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Abstract. A cyclotomic polynomial $\Phi_n(x)$ is called ternary if $n$ is the product of three distinct primes. In this paper, we derive explicit expressions for a ternary cyclotomic polynomial under certain condition. Moreover, we use these expressions to study the density of these polynomials and provide an upper bound. We also investigate the number of nonzero terms (the Hamming weight) and derive formulas for the Hamming weight under certain conditions.

*Keywords.* Cyclotomic polynomials, Hamming weight.
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## 1. Introduction and preliminaries

The $n$-th cyclotomic polynomial $\Phi_n(x)$ is defined as the monic polynomial in $\mathbb{Z}[x]$ whose complex roots are the primitive $n$-th roots of unity, the polynomial $\Phi_n(x)$ is of degree $\varphi(n)$, where $\varphi(n)$ is the Euler's totient function. Due to its importance in many branches of mathematics, its properties have been extensively studied. In particular, several properties have been investigated by many authors; for example, the height (the maximum absolute value of its coefficients) was motivated by the observation that it can exceed 1. This line of research has produced numerous results; see, for instance, [1, 2, 6, 7, 9, 8].

The investigation of the number of nonzero terms in $\Phi_n(x)$ (also called the Hamming weight $\mathrm{hw}(\Phi_n)$) was initiated in 1965 by Carlitz [11], who gave an explicit formula for $\mathrm{hw}(\Phi_{pq})$, where $p < q$ are distinct odd prime numbers. Hence, a natural question is whether there exists a formula for the number of nonzero terms of ternary cyclotomic polynomials $\Phi_{pqr}$, where $p < q < r$ are three odd primes, and more generally, for arbitrary cyclotomic polynomials.

In 2014, Bezdga [10] proved that the Hamming weight of the cyclotomic polynomial $\Phi_n(x)$ is greater than or equal to $n^{1/3}$. In 2016, A. Al-Kateeb [1] investigated the number of terms of ternary cyclotomic polynomials, and the following theorem was given (Theorem 7.1).

**Theorem 1.1** (Theorem 7.1 [1]). *Let $p_1 < p_2 < p_3$ be odd prime numbers. Then*

$$\mathrm{hw}(\Phi_{p_1p_2p_3}) = \begin{cases} N(p_3-1)+1, & r_3 = 1 \\ N(p_3+1)-1, & r_3 = p_2 - 1 \end{cases}$$

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*where* $N = \frac{2}{3}\left(\frac{(p_1-1)((p_1+4)(p_2-1)-(r_2-1))}{p_1p_2}\right)$ *and* $r_3 = \text{rem}(p_3, 3p_2)$.

Another result from [3] for the case $p_1 = 3$ and $p_3 \equiv \pm 2 \mod 3p_2$.

**Theorem 1.2** ([3])**.** *Let* $3 < p_2 < p_3$ *be odd prime numbers such that* $p_2 \equiv 1 \mod 3$ *and* $p_3 > 3p_2$*. Then*

$$\text{hw}(\Phi_{3p_2p_3}) = \begin{cases} N(p_3-2) + \left(\frac{4p_2-1}{3}\right), & \text{if } r_3 = 2 \\ N(p_3+2) - \left(\frac{4p_2-1}{3}\right), & \text{if } r_3 = 3p_2 - 2 \end{cases},$$

*where* $N = \frac{7(p_2^2-1)}{9p_2}$.

Let $p_1 < p_2$ and $p_3$ be three prime numbers such that $p_3 > p_1 \cdot p_2$. Throughout this paper, we denote

$$r_i := \text{rem}\left(p_i, p_1 \cdots p_{i-1}\right), \quad q_i := \text{quo}(p_i, p_1 \cdots p_{i-1})$$

where quo and rem denote the quotient and remainder, respectively. In this paper, we study the family of cyclotomic polynomials $\Phi_{p_1p_2p_3}(x)$, where $p_3 \equiv 1 \pmod{p_1p_2}$. It is well known that this family is flat. Here, we derive explicit expressions for these polynomials. The expressions given in this paper follow similar ideas to those in [4]; however, they are simpler.

The obtained expressions are then used to study the density of these polynomials, and we provide an upper bound for the density in the following theorem.

**Theorem 1.3.** *Let* $m = p_1 \cdot p_2$ *be a square free odd composite integer with* $p_1 < p_2$ *and* $p > m$ *be an odd prime. Then we have*

$$d(\Phi_{mp}) \leq \frac{\psi(m)}{m} \quad \text{if } p \equiv 1 \mod m,$$

*where* $\psi(m) = m - \varphi(m)$.

Then we use the found expressions to prove the next two theorems regarding the number of terms of the cyclotomic polynomials $\Phi_{5p_2p_3}$ such that $p_2 \equiv \pm 2 \mod 5$ and $p_3 \equiv \pm 1 \mod 5p_2$.

**Theorem 1.4.** *Let* $5 < p_2 < p_3$ *be odd prime numbers such that* $p_2 \equiv 2 \mod 5$ *and* $p_3 \equiv \pm 1 \mod 5p_2$*. Then*

$$\text{hw}(\Phi_{5p_2p_3}) = \begin{cases} N(p_3-1) + 1, & \text{if } r_3 = 1 \\ N(p_3+1) - 1, & \text{if } r_3 = 5p_2 - 1 \end{cases},$$

*where* $N = \frac{2}{5p_2}(12p_2 - 14)$.

**Theorem 1.5.** *Let* $5 < p_2 < p_3$ *be odd prime numbers such that* $p_2 \equiv 3 \mod 5$*. Then*

$$\text{hw}(\Phi_{5p_2p_3}) = \begin{cases} N(p_3-1) + 1, & \text{if } r_3 = 1 \\ N(p_3+1) - 1, & r_3 = 5p_2 - 1 \end{cases},$$

*where* $N = \frac{4}{25p_2}(29p_2 - 22)$.

