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KANNAN DISTORTION PROFILES AND INTRINSIC THRESHOLD GEOMETRY

LEDIA SUBASHI[1] and FLORION CELA[1]

Abstract. We introduce the finite-distortion Kannan profile, which measures the best Kannan coefficient achievable under a prescribed bi-Lipschitz change of metric. Two lower bounds distinguish geometric reweighting from the asymptotic rate of iteration. For threshold extensions, we obtain an exact decomposition into a switching contribution and the profile of the base dynamics. This yields an optimal piecewise affine remetrization and the sharp distortion threshold for genuine Kannan contractivity. We also compute the intrinsic chain geometry generated by a weighted directed excess, clarifying when a natural local-cost geometry is quantitatively close to the optimal one.

Keywords. Kannan contraction; distortion profile; threshold system; intrinsic chain geometry; bi-Lipschitz remetrization
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1. Introduction

Kannan's fixed point theorem [11] replaces the Banach inequality [1] by

$$d(Tx,Ty) \le \lambda\big(d(x,Tx) + d(y,Ty)\big), \qquad \lambda < \frac{1}{2}.$$

Because a Kannan contraction need not be continuous, this condition is well suited to dynamics with abrupt switching. It also has a strong relation with completeness [19]. Recent developments include paired-Kannan contractions, which extend the classical Kannan condition while retaining its fixed-point character [5]. Reich's three-parameter condition [17] provides a broader comparison class and is used below to identify a native obstruction created by the switch.

A different line of fixed point theory asks whether the metric can be changed so that a given iteration becomes contractive. Classical converse theorems of Bessaga and Meyers, and modern variants, are primarily existence results [2, 13, 7]. Here the native metric remains fixed as a reference, and the admissible changes are constrained by a finite bi-Lipschitz distortion. This leads to the profile $\kappa_F(D)$, the best Kannan coefficient attainable with distortion at most D.

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The native modulus first gives the budget bound

$$\kappa_F(D) \geq \frac{\kappa_F(1)}{D}.$$

If F has a fixed point p, its orbital root rate

$$\varrho_p(F) = \sup_y \limsup_{n\to\infty} d_0(F^n y, p)^{1/n}$$

is unchanged under every bi-Lipschitz change of metric and yields the additional bound

$$\kappa_F(D) \geq \frac{\varrho_p(F)}{1+\varrho_p(F)}.$$

The budget term may decrease when larger distortions are allowed. By contrast, the orbital term is invariant under every bi-Lipschitz remetrization. This variational picture is related to norm-minimization formulas in spectral-radius theory, including the Rota–Strang formulation of the joint spectral radius [18]; here the orbital rate identifies a part of the iteration that no finite-distortion change of metric can remove.

We compute the profile for a class of threshold extensions. Let (Y, d) be a complete metric space, let $\Phi : Y \to Y$ be a Kannan contraction, and set

$$T(r, y) = (S(r), \Phi y), \qquad X = [0, \infty) \times Y,$$

where S takes the value s below a threshold t and the value b at and above it, with $0 < t < s < b$. The relevant quantity is

$$\mathcal{R} = \frac{b-s}{s-t},$$

the ratio between the image jump and the shortest displacement of a point approaching the threshold from below. Such two-level rules occur as elementary components of relay and on–off control, regime-dependent intervention, congestion rules, and active-set algorithms. In these settings the update inside each regime may be stable, while the switching law introduces a discontinuity that prevents a direct application of a classical contraction theorem.

The purpose of the profile is not merely to attach a number to this jump. A contractive estimate is often the step that turns an iterative rule into a convergence and uniqueness result. When the native metric fails to provide such an estimate, the profile distinguishes a genuine dynamical obstruction from one caused by the relative weighting of the variables. It gives the least geometric distortion needed to reach a prescribed Kannan coefficient, while the phase transition identifies the point at which the fixed-point theorem becomes available. Thus, for threshold algorithms and hybrid update rules, the profile can be read as a quantitative guide to rescaling or preconditioning without changing the underlying topology or completeness.

The threshold coordinate reaches its fixed value after at most two steps, so its orbital root rate is zero. Its obstruction is therefore transient and can be reduced by geometric distortion. The base dynamics may retain a positive orbital rate.

