy solutions for some nonlinear and noncoercive parabolic problems in the anisotropic Sobolev spaces. *Gulf Journal of Mathematics*, *19*(1), 285–311. https://doi.org/10.56947/gjom.v19i1.2600

[16] Di Nardo, R., Feo, F., & Guibé, O. (2011). Existence result for nonlinear parabolic equations with lower order terms. *Analysis and Applications*, *9*(2), 161–186.https://doi.org/10.1142/S0219530511001790

[17] Di Nardo, R., Feo, F., & Guibé, O. (2013). Uniqueness of renormalized solutions to nonlinear parabolic problems with lower order terms. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, *143*(6), 1185–1208.https://doi.org/10.1017/s0308210511001831

[18] Do, T. D., Truong, L. X., & Trong, N. N. (2024). Renormalized solutions to a parabolic equation with mixed boundary constraints. *Dissertationes Mathematicae*, *593*, 1–46.DOI:10.4064/dm230316-21-3

[19] Donato, P., & Raimondi, F. (2017). Existence and uniqueness results for a class of singular elliptic problems in two-component domains. In C. Constanda, A. Kirsch, & S. Mikilyan (Eds.), *Integral methods in science and engineering* (Vol. 1, pp. 83–93). Birkhäuser.https://doi.org/10.1007/978-3-319-59384-5_8

[20] Donato, P., Fulgencio, R., & Guibé, O. (2017). Homogenization of a quasilinear elliptic problem in a two-component domain with L^1 data. *Journal de Mathématiques Pures et Appliquées*, *108*(6), 91–129.https://doi.org/10.1007/s10231-021-01150-1

[21] Donato, P., Guibé, O., & Oropeza, A. (2018). Homogenization of quasilinear elliptic problems with nonlinear Robin conditions and L^1 data. *Journal de Mathématiques Pures et Appliquées*, *120*, 91–129.https://doi.org/10.1016/j.matpur.2017.10.002.

[22] Donato, P., Monsurrò, S., & Raimondi, F. (2017). Existence and uniqueness results for a class of singular elliptic problems in perforated domains. *Ricerche mat*, *66*, 333–360. https://doi.org/10.1007/s11587-016-0303-y.

[23] El Minsari, A., El Alami, O., & Ourraoui, A. (2026). Some results of the existence and behavior of solutions for p-Laplacian parabolic-type equations on networks. *Gulf Journal of Mathematics, 22*(1), 1–15. https://doi.org/10.56947/gjom.v22i1.3926.

[24] Every, A. G., Tzou, Y., Hasselman, D. P. H., & Raj, R. (1992). The effect of particle size on the thermal conductivity of ZnS/diamond composites. *Acta Metallurgica et Materialia*, *40*(1), 123–129.https://doi.org/10.1016/0956-7151(92)90205-S.

[25] Fadale, T. D., & Taya, M. (1991). Effective thermal conductivity of composites with fibre-matrix debonding. *Journal of Materials Science Letters*, *10*(11), 682–684.https://doi.org/10.1007/BF00722767

[26] Fulgencio, R., & Guibé, O. (2021). Quasilinear elliptic problems in a two-component domain with L^1 data. In P. Donato & M. Luna-Laynez (Eds.), *Emerging problems in the homogenization of partial differential equations* (SEMA SIMAI Springer Series, Vol. 10, pp. 59–83). Springer.https://doi.org/10.1007/978-3-030-62030-1_4

[27] Fulgencio, R., & Guibé, O. (2021). Uniqueness for quasilinear elliptic problems in a two-component domain with L^1 data. *Nonlinear Analysis*, *211*, Article 112442.https://doi.org/10.1016/j.na.2021.112406.

[28] Guibé, O., & Oropeza, A. (2017). Renormalized solutions of equations with Robin boundary conditions. *Acta Mathematica Scientia*, *37*(4), 889–910.https://doi.org/10.1016/S0252-9602(17)30046-2.