## 2. Repeating patterns in cyclotomic polynomials $\Phi_{mp}$ if $p \equiv 1 \mod m$

Let $m$ be a square-free integer and $p > m$ be a prime such that $p \equiv 1 \mod m$. We will partition $\Phi_{mp}$ into several parts. We found that the following repeated partitioning of $\Phi_{mp}$ is elegant, enlightening and useful.

(1) We partition $\Phi_{mp}$ under the radix $p-1$, obtaining

$$\Phi_{mp} = \sum_{i=0}^{\varphi(m)-1} f_i \; x^{i(p-1)} + x^{\varphi(mp)}, \qquad \deg f_i < p-1$$

since

$$\frac{\deg \Phi_{mp}}{p-1} = \frac{\varphi(m)\,(p-1)}{p-1} = \varphi(m) \quad \text{and} \quad f_{\varphi(m)} = 1$$

(2) We partition $f_i$ under the radix $m$, obtaining

$$f_i = \sum_{j=0}^{q-1} f_{i,j} \; x^{jm}, \qquad \deg f_{i,j} < m$$

where $q = \frac{p-1}{m}$. Since

$$\frac{\deg f_i}{m} < \frac{p-1}{m} = q$$

put together we have

$$\Phi_{mp} = \sum_{\substack{0 \le i \le \varphi(m)-1 \\ 0 \le j \le q-1}} f_{i,j} x^{i(p-1)+jm} + x^{\varphi(mp)}, \qquad \deg f_{i,j} < m$$

**Notation 2.1.** *We have*

(1) $\Phi_{mp} = \sum_{i=0}^{\varphi(m)-1} f_i \; x^{i(p-1)} \quad + \quad x^{\varphi(mp)}, \qquad \deg f_i < p-1$
(2) $f_i = \sum_{j=0}^{q-1} f_{i,j} \; x^{jm}, \qquad \deg f_{i,j} < m$

Next we write $f_{i,j}$ in terms of $\Phi_m$.

**Lemma 2.2.** *Let* $0 \le i \le \varphi(m)$ *and* $0 \le j \le q-1$. *Then we have*

$$f_{i,j} = -\Psi_m \cdot \mathcal{T}_{i+1} \Phi_m,$$

*where* $\Psi_m(x) = \frac{x^m-1}{\Phi_m(x)}$ *and* $\mathcal{T}_s(\cdot) = \operatorname{rem}(\cdot, x^s)$

*Proof.* Note

$$\begin{aligned}
\Phi_{mp} &= \frac{\Phi_m(x^p)}{\Phi_m} \\
&= -\,\Phi_m(x^p)\;\Psi_m(x)\;\frac{1}{1-x^m} \\
&= -\,\Phi_m(x^p)\;\Psi_m(x)\;\sum_{t \ge 0} x^{tm} \qquad \text{by carrying out} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{a formal expansion of } \frac{1}{1-x^m}
\end{aligned}$$

$$= -\sum_{s\geq 0} a_s x^{sp}\ \Psi_m(x)\ \sum_{t\geq 0} x^{tm} \qquad \text{where } \Phi_m(x) = \sum_{s\geq 0} a_s x^s,\ \ a_s = 0 \text{ for } s > \varphi(m)$$

$$= -\sum_{s\geq 0} a_s x^{sqm} x^s\ \Psi_m(x)\ \sum_{t\geq 0} x^{tm} \qquad \text{since } p = qm+1$$

$$= -\sum_{s\geq 0} a_s x^{sqm} x^s\ \Psi_m(x) \sum_{i\geq 0}\sum_{j=0}^{q-1} x^{(iq+j)m} \qquad \text{where } i = \mathrm{quo}(t,q),\ \ j = \mathrm{rem}(t,q)$$

$$= -\sum_{j=0}^{q-1} \Psi_m(x) \sum_{i\geq 0,\ s\geq 0} a_s x^s\ x^{(i+s)qm+jm} \qquad \text{by reordering and combining}$$

$$= -\sum_{j=0}^{q-1} \Psi_m(x) \sum_{i\geq s,\ s\geq 0} a_s x^s\ x^{iqm+jm} \qquad \text{writing } i+s \text{ as } i$$

$$= -\sum_{j=0}^{q-1} \Psi_m(x) \sum_{i\geq 0}\sum_{s=0}^{i} a_s x^s\ x^{iqm+jm} \qquad \text{by rewriting } \sum_{i\geq s,\ s\geq 0} \text{ into an iterated sum}$$

$$= \sum_{i\geq 0}\sum_{j=0}^{q-1} \left( -\ \Psi_m(x) \sum_{s=0}^{i} a_s x^s \right)\ x^{iqm+jm} \qquad \text{by reordering}$$

Let

$$h_i = -\Psi_m(x) \sum_{s=0}^{i} a_s x^s$$

For $i \geq \varphi(m)$, we have

$$h_i = -\Psi_m(x)\ \sum_{s=0}^{\varphi(m)} a_s x^s = -\Psi_m \Phi_m = 1 - x^m$$

Therefore we have

$$\Phi_{mp} = -\sum_{u=i}^{\varphi(m)-1}\sum_{j=0}^{q-1} h_i x^{iqm+jm} + \sum_{i\geq\varphi(m)}\sum_{j=0}^{q-1} (1-x^m) x^{iqm+jm}$$

$$= -\sum_{i=0}^{\varphi(m)-1}\sum_{j=0}^{q-1} h_i x^{iqm+jm} + \sum_{i\geq\varphi(m)} x^{iqm} \sum_{j=0}^{q-1} \left(x^{jm} - x^{(j+1)m}\right) \qquad \text{by rearranging}$$