These two mechanisms separate exactly:

$$\boxed{\kappa_T(D) = \max\left\{\frac{\mathcal{R}}{D}, \kappa_\Phi(D)\right\}}, \qquad D \geq 1.$$

If Φ is ν-Kannan in the native metric, then $\kappa_T(D) = \mathcal{R}/D$ for $1 \leq D \leq \mathcal{R}/\nu$; the minimum is realized by an explicit metric, and the transition to a genuine Kannan contraction occurs at $D = 2\mathcal{R}$. When $\mathcal{R} \geq 1$, the same jump-to-approach ratio rules out every native Reich inequality whose coefficient sum is below one.

The threshold mechanism also admits an intrinsic description. For $\gamma > 0$, consider the directed excess

$$E_\gamma((r,y),(q,z)) = e^{-\gamma r}\,(q-r)^+ + d(y,z).$$

Its chain envelope is

$$\rho_\gamma((r,y),(q,z)) = (H_\gamma(q) - H_\gamma(r))^+ + d(y,z), \qquad H_\gamma(r) = \frac{1-e^{-\gamma r}}{\gamma}.$$

Thus the coordinate change H_γ is generated by the local cost itself. This construction is useful when the primitive data of a model are local and possibly directional transition costs rather than a global metric. The chain envelope aggregates those costs along finite paths and produces the geometry that they induce intrinsically. Comparing this canonical geometry with the optimal finite-distortion metric shows whether a natural cost-based modeling choice is already efficient for contraction analysis or whether substantial reweighting is unavoidable. The comparison between the raw and intrinsic Kannan coefficients is controlled by $\mathcal{R}$, and on bounded strips the associated symmetric metric has an exact distortion.

The path infimum defining ρ_E is the directed counterpart of the shortest-path construction used in the metrization of weighted graphs. In the symmetric graph setting, Dovgoshey, Martio and Vuorinen characterize the shortest-path pseudometric through an extremal extension property [8]. Petrov obtains related maximal continuations for several classes of semimetrics [14]. Here the starting cost is defined on all ordered pairs and need not satisfy a triangle function; the envelope extracts the largest quasi-pseudometric below that cost. Fixed-point principles based directly on triangle functions, including Kannan and Reich forms, may be found in [16]. For further background on intrinsic quasimetrics, weighted generalized quasi-metrics, distance spaces, generalized metrics, semimetric contractions, and metric-reduction questions, see [20, 4, 12, 10, 3, 6, 9, 15]. The results below add a quantitative remetrization problem to this setting: they combine a distortion profile with an orbital lower bound and compare the chain-generated geometry with an optimal finite-distortion metric.

Section 2 develops the two general profile bounds and the threshold obstruction. Section 3 computes the intrinsic chain geometry. Section 4 proves the exact remetrization formula and compares the canonical and optimal metrics. A linear base map illustrates the orbital floor at the end of that section.

2. Distortion profiles and threshold extensions

Let (Z, d_0) be a metric space and $F : Z \to Z$. For $D \geq 1$, let $\mathcal{M}_D(d_0)$ be the class of metrics σ for which

$$c_1 d_0 \leq \sigma \leq c_2 d_0, \qquad 0 < c_1 \leq c_2, \qquad \frac{c_2}{c_1} \leq D.$$

Define

$$\kappa_F(D) = \inf \left\{ \lambda \geq 0 : \begin{array}{l} \text{some } \sigma \in \mathcal{M}_D(d_0) \text{ satisfies} \\ \sigma(Fx, Fy) \leq \lambda\big(\sigma(x, Fx) + \sigma(y, Fy)\big) \text{ for all } x, y \end{array} \right\}.$$

The infimum of the empty set is $+\infty$. The function κ_F is nonincreasing. When $D = 1$, the relations $0 < c_1 \leq c_2$ and $c_2/c_1 \leq 1$ force $c_1 = c_2$, so every metric in $\mathcal{M}_1(d_0)$ is a positive scalar multiple of d_0. Hence $\kappa_F(1)$ is the optimal coefficient in the native metric. If d_0 is complete, every member of $\mathcal{M}_D(d_0)$ is complete.