[29] Hajji, Y., & Hjiaj, H. (2023). A class of quasilinear and noncoercive elliptic problem in a two-component domain with L^1 data. *Advances in Operator Theory*, *8*(3), Article 52.https://doi.org/10.1007/s43036-023-00281-x

[30] Hajji, Y., & Hjiaj, H. (2024). Existence and uniqueness solutions for some strongly quasilinear parabolic problems in anisotropic Sobolev spaces. *Results in Mathematics*, *79*(4), Article 175.https://doi.org/10.1007/s00025-024-02191-7.

[31] Hajji, Y., Hjiaj, H., Jamiai, I., & Mohamed, Y. (2026). Existence of entropy solutions for nonlinear elliptic problem in a two-component domain. *Electronic Journal of Differential Equations*, *2026*(42), 1–14.https://doi.org/10.58997/ejde.2026.42

[32] Haller-Dintelmann, R., Jonsson, A., Knees, D., & Rehberg, J. (2016). Elliptic and parabolic regularity for second-order divergence operators with mixed boundary conditions. *Mathematical Methods in the Applied Sciences*, *39*(17), 5007–5026.https://doi.org/10.1002/mma.3484

[33] Hasselman, D. P. H., & Johnson, L. (1987). Effective thermal conductivity of composites with interfacial thermal barrier resistance. *Journal of Composite Materials*, *21*(6), 508–515. https://doi.org/10.1177/002199838702100602

[34] Hasselman, D. P. H., Donaldson, K. Y., & Thomas, J. R., Jr. (1993). Effective thermal conductivity of uniaxial composite with cylindrically orthotropic carbon fibers and interfacial thermal barrier. *Journal of Composite Materials*, *27*(6), 637–644.https://doi.org/10.1177/00219983930270060

[35] Leone, C., & Porretta, A. (1998). Entropy solutions for nonlinear elliptic equations in L^1. *Nonlinear Analysis: Theory, Methods & Applications*, *32*(3), 325–334.https://doi.org/10.1016/S0362-546X(96)00323-9.

[36] Lions, J.-L. (1969). *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod; Gauthier-Villars.

[37] Monsurrò, S. (2003). Homogenization of a two-component composite with interfacial thermal barrier. *Advances in Mathematical Sciences and Applications*, *13*(1), 43–63.

[38] Omari, B. E., Hajji, Y., Hjiaj, H., & Jamiai, I. (2026). Entropy solutions for some elliptic unilateral problems with degenerate coercivity. *Gulf Journal of Mathematics*, *22*(2). https://doi.org/10.56947/gjom.v22i2.4111

[39] Santiago, J. G. H., & Cabarrubias, B. C. (2021). On the well-posedness of a quasilinear elliptic problem with semilinear terms in a two-component domain. *Matimyaś Matematika*, *44*(2), 1–18.

[40] Santiago, J. G. H., & Cabarrubias, B. C. (2023). Homogenization of a quasilinear problem with semilinear terms in a two-component domain. *Applicable Analysis*, *102*(14), 3886–3908.https://doi.org/10.1080/00036811.2022.2101454

[41] Taylor, J. L., Ott, K. A., & Brown, R. M. (2012). The mixed problem in Lipschitz domains with general decompositions of the boundary. *Transactions of the American Mathematical Society*, *365*(6), 2895–2930.http://www.jstor.org/stable/23513528

[42] Vitillaro, E. (2002). Global existence for the wave equations with nonlinear boundary damping and source terms. *Journal of Differential Equations*, *186*(1), 259–298.https://doi.org/10.1016/S0022-0396(02)00023-2

[1] Mohamed Badr Benboubker
Higher School of Technology of Fez, Sidi Mohamed Ben Abdellah University, Fez, Morocco
Email address: simo.ben@hotmail.com

[2] Youssef HAJJI
Department of Mathematics, Faculty of Sciences Tétouan, University Abdelmalek Essaadi, BP 2121, Tétouan, Morocco
Email address: youssef.hajji@etu.uae.ac.ma

[3] Hassane HJIAJ
Department of Mathematics, Faculty of Sciences Tétouan, University Abdelmalek Essaadi, BP 2121, Tétouan, Morocco
Email address: hjiajhassane@yahoo.fr

[4] Oussama Messbahi
Department of Mathematics, Faculty of Sciences Tétouan, University Abdelmalek Essaadi, BP 2121, Tétouan, Morocco
Email address: messbahi.oussama@gmail.com