$$= -\sum_{i=0}^{\varphi(m)-1}\sum_{j=0}^{q-1} h_i x^{iqm+jm} + \sum_{i\geq\varphi(m)} x^{iqm}(1-x^{qm}) \qquad \text{by telescoping sum}$$

$$= -\sum_{i=0}^{\varphi(m)-1}\sum_{j=0}^{q-1} h_i x^{iqm+jm} + \sum_{i\geq\varphi(m)} \left(x^{iqm} - x^{(i+1)qm}\right) \qquad \text{by rearranging}$$

$$= -\sum_{i=0}^{\varphi(m)-1}\sum_{j=0}^{q-1} h_i x^{iqm+jm} + x^{\varphi(m)mq} \qquad \text{by telescoping sum}$$

$$= - \sum_{i=0}^{\varphi(m)-1} \sum_{j=0}^{q-1} h_i x^{i(p-1)+jm} + x^{\varphi(mp)} \qquad \text{since } qm = p-1$$

$$= - \sum_{i=0}^{\varphi(m)-1} f_i x^{i(p-1)} + x^{\varphi(mp)} \qquad f_{i,j} = h_i$$

Thus

$$f_{i,j} = -\Psi_m \cdot \mathcal{T}_{i+1} \Phi_m,$$

□

**Theorem 2.3.** *Let $m$ be a square-free integer and let $p > m$ be such that $p \equiv 1 \mod m$. Then*

$$\Phi_{mp} = \sum_{i=0}^{\varphi(m)-1} f_i x^{i(p-1)} + x^{\varphi(mp)}$$

*where*

$$f_i = \sum_{j=0}^{q-1} f_{i,j}\ x^{jm}$$

*and*

$$f_{i,0} = f_{i,1} = \cdots = f_{i,j} = -\Psi_m \cdot \mathcal{T}_{i+1} \Phi_m$$

*Proof.* Immediate from Notation 2.1 and Lemma 2.2. □

**Lemma 2.4.** $f_{i+1,0} = -a_{i+1} x^{i+1} \Psi_m + f_{i,j}$

*Proof.* Immediate from the following.

$$\begin{aligned} f_{i+1,j} &= -\Psi_m \cdot \mathcal{T}_{i+2} \Phi_m && \text{from Lemma 2.2} \\ &= -a_{i+1} x^{i+1} \Psi_m + f_{i,j} \end{aligned}$$

□

The next theorem shows a symmetry relationship between the polynomials $f_{i,j}$.

**Theorem 2.5.** *Let $0 \le i \le \varphi(p_1 p_2) - 1$. Then*

$$f_{\varphi(p_1p_2)-i-1,j} = \operatorname{rem}(x^{p_1p_2} f_{i,j}(x^{-1}), x^{p_1p_2} - 1)$$

*Proof.* Note

$$\begin{aligned} f_{\varphi(p_1p_2)-i-1,j} &= -\Psi_{p_1p_2} \cdot \mathcal{T}_{\varphi(p_1p_2)-i} \Phi_{p_1p_2} \\ &= -\Psi_{p_1p_2} \cdot \sum_{s=0}^{\varphi(p_1p_2)-i-1} a_s x^s \\ &= -\Psi_{p_1p_2} \cdot \left( \Phi_{p_1p_2} - \sum_{s=\varphi(p_1p_2)-i}^{\varphi(p_1p_2)} a_s x^s \right) \end{aligned}$$

$$
\begin{aligned}
&= -\Psi_{p_1p_2} \cdot \left( \Phi_{p_1p_2} - \sum_{s=\varphi(p_1p_2)-i}^{\varphi(p_1p_2)} a_{\varphi(p_1p_2)-s} \ x^s \right) && a_s = a_{\varphi(p_1p_2)-s} \\
&= -\Psi_{p_1p_2} \cdot \left( \Phi_{p_1p_2} - \sum_{s=i}^{0} a_s x^{\varphi(p_1p_2)-s} \right) && \text{By reindexing} \\
& && s \text{ with } \varphi(p_1p_2) - s \\
&= -\Psi_{p_1p_2} \cdot \left( \Phi_{p_1p_2} + \sum_{s=0}^{i} a_s x^{\varphi(p_1p_2)-s} \right) \\
&= -\mathrm{rem}\left( (x^{p_1p_2} - 1) + \Psi_{p_1p_2} \ \sum_{s=0}^{i} a_s x^{\varphi(p_1p_2)-s}, x^{p_1p_2} - 1 \right) && \deg(f_{i,j}) < p_1p_2 \\
&= \mathrm{rem}\left( x^{\psi(p_1p_2)} \Psi_{p_1p_2}(x^{-1}) \cdot \ \sum_{s=0}^{i} a_s x^{\varphi(p_1p_2)-s}, x^{p_1p_2} - 1 \right) && \Psi_{p_1p_2} = -x^{\psi(p_1p_2)} \Psi_{p_1p_2}(x^{-1}) \\
&= \mathrm{rem}\left( x^{p_1p_2} \Psi_{p_1p_2}(x^{-1}) \cdot \ \sum_{s=0}^{i} a_s x^{-s}, x^{p_1p_2} - 1 \right) && p_1p_2 = \varphi(p_1p_2) + \psi(p_1p_2) \\
&= \mathrm{rem}\left( x^{p_1p_2} f_{i,j}(x^{-1}), x^{p_1p_2} - 1 \right)
\end{aligned}
$$

□

## 3. Density of $\Phi_{p_1p_2p_3}$ with $p_3 \equiv 1 \mod p_1p_2$

In this section we work on the density of ternary cyclotomic polynomials $\Phi_{p_1p_2p_3}$ with $r_3 = 1$. Let $d(\Phi_{mp}) = \frac{\mathrm{hw}(\Phi_{mp})}{\deg(\Phi_{mp})}$, we find an upper bound for the density when $p_3 \equiv 1 \mod p_1p_2$. Let $\mathrm{tdeg}(f)$ denotes the tail degree of a polynomial $f$, that is the lowest degree of a non-zero term in $f$.