Proposition 2.1. *For every self-map F and every $D \geq 1$,*

$$\kappa_F(D) \geq \frac{\kappa_F(1)}{D}.$$

Proof. If $c_1 d_0 \leq \sigma \leq c_2 d_0$ and F is λ-Kannan in σ, then

$$d_0(Fx, Fy) \leq \lambda \frac{c_2}{c_1}\big(d_0(x, Fx) + d_0(y, Fy)\big).$$

Hence $\kappa_F(1) \leq \lambda D$; take the infimum. □

A second lower bound comes from the asymptotic rate of the iterates. Suppose that F has a fixed point p and define

$$\varrho_p(F) = \sup_{y \in Z} \limsup_{n \to \infty} d_0(F^n y, p)^{1/n}.$$

The value is unchanged if d_0 is replaced by a bi-Lipschitz equivalent metric, because the comparison constants disappear after taking n-th roots.

Theorem 2.2. *For every $D \geq 1$,*

$$\kappa_F(D) \geq \frac{\varrho_p(F)}{1 + \varrho_p(F)},$$

where the right-hand side is interpreted as 1 when $\varrho_p(F) = +\infty$.

Proof. Let $\sigma \in \mathcal{M}_D(d_0)$ and suppose that F is λ-Kannan in σ. The conclusion is immediate when $\lambda \geq 1$. If $\lambda < 1$, then, since $Fp = p$,

$$\sigma(Fy, p) \leq \lambda \sigma(y, Fy) \leq \lambda\big(\sigma(y, p) + \sigma(Fy, p)\big).$$

Hence

$$\sigma(Fy, p) \leq K \sigma(y, p), \qquad K = \frac{\lambda}{1 - \lambda}.$$

Iteration and the comparison $c_1 d_0 \leq \sigma \leq c_2 d_0$ give

$$d_0(F^n y, p) \leq D K^n d_0(y, p).$$

Taking n-th roots, then the upper limit and the supremum over y, yields $\varrho_p(F) \leq K$. Solving for λ and taking the infimum proves the claim. □

Together with Proposition 2.1, this gives

$$\kappa_F(D) \geq \max\left\{\frac{\kappa_F(1)}{D}, \frac{\varrho_p(F)}{1+\varrho_p(F)}\right\}. \tag{2.1}$$

The two terms have different origins: one is limited by the available distortion, and the other comes from the asymptotic rate of the orbit.

Now let (Y,d) be complete and suppose that $\Phi : Y \to Y$ satisfies

$$d(\Phi y, \Phi z) \leq \nu\big(d(y,\Phi y) + d(z,\Phi z)\big), \qquad 0 < \nu < \frac{1}{2}. \tag{2.2}$$

Let p be its unique fixed point and write $\kappa_\Phi(D)$ for its profile. Fix

$$0 < t < s < b, \qquad A = s - t, \qquad J = b - s, \qquad \mathcal{R} = \frac{J}{A},$$

and define

$$S(r) = \begin{cases} s, & 0 \leq r < t, \\ b, & r \geq t, \end{cases} \qquad T(r,y) = (S(r), \Phi y).$$

The native metric on $X = [0,\infty) \times Y$ is

$$\rho_0((r,y),(q,z)) = |q-r| + d(y,z),$$

and $u = (b,p)$ is the unique fixed point of T. Since $S^n r = b$ for every $n \geq 2$,

$$\varrho_u(T) = \varrho_p(\Phi). \tag{2.3}$$

Thus the threshold coordinate contributes no asymptotic floor; any positive floor of the product profile comes from the base dynamics.

Proposition 2.3. *The optimal native Kannan coefficient is*

$$\kappa_T(1) = \max\{\mathcal{R}, \kappa_\Phi(1)\}.$$

If $\mathcal{R} \geq 1$, no $\alpha, \beta, \chi \geq 0$ with $\alpha + \beta + \chi < 1$ satisfy the Reich inequality

$$\rho_0(Tx,Ty) \leq \alpha\rho_0(x,y) + \beta\rho_0(x,Tx) + \chi\rho_0(y,Ty) \qquad (x,y \in X).$$

Proof. For $r < t$, use $x = (r,p)$ and $y = (b,p)$. Letting $r \to t^-$ shows that every Kannan coefficient is at least $J/A = \mathcal{R}$. The invariant slice $\{b\} \times Y$ gives the lower bound $\kappa_\Phi(1)$. Conversely, a crossing of the threshold produces an image jump J, while the displacement of the point below the threshold exceeds A; the first coordinate is therefore $\mathcal{R}$-Kannan. For every $\eta > \kappa_\Phi(1)$, the base map is η-Kannan in d; adding the two coordinate estimates and letting $\eta \downarrow \kappa_\Phi(1)$ gives the formula.

For the Reich statement, first pair (r,p) with (t,p) and then with (b,p), and let $r \to t^-$. The two inequalities obtained are

$$J \leq \beta A + \chi(J+A), \qquad J \leq \alpha(J+A) + \beta A.$$

Since $J \geq A$, their sum gives

$$2J \leq (\alpha+\beta+\chi)(J+A) < J + A \leq 2J,$$

a contradiction. $\square$

The map T is discontinuous on $\{t\} \times Y$. Consequently no metric inducing the product topology, and in particular no metric bi-Lipschitz equivalent to ρ_0, can make it a Banach contraction. Kannan remetrization preserves the topology while allowing this discontinuity.

3. Directed excess and intrinsic chain geometry

A *directed premetric* is a function $E : Z \times Z \to [0, \infty)$ with $E(z, z) = 0$. Its *intrinsic chain envelope* is

$$\rho_E(x, y) = \inf\Big\{ \sum_{i=0}^{k-1} E(z_i, z_{i+1}) : \ k \geq 1, \ z_0 = x, \\ z_k = y \Big\}.$$

It is a quasi-pseudometric satisfying $\rho_E \leq E$. Moreover, it is the largest quasi-pseudometric dominated by E: concatenation gives the triangle inequality, and every quasi-pseudometric $q \leq E$ satisfies

$$q(x, y) \leq \sum_i E(z_i, z_{i+1})$$

along each chain.

The passage to ρ_E need not preserve a Kannan estimate. A three-point example shows the mechanism directly.

Proposition 3.1. *Let $Z_0 = \{\xi_0, \xi_1, \xi_2\}$, define*

E_0	ξ_0	ξ_1	ξ_2
ξ_0	0	1	$\frac{1}{10}$
ξ_1	$\frac{1}{10}$	0	1
ξ_2	$\frac{7}{20}$	1	0

and set $F\xi_1 = \xi_2$, $F\xi_2 = \xi_0$, $F\xi_0 = \xi_0$. Its chain envelope is

ρ_{E_0}	ξ_0	ξ_1	ξ_2
ξ_0	0	1	$\frac{1}{10}$
ξ_1	$\frac{1}{10}$	0	$\frac{1}{5}$
ξ_2	$\frac{7}{20}$	1	0,

so $\rho_{E_0} \leq E_0 \leq 5\rho_{E_0}$. The map F is $7/20$-Kannan in E_0, with optimal coefficient, but no Reich inequality for ρ_{E_0} has nonnegative coefficients whose sum is at most one.

Proof. Only the edge $\xi_1 \to \xi_2$ is shortened by a detour: $E_0(\xi_1, \xi_0) + E_0(\xi_0, \xi_2) = 1/5 < 1$. This gives the matrix and the optimal comparison constant 5. The four pairs with nonzero image distance verify the $7/20$-Kannan inequality; the pair (ξ_1, ξ_0) gives equality. For the same pair, a Reich inequality in ρ_{E_0} would imply

$$\frac{7}{20} \leq \frac{1}{10}a + \frac{1}{5}b \leq \frac{1}{5},$$

which is impossible. $\square$

For the threshold system, consider

$$E_\gamma((r,y),(q,z)) = e^{-\gamma r}\,(q-r)^+ + d(y,z), \qquad \gamma > 0. \tag{3.1}$$

Although it may vanish in one direction for distinct points, it satisfies $E_\gamma(x,y) = E_\gamma(y,x) = 0$ only when $x = y$; however, it does not satisfy the triangle inequality. For example,

$$E_1((0,y),(2,y)) = 2 > 1 + e^{-1} = E_1((0,y),(1,y)) + E_1((1,y),(2,y)).$$

We use E_γ as a local edge cost. The direct Kannan coefficient is the estimate obtained before paths are introduced; the chain envelope then gives the associated geometry and allows the two coefficients to be compared.