**Lemma 3.1.** *For $i > 0$. We have $\mathrm{tdeg}(f_{i,j}) = 0$.*

*Proof.*

$$
\begin{aligned}
f_{i,j} &= -\Psi_m \cdot \mathcal{T}_{i+1} \Phi_m \\
&= ((1 - x^{p_2}) \Phi_{p_1}) \cdot (1 + a_1 x^1 + \cdots + c_i x^i) \\
&= 1 + xH(x)
\end{aligned}
$$

so $\mathrm{tdeg}(f_{i,j}) = 0$. □

Let $i' = \max\{k \mid a_k \neq 0, 0 \leq k \leq i\}$

**Lemma 3.2.** *Let $p \equiv +1 \bmod m$ and $0 \leq i \leq \varphi(m) - 1$. Then $\deg f_{i,0} = \psi(m) + i'$.*

*Proof.* for

$$
\begin{aligned}
f_{i,j} &= -\Psi_m \cdot \mathcal{T}_{i+1} \Phi_m \\
&= ((1 - x^{p_2}) \Phi_{p_1}) \cdot (1 + a_1 x^1 + \cdots + c_{i'} x^{i'}) \\
&= H(x) - x^{p_2 + p_1 - 1 + i'},
\end{aligned}
$$

so $\deg(f_{i,j}) = \psi(m) + i^{'}$. □

Let $g(f(x))$ be the maximum gap inside the polynomial $f(x)$is the maximum of the differences between any two consecutive exponents, for example $f(x) = 1 + x^2 + x^6 + x^9$ here $g(f) = 2 - 0 - 1 = 1$.

**Lemma 3.3.** *For $1 \le i \le \varphi(m) - 1$ we have $g(f_{i,j}) \ge i^{'} + 1$.*

*Proof.* We claim that for $i > 0$, $f_{i,j} = 1 \pm x^{p_1+w}H(x)$ where $w \ge i^{'} - 1$. We prove the claim by induction. At first $f_{1,j} = -a_1 x\Psi_m - \Psi_m = x\Psi_m - \Psi_m = 1 - x^{p_1} - x^{p_2} + x^{p_2+p_1}$, here $w = 0$ . Assume that $f_{i,j} = 1 \pm x^{p_1+w}$ where $w \ge i' - 1$. Consider

$$f_{i+1,j} = -a_{i+1}x^{i+1}\Psi_m + f_{i,j} = \begin{cases} f_{i,j}, & \text{if } a_{i+1} = 0 \\ 1 - a_{i+1}x^{i+1}\Psi_m \pm x^{p_1+w}H(x), & \text{otherwise} \end{cases}$$

notice that if $1 \le i \le p_1 - 2$, then $a_{i+1} = 0$, since $a_{i+1} \ne 0$ we have $i \ge p_1 - 1$, which means that $f_{i+1,j} = 1 \pm x^{p_1+w}H(x)$. By induction we have $f_{i,j} = 1 \pm x^{p_1+w}H(x)$. Thus, $g(f_{i,j}) \ge p_1 + w - 1 \ge p_1 + i' - 1 \ge i' + 1$ as desired. □

**Lemma 3.4.** *If $p \equiv +1 \bmod m$, then we have $\mathrm{hw}(f_{i,j}) \le \psi(m)$.*

*Proof.*

$$\begin{aligned} \mathrm{hw}(f_{i,j}) \le & \deg(f_{i,j}) + 1 - \mathrm{tdeg}(f_{i,j}) - g(f_{i,j}) \\ \le & \psi(m) + i^{'} + 1 - \mathrm{tdeg}(f_{i,j}) - g(f_{i,j}) && \text{by Lemma 3.2} \\ \le & \psi(m) + i^{'} + 1 - 0 - i^{'} + 1 && \text{by Lemma 3.3} \\ = & \psi(m) \end{aligned}$$

□

**Theorem 3.5.** *Let $m = p_1 \cdot p_2$ be a square free odd composite integer with $p_1 < p_2$ and $p > m$ be an odd prime. Then we have*

$$d(\Phi_{mp}) \le \tfrac{\psi(m)}{m} \quad \text{if } p \equiv 1 \mod m$$

*Proof.* Note

$$\begin{aligned} \mathrm{hw}(\Phi_{mp}) = & \sum_{0 \le i \le \varphi(m)-1} \mathrm{hw}(f_i) \ + 1 \\ = & \, q \sum_{i=0}^{\varphi(m)-1} \mathrm{hw}(f_{i,j}) \ + \ 1 \\ \le & \, q \left( -2 + \sum_{i=0}^{\varphi(m)-1} \psi(m) \right) + 1 \qquad \text{by symmetry and } \mathrm{hw}(f_{0,0}) \le \psi(m) \\ \le & \, \frac{p-1}{m} \left( -2 + \psi(m)\varphi(m) \right) + 1 \\ \le & \, \frac{\varphi(mp)\psi(m)}{m} \end{aligned}$$

Hence

$$d(\Phi_{mp}) = \frac{\mathrm{hw}\left(\Phi_{mp}\right)}{\deg\left(\Phi_{mp}\right)} \le \frac{\frac{\varphi(mp)\psi(m)}{m}}{\varphi(mp)} = \frac{\psi(m)}{m}$$

□

## 4. The Hamming weight for $\Phi_{5p_2p_3}$ with $p_2 \equiv 2 \mod 5$ and $p_3 \equiv \pm 1 \mod 5p_2$

By the formulas found in [5], we see that if $p_2 \equiv 2 \mod 5$

$$\Phi_{5p_2} = \sum_{i=0}^{q_2} x^{5j}(1-x) + (1-x+x^3-x^4)x^{p_2} \sum_{j=0}^{q_2-1} x^{5j} + G(x), \qquad (4.1)$$

Notice that the polynomial $G(x)$ can be found by symmetry. However we don't need it in our proofs. In order to prove the main theorem in this section we firstly need the following results about $f_{i,j}$.

**Lemma 4.1.** *Let $i = 5(uq_2 + v)$, where $0 \le u \le 1$, $0 \le v \le q_2 - 1$. Then* $\mathrm{hw}(f_{i,j}) = 2(5-u)$.

*Proof.* Let $i = 5(uq_2 + v)$. We claim that $f_{i,0} = 1 + x^i g_u - x^{i+p_2} g_u - x^{(u+1)p_2}$, where

$$g_u = 1 + x^3 + \sum_{j=u+1}^{2} x^{2j}$$

We consider the following cases

(1) If $v = 0$, then the result is trivial for $f_{0,0}$. Now

$$\begin{aligned} f_{5q_2,0} &= -\Psi_{5p_2}\left(\sum_{j=0}^{q_2-1} x^{5j}(1-x) + x^{5q_2}\right) \\ &= (1-x^{p_2})(1-x^5)\sum_{j=0}^{q_2-1} x^{5j} - x^{5q_2}\Psi_{5p_2} \\ &= (1-x^{p_2})(1-x^{5q_2}) - x^{5q_2}(x^{p_2}-1)\Phi_5 \\ &= 1 + x^{5q_2}g_1 - x^{5q_2+p_2}g_1 - x^{2p_2} \end{aligned}$$

which has 8 terms.

(2) If $u = 0$ and $v \ge 1$, then

$$\begin{aligned} f_{5v,0} &= -\Psi_{5p_2}\left(\sum_{j=0}^{v-1} x^{5j}(1-x) + x^v\right) \\ &= (1-x^{p_2})(1-x^5)\sum_{j=0}^{v-1} x^{5j} - x^v\Psi_{5p_2} \\ &= (1-x^{p_2})(1-x^{5v}) - x^v\Psi_{5p_2} \qquad \text{using direct calculations} \\ &= 1 + x^{5v}g_0 - x^{5v+p_2}g_0 - x^{p_2} \end{aligned}$$

which has 10 terms.

(3) If $u = 1$ and $v \geq 1$, then

$$\begin{aligned} f_{5(q_2+v),0} &= -\Psi_{5p_2}\left(\sum_{j=0}^{q_2} x^{5j}(1-x) + x^{p_2}(1-x) + x^{p_2+3}\right) \\ &= 1 + x^{5(q_2+v)}g_1 - x^{5(q_2+v)+p_2}g_1 - x^{(5(q_2+v)+1)p_2} \qquad \text{direct calculations} \end{aligned}$$

which has 8 terms.

□

**Lemma 4.2.** *Let $i = 5u + v$, where $1 \leq v \leq 4$ and $0 \leq u \leq q_2 - 1$. Then* $\mathrm{hw}(f_{i,0}) = 4$.

*Proof.* Notice that $i = 5u+v \leq 5(q_2-1)+4 = 5q_2-5+5-1 = p_2-2-1 < p_2-1$, so by equation 4.1 we have

$$\begin{aligned} f_{i,j} &= -\Psi_m \cdot \mathcal{T}_{i+1}\Phi_m \\ &= -\Psi_m \sum_{a=0}^{w} x^{ap_1}(1-x) \\ &= (1-x^{p_2})(x-1)\sum_{k=0}^{p_1-1} x^k \cdot \sum_{a=0}^{w} x^{ap_1} \\ &= (x^{p_2}-1)(1-x^{p_1}) \cdot \sum_{a=0}^{w} x^{ap_1} \\ &= (x^{p_2}-1)(1-x^{(w+1)p_1}) \\ &= 1 - x^{(w+1)p_1} - x^{p_2} + x^{p_2+(w+1)p_1} \end{aligned}$$

which has 4 non-zero terms. □

**Lemma 4.3.** *Let $i = 5(q_2 + v) + 2$, where $0 \leq v \leq q_2 - 1$. Then* $\mathrm{hw}(f_{i,0}) = 8$.

*Proof.* By equation 4.1 we have

$$\begin{aligned} f_{i,j} &= -\Psi_m \cdot \mathcal{T}_{i+1}\Phi_m \\ &= -\Psi_m\left(\sum_{j=0}^{q_2} x^{5j}(1-x) + x^{p_2}(1-x)(1+x^3)\sum_{j=0}^{v-1} x^{5j} + x^{5(q_2+v)+2}\right) \\ &= (1-x^{p_2})(1-x^5)\sum_{j=0}^{q_2} x^{5j} + x^{p_2}(1+x^3)(1-x^{p_2})(1-x^5)\sum_{j=0}^{v-1} x^{5j} - x^{5(q_2+v)+2}\Psi_m \\ &= (1-x^{p_2})(1-x^{5(q_2+1)}) + x^{p_2}(1+x^3)(1-x^{p_2})(1-x^{5v}) - x^{5(q_2+v)+2}\Psi_m \\ &= (1-x^{p_2})\left(1 - x^{5(q_2+1)} + x^{p_2} - x^{p_2+5v} + x^{p_2+3} - x^{p_2+5v+3} + x^{5(q_2+v)+2}\Phi_5\right) \\ &= (1-x^{p_2})\left(1 + x^{p_2} + x^{5(q_2+v)+2}(x+x^2+x^4)\right) \end{aligned}$$