Theorem 3.2. *Let $H_\gamma(r) = (1-e^{-\gamma r})/\gamma$. The intrinsic chain envelope of E_γ is*

$$\rho_\gamma((r,y),(q,z)) = (H_\gamma(q) - H_\gamma(r))^+ + d(y,z).$$

Proof. For $a, c \geq 0$,

$$H_\gamma(c) - H_\gamma(a) \leq e^{-\gamma a}\,(c-a)^+\,.$$

Summing this inequality along a chain, together with the triangle inequality in Y, gives the lower bound for the envelope. If $q > r$, monotone refinements of $[r,q]$ produce left Riemann sums converging to $\int_r^q e^{-\gamma u}\,du$; if $q \leq r$, the first coordinate moves downward at zero cost. This proves the formula. □

The quasi-pseudometric ρ_γ likewise satisfies $\rho_\gamma(x,y) = \rho_\gamma(y,x) = 0$ only when $x = y$, although downward motion in the first coordinate has zero cost in one orientation. Its symmetrization on X is the metric

$$d_\gamma((r,y),(q,z)) = |H_\gamma(q) - H_\gamma(r)| + d(y,z).$$

For $M \geq b$, write

$$X_M = [0,M] \times Y.$$

Since $S([0,M]) \subset \{s,b\}$, the strip X_M is invariant under T, and the restriction of d_γ to X_M is complete.

Proposition 3.3. *Set*

$$\lambda_\gamma = \max\{\mathcal{R}e^{-\gamma A}, \nu\}, \qquad \theta_\gamma = \frac{H_\gamma(b) - H_\gamma(s)}{H_\gamma(s) - H_\gamma(t)}, \qquad \widehat{\lambda}_\gamma = \max\{\theta_\gamma, \nu\}.$$

Then T is λ_γ-Kannan in E_γ and $\widehat{\lambda}_\gamma$-Kannan in both ρ_γ and d_γ on X. The threshold coefficient θ_γ is sharp in each of the two canonical geometries. Moreover,

$$\frac{\theta_\gamma}{\mathcal{R}e^{-\gamma A}} = \frac{(1-e^{-\gamma J})/J}{(1-e^{-\gamma A})/A}.$$

Consequently $\widehat{\lambda}_\gamma \leq \lambda_\gamma$ when $\mathcal{R} \geq 1$, while the reverse inequality holds when $0 < \mathcal{R} \leq 1$.

Proof. Suppose first that $r < t \leq q$. The first-coordinate image term in

$$E_\gamma(T(r,y), T(q,z))$$

is $e^{-\gamma s}J$, whereas the displacement of the point below the threshold is at least $e^{-\gamma t}A$. This gives the threshold coefficient $\mathcal{R}e^{-\gamma A}$. If $q < t \leq r$, the directed image term is $(s-b)^+=0$, and if the two points lie on the same side of the threshold, it is again zero. The second coordinate is controlled by ν.

For ρ_γ, only the first crossing orientation has a nonzero directed image term. For the symmetric metric d_γ, the reverse orientation is controlled by the displacement of the point below the threshold, since

$$H_\gamma(s)-H_\gamma(q)\geq H_\gamma(s)-H_\gamma(t).$$

Thus the threshold coefficient in either canonical geometry is θ_γ. It is also necessary: with $x=(r,p)$, $y=(b,p)$, and $r\to t^-$, the image distance is $H_\gamma(b)-H_\gamma(s)$, the displacement of y is zero, and that of x tends to $H_\gamma(s)-H_\gamma(t)$. The displayed ratio follows from $J=\mathcal{R}A$. Since $x\mapsto(1-e^{-\gamma x})/x$ is decreasing, the comparison is determined by whether $J\geq A$ or $J\leq A$. □

4. Exact remetrization and comparison of geometries

For $\mu>0$, set

$$\alpha_\mu=\min\left\{1,\frac{\mu}{\mathcal{R}}\right\}.$$

Let $G_\mu:[0,\infty)\to[0,\infty)$ be continuous and piecewise affine, with $G_\mu(0)=0$, slope 1 on $[0,s]\cup[b,\infty)$, and slope α_μ on $[s,b]$. Put

$$h_\mu(r,q)=|G_\mu(q)-G_\mu(r)|.$$

Then

$$\alpha_\mu|q-r|\leq h_\mu(r,q)\leq|q-r|, \tag{4.1}$$

and the transformed jump satisfies

$$h_\mu(Sr,Sq)\leq\mu\big(h_\mu(r,Sr)+h_\mu(q,Sq)\big). \tag{4.2}$$

Indeed, a crossing has image jump

$$\alpha_\mu J=\min\left\{1,\frac{\mu}{\mathcal{R}}\right\}\mathcal{R}A=\min\{\mathcal{R},\mu\}A\leq\mu A,$$

while the displacement from below is at least A.

Theorem 4.1. *For every $D\geq 1$,*

$$\boxed{\kappa_T(D)=\max\left\{\frac{\mathcal{R}}{D},\kappa_\Phi(D)\right\}}.$$

Proof. The universal bound and Proposition 2.3 give $\kappa_T(D)\geq\mathcal{R}/D$. Restricting any admissible metric to the invariant slice $\{b\}\times Y$ gives a metric of distortion at most D for Φ; hence $\kappa_T(D)\geq\kappa_\Phi(D)$.

For the reverse inequality, choose

$$\mu>\max\left\{\frac{\mathcal{R}}{D},\kappa_\Phi(D)\right\}.$$

There is a metric d' on Y such that $md\leq d'\leq Md$, $M/m\leq D$, and Φ is η-Kannan in d' for some $\eta<\mu$. With $\alpha=\alpha_\mu$, define $\widetilde{d}=(\alpha/m)d'$ and

$$\sigma_\mu((r,y),(q,z))=h_\mu(r,q)+\widetilde{d}(y,z).$$

Since $\mu > \mathcal{R}/D$, one has $\alpha D \geq 1$. Thus

$$\alpha\rho_0 \leq \sigma_\mu \leq \alpha D\rho_0.$$

Equations (4.2) and the base estimate show that T is μ-Kannan in σ_μ. Let μ decrease to the stated maximum. □

The same formula holds for the restriction of T to every invariant strip X_M, with the product metric ρ_0 restricted to X_M.

Combining Theorem 4.1, Theorem 2.2, and (2.3) gives the explicit estimate

$$\kappa_T(D) \geq \max\left\{\frac{\mathcal{R}}{D}, \frac{\varrho_p(\Phi)}{1+\varrho_p(\Phi)}\right\}. \tag{4.3}$$

The switching term can be reduced at the rate D^{-1}, whereas a positive orbital rate of the base map produces a distortion-independent floor.

Corollary 4.2. *Assume $\mathcal{R} \geq \nu$ (so that the interval below is nonempty). If $1 \leq D \leq \mathcal{R}/\nu$, then*

$$\kappa_T(D) = \frac{\mathcal{R}}{D},$$

and the minimum is attained by

$$\sigma_D((r,y),(q,z)) = |G_{\mathcal{R}/D}(q) - G_{\mathcal{R}/D}(r)| + d(y,z), \qquad \frac{1}{D}\rho_0 \leq \sigma_D \leq \rho_0.$$

Proof. For $\mu = \mathcal{R}/D$, $\alpha_\mu = 1/D$. The threshold coefficient is $\mathcal{R}/D$ and $\nu \leq \mathcal{R}/D$, so σ_D realizes the minimum. □

Corollary 4.3. *For every $D \geq 1$, a metric of distortion at most D makes T a genuine Kannan contraction if and only if*

$$D > 2\mathcal{R}.$$

Proof. If $D \leq 2\mathcal{R}$, then Theorem 4.1 gives $\kappa_T(D) \geq \mathcal{R}/D \geq 1/2$. If $D > 2\mathcal{R}$, then $\mathcal{R}/D < 1/2$ and $\kappa_\Phi(D) \leq \nu < 1/2$, so Theorem 4.1 gives $\kappa_T(D) < 1/2$. □

The chain metric and the optimizing metric allocate distortion differently. This can be measured exactly on a bounded invariant strip.