which has 8 non-zero terms. □

**Lemma 4.4.** *Let $i = 5(q_2 + v) + 1$, where $0 \le v \le q_2 - 1$. Then*

$$\mathrm{hw}(f_{i,0}) = \begin{cases} 4, & v = 0 \\ 6, & v > 0 \end{cases}.$$

*Proof.* By equation 4.1 we have

$$\begin{aligned} f_{i,j} &= -\Psi_m \cdot \mathcal{T}_{i+1}\Phi_m \\ &= -\Psi_m \left( \sum_{j=0}^{q_2} x^{5j}(1-x) + x^{p_2}(1-x)(1+x^3) \sum_{j=0}^{v-1} x^{5j} \right) \\ &= (1-x^{p_2})(1-x^5) \sum_{j=0}^{q_2} x^{5j} + x^{p_2}(1+x^3)(1-x^{p_2})(1-x^5) \sum_{j=0}^{v-1} x^{5j} \\ &= (1-x^{p_2})(1-x^{5(q_2+1)}) + x^{p_2}(1+x^3)(1-x^{p_2})(1-x^{5v}) \\ &= 1 + x^{p_2+5(q_2+1)} - x^{2p_2} - x^{p_2+5v} + x^{2p_2+5v} + x^{2p_2+5v+3} \end{aligned}$$

which has 4 terms if $v = 0$ (because of the cancelation of $-x^{2p_2}$ and $x^{2p_2+5v}$), and 6 terms otherwise. □

**Lemma 4.5.** *Let $i = 5(q_2 + v) + t$, where $0 \le v \le q_2 - 1, t = 3, 4$. Then* $\mathrm{hw}(f_{i,0}) = 6$.

*Proof.* By Lemma 2.4,

$$\begin{aligned} f_{5(q_2+v)+3,j} &= -a_{5(q_2+v)+3}x^{5(q_2+v)+3}\Psi_{5p_2} + f_{5(q_2+v)+2,j} \\ &= -a_{5(q_2+v)+3}x^{5(q_2+v)+3}\Psi_{5p_2} + 1 + x^{5(q_2+v)}(x^3 + x^4 + x^6) \\ &\quad - x^{2p_2} - x^{p_2+(5(q_2+v))}(x^3 + x^4 + x^6) \\ &= x^{5(q_2+v)+3}\Psi_{5p_2} + 1 + x^{5(q_2+v)+3}(1 + x + x^3) \\ &\quad - x^{2p_2} - x^{p_2+5(q_2+v)+3}(1 + x + x^3) \qquad\qquad a_{5(q_2+v)+3} = -1 \\ &= 1 - x^{2p_2} - x^{5(q_2+v)+3}(x^2 + x^4) - x^{p_2+5(q_2+v)+3}(x^2 + x^4) \end{aligned}$$

which has 6 terms. Now, for $f_{5(q_2+v)+4,j}$ we have $a_{5(q_2+v)+4} = 0$ so $f_{5(q_2+v)+4,j} = f_{5(q_2+v)+3,j}$ and both have the same Hamming weight. □

**Lemma 4.6.** *Let $i = 10q_2 + t$, where $0 \le t \le 2$. Then* $\mathrm{hw}(f_{i,0}) = 6$.

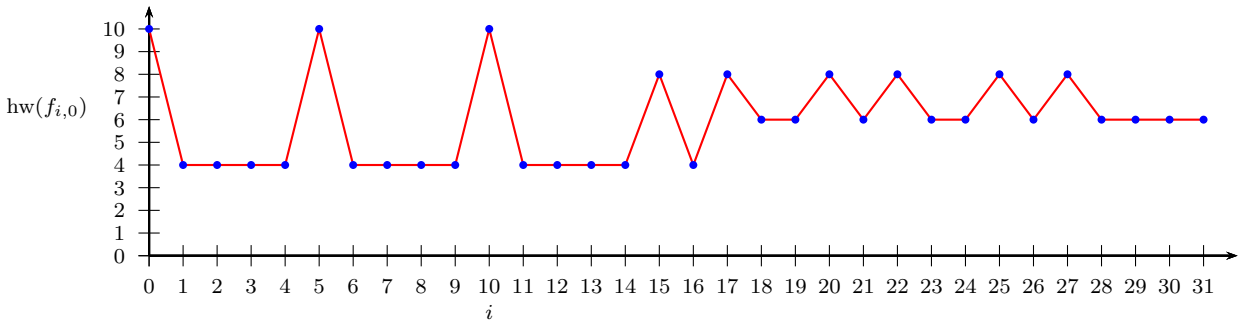
*Proof.* The proof is analogous to those of Lemma 4.1 (Part 1) and Lemma 4.4, and is therefore omitted. □

The next Lemma summarizes the information which we have about $f_{i,j}$ and its terms.

**Lemma 4.7.** *Let $i = 5(uq_2 + v) + t$, where $0 \le u \le 1, 0 \le v \le q_2$ and $0 \le t \le p_1 - 1$. Then*

$$\mathrm{hw}(f_{i,0}) = \begin{cases} 2(5-u), & t \le 2u \text{ and even} \\ 2(u+1), & v = 0 \text{ and } t \le 2u \text{ and odd} \\ 2(u+2), & t > 2u \text{ or } v \ge 1 \end{cases}$$

**Example 4.8.** This example illustrates the results in the last Lemma. Let $p_1 = 5, p_2 = 17$ and $p_3 = 1021$. Note that $r_2 = 2$ and $r_3 = 1$. The following figure presents the relationship between $i$, $\mathrm{hw}(f_{i,0})$.




Now we are ready to prove the main result in this section.