Proposition 4.4. *Let $M \geq b$ and $D_\gamma = e^{\gamma M}$. On X_M, the metric d_γ has exact distortion D_γ relative to ρ_0. For the threshold map S,*

$$\kappa_S(D_\gamma) = \mathcal{R}e^{-\gamma M},$$

whereas its optimal coefficient in the canonical metric is θ_γ. Hence

$$\frac{\theta_\gamma}{\kappa_S(D_\gamma)} = \frac{e^{\gamma(M-A)}}{\mathcal{R}}\frac{1-e^{-\gamma J}}{1-e^{-\gamma A}} \sim \frac{1}{\mathcal{R}}e^{\gamma(M-A)} \qquad (\gamma \to \infty).$$

Proof. Since $H'_\gamma(r) = e^{-\gamma r}$ on $[0, M]$,

$$e^{-\gamma M}\rho_0 \leq d_\gamma \leq \rho_0,$$

and the extreme first-coordinate ratios are approached near M and 0. Thus the distortion is exactly $e^{\gamma M}$. Applying Theorem 4.1 with a one-point base gives $\kappa_S(D) = \mathcal{R}/D$; for a one-point space the base inequality is trivial and $\kappa_\Phi \equiv 0$.

Proposition 3.3 shows that θ_γ is the optimal coefficient in the canonical metric. The remaining formula follows from its definition. □

The canonical metric integrates the local weight over the whole strip. The optimizer spends the same distortion budget only on the jump interval. For the full product map,

$$\widehat{\lambda}_\gamma = \max\{\theta_\gamma, \nu\}, \qquad \kappa_T(D_\gamma) = \max\{\mathcal{R}e^{-\gamma M}, \kappa_\Phi(D_\gamma)\},$$

so the threshold comparison remains explicit and the remaining contribution is exactly the base profile.

Example 4.5. Let $Y = [0, \infty)$ with the usual metric and let

$$\Phi(y) = cy, \qquad 0 < c < \frac{1}{3}.$$

The fixed point is $p = 0$, the orbital root rate is $\varrho_0(\Phi) = c$, and the optimal native Kannan coefficient is

$$\kappa_\Phi(1) = \frac{c}{1-c}.$$

Indeed, $|y - \Phi y| + |z - \Phi z| = (1-c)(y+z)$ and $|y - z| \le y + z$, while the pair $(y, 0)$ gives equality. Therefore, for every $D \ge 1$,

$$\frac{c}{1+c} \le \kappa_\Phi(D) \le \frac{c}{1-c}.$$

For the threshold extension,

$$\max\left\{\frac{\mathcal{R}}{D}, \frac{c}{1+c}\right\} \le \kappa_T(D) \le \max\left\{\frac{\mathcal{R}}{D}, \frac{c}{1-c}\right\}.$$

In particular,

$$\kappa_T(D) = \frac{\mathcal{R}}{D} \qquad \left(1 \le D \le \frac{\mathcal{R}(1-c)}{c}\right).$$

Hence the explicit threshold term determines the profile on the stated initial range. The geometric rate of the base map, however, prevents the profile from tending to zero as $D \to \infty$.

5. Conclusion

The finite-distortion profile distinguishes improvement obtained by changing the metric from limitations imposed by the long-term behavior of the orbit. The general estimates make both effects visible, and the threshold product admits an exact formula for every distortion budget. The resulting phase transition identifies precisely when the discontinuous system can be placed within the classical Kannan range.

The chain construction gives a second description of the switching geometry. Starting from the directed local cost, it produces the potential H_γ without an externally chosen coordinate change. On bounded invariant strips, the comparison with the piecewise affine optimizer shows quantitatively how a path-generated metric differs from a metric designed for the distortion problem.

Natural next questions are whether the orbital floor is attained for basic nonlinear and linear maps, and how distortion profiles behave for several thresholds or coupled switching coordinates, where the product separation is no longer available.

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[1] Department of Mathematics, Faculty of Natural Sciences, University of Tirana, Tirana, Albania.