**Theorem 4.9.** *Let $5 < p_2 < p_3$ be odd prime numbers such that $p_2 \equiv 2 \mod 5$ and $p_3 \equiv \pm 1 \mod 5p_2$. Then*

$$\mathrm{hw}(\Phi_{5p_2p_3}) = \begin{cases} N(p_3-1)+1 & r_3 = 1 \\ N(p_3+1)-1 & r_3 = 5p_2-1 \end{cases}$$

*where $N = \frac{2}{5p_2}(12p_2 - 14)$*

*Proof.* Note

$$\begin{aligned} \mathrm{hw}(\Phi_{5p_2p_3}) &= 1 + \sum_{0 \le i \le \varphi(m)-1} \mathrm{hw}(f_i) \\ &= q_3 \sum_{i=0}^{\varphi(5p_2)-1} \mathrm{hw}(f_{i,0}) \ + \ 1 && \text{by the partition} \\ &= 2q_3 \sum_{i=0}^{\frac{\varphi(5p_2)}{2}-1} \mathrm{hw}(f_{i,0}) \ + 1 && \text{by Theorem 2.5} \end{aligned}$$

We have $\frac{\varphi(5p_2)}{2} - 3 = ((\frac{5-1}{2} - 1)q_2 + (q_2 - 1))p_1 + (p_1 - 1)$. Thus $\{i : 0 \le i \le \frac{\varphi(5p_2)}{2} - 3\} = \{5(uq_2)v + t : 0 \le u \le 1, 0 \le v \le q_2 - 1, 0 \le t \le p_1 - 1\}$ and $\frac{\varphi(5p_2)}{2} - 2 = 10q_2$ and $\frac{\varphi(5p_2)}{2} - 1 = 10q_2 + 1$.

$$\begin{aligned} \mathrm{hw}(\Phi_{5p_2p_3}) &= 2q_3 \left( 16q_2 + \sum_{u=0}^{1} 2(u+1)q_2(5-u) + \frac{24}{2} \right) + 1 \\ &\quad + 2q_3 \left(2(2+1)((4-u)q_2 - u) + 6\right) + 1 && \text{by Lemma 4.7} \\ &= 2q_3 \left( \frac{(p_1-1)(4p_1^2q_2 + 16p_1q_2 + 3p_1 + 15)}{12} \right) + 1 \\ &= \frac{4(p_3-1)}{5p_2}(6p_2 - 7) + 1. \end{aligned}$$

Now, for $r_3 \equiv p_3 - 1$. By Theorem 6.1 in [1] we have $\mathrm{hw}(\Phi_{5p_2p_3}) = Np_3 - (-N + 1) = N(p_3 + 1) - 1$ □

## 5. The Hamming weight for $\Phi_{5p_2p_3}, p_2 \equiv 3 \mod 5$ and $p_3 \equiv \pm 1 \mod 5p_2$

By the formulas found in [5], we see that if $p_2 \equiv 3 \mod 5$, then

$$\Phi_{5p_2} = (1-x)\sum_{j=0}^{q_2} x^{5j} + x^{p_2}(1 - x + x^2 - x^3)\sum_{j=0}^{q_2} x^{5j} + G(x), \tag{5.1}$$

Notice that the polynomial $G(x)$ can be found by symmetry. However we don't need it in our proofs. In order to prove the main theorem in this section we need the following results about $f_{i,j}$. In this section, we omit certain details in the proofs due to their similarity to those presented in the previous section.

**Lemma 5.1.** *Let $i = up_2 + 5v$, where $0 \le u \le 1, 0 \le v \le q_2$. Then*

$$\mathrm{hw}(f_{i,0}) = \begin{cases} 2(5-u), & \text{if } v < q_2 - 1 \\ 2(4-u), & \text{if } v = q_2 \end{cases}$$

*Proof.* When $u = 0$, by Equation 5.1 and Lemma 2.2 we have,

$$\begin{aligned} f_{5v,0} &= -\Psi_{5p_2} \cdot \mathcal{T}_{5v+1}\Phi_m \\ &= -\Psi_{5p_2} \cdot ((1-x)\sum_{j=0}^{v-1} x^{5j} + x^{5v}) \\ &= -(x^{p_2} - 1)(1 - x^{5v}) - x^{5v}\Psi_{5p_2} \\ &= x^{p_2+5v} - x^{p_2} + 1 - x^{5v} + x^{5v}\sum_{s=0}^{4} x^s - x^{p_2+5v}\sum_{s=0}^{4} x^s \\ &= -x^{p_2} + 1 + x^{5v}\sum_{s=1}^{4} x^s - x^{p_2+5v}\sum_{s=1}^{4} x^s \end{aligned}$$

which has 10 terms unless $v = q_2$, where $f_{5v,0}$ has 8 terms. When $u = 1$, by Equation 5.1 and Lemma 2.2 we have,

$$\begin{aligned} f_{p_2+5v,0} &= -\Psi_{5p_2} \cdot \mathcal{T}_{p_2+5v+1}\Phi_m \\ &= -\Psi_{5p_2} \cdot ((1-x)\sum_{j=0}^{q_2} x^{5j} + x^{p_2}(1 - x + x^2 - x^3)\sum_{s=0}^{v-1} x^{5s} + x^{p_2+5v}) \\ &= -(x^{p_2} - 1)(1 - x^{5(q_2+1)}) - x^{p_2}(x^{p_2} - 1)(1 - x^5)(1 + x^2)\sum_{s=0}^{v-1} x^{5s} - x^{p_2+5v}\Psi_{5p_2} \\ &= (x^{p_2+5q_2+5} - x^{p_2} + 1 - x^{5q_2+5}) - x^{p_2}(x^{p_2} - 1)(1 - x^{5v})(1 + x^2) - x^{p_2+5v}\Psi_{5p_2} \\ &= 1 - x^{2p_2} + x^{p_2+5v+1} + x^{p_2+5v+3} + x^{p_2+5v+4} - x^{2p_2+5v+1} - x^{2p_2+5v+3} - x^{2p_2+5v+4} \end{aligned}$$

which has 8 terms unless $v = q_2$, where $f_{5v,0}$ has 6 terms. □

**Lemma 5.2.** *Let $i = 5q_2 + t$, where $1 \le t \le 2$. Then* $\mathrm{hw}(f_{i,0}) = 4$.

*Proof.*

$$\begin{aligned} f_{5q_2+t,0} &= -\Psi_{5p_2} \cdot \mathcal{T}_{5q_2+t+1}\Phi_m \\ &= -\Psi_{5p_2} \cdot (1-x)\sum_{j=0}^{q_2} x^{5j} \\ &= (1-x^{p_2})(1-x^{5(q_2+1)}) \end{aligned}$$

which has four terms. □

**Lemma 5.3.** *Let $i = up_2+5v+j$, where $0 \le u \le 1, 0 \le v \le q_2-1$ and $1 \le j \le 4$. Then*

$$\mathrm{hw}(f_{i,0}) = \begin{cases} 2(5-u), & \text{if } j \le u+1 \text{ and even} \\ 2(2+u), & \text{if } j \text{ is odd or } \ j \ge u+1 \end{cases}$$

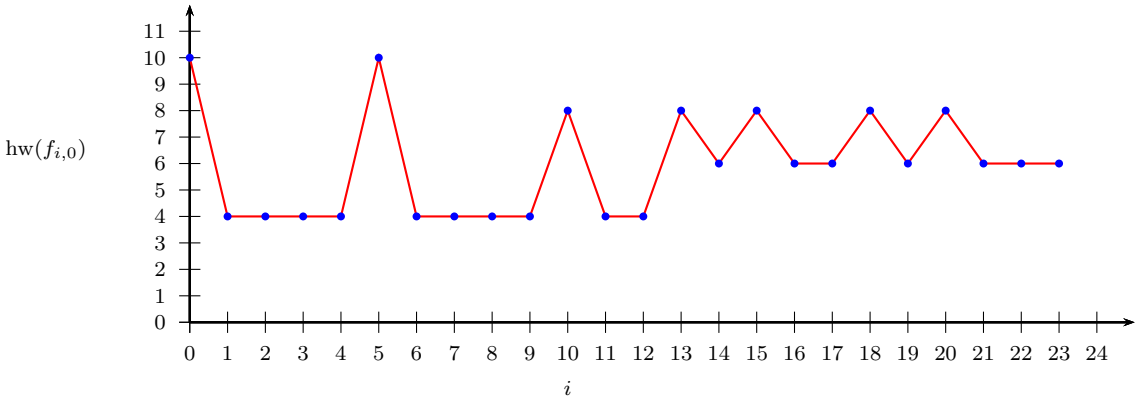
*Proof.* Here we handle the cases of $\mathcal{T}_{i+1}\Phi_{5p_2}$ and then determine the number of terms in $f_{i,0} = -\Psi_{5p_2} \cdot \mathcal{T}_{i+1}\Phi_{5p_2}$ using direct calculations, as in the preceding proofs. □

The next Lemma summarizes the information which we have about $f_{i,j}$ and its terms.

**Lemma 5.4.** *Let $i = up_2+5v+t$, where $0 \le u \le 1, 0 \le v \le q_2$ and $0 \le t \le p_1-1$. Then*

$$\mathrm{hw}(f_{i,0}) = \begin{cases} 2(5-u), & t < u+1 \text{ and even and } v < q_2 \\ 2(4-u), & t = 0 \text{ and } v = q_2 \\ 2(u+1), & \ t > 2u \text{ or odd} \\ 4, & u = 0, v = q_2, t = 1, 2 \end{cases}$$

**Example 5.5.** This example illustrates the results in the last Lemma. Let $p_1 = 5, p_2 = 13, p_2 = 131$. Note that $r_2 = 3$ and $r_3 = 1$. The following figure presents the relationship between $i$, $\mathrm{hw}(f_{i,0})$




Now we are ready to state the main result in this section.

**Theorem 5.6.** *Let $5 < p_2 < p_3$ be odd prime numbers such that $p_2 \equiv 3 \mod 5$. Then*

$$\mathrm{hw}(\Phi_{p_1p_2p_3}) = \begin{cases} N(p_3-1)+1 & r_3 = 1 \\ N(p_3+1)-1 & r_3 = 5p_2-1 \end{cases}$$

*where $N = \frac{4}{25p_2}(29p_2 - 22)$*

*Proof.* We have

$$\begin{aligned} \mathrm{hw}(\Phi_{5p_2p_3}) &= 1 + \sum_{i=0}^{\varphi(5p_2)-1} \mathrm{hw}(f_{i,j}) \\ &= 1 + q_3 \left( \sum_{i=0}^{\varphi(5p_2)-1} \mathrm{hw}\left(f_{i,0}\right) \right) && \text{by the partition} \\ &= 2q_3 \left( \sum_{i=0}^{\frac{\varphi(5p_2)}{2}-1} \mathrm{hw}\left(f_{i,0}\right) \right) + 1 && \text{by Theorem 2.5} \end{aligned}$$

Note that for

$$\left\{ i : 0 \le i \le \frac{\varphi(5p_2)}{2} - 1 \right\} = \{ up_2 + 5v + t : 0 \le u \le 1, \\ 0 \le v \le q_2, 0 \le t \le 4 \}$$

Hence

$$\begin{aligned} \mathrm{hw}(\Phi_{5p_2p_3}) &= 2q_3 \left( \sum_{u=0}^{\frac{p_1-1}{2}-1} \sum_{v=0}^{q_2} \sum_{t=0}^{4} \mathrm{hw}\left(f_{up_2+5v+t,0}\right) \right) + 1 \\ &= 2q_3 \left( 2q_2 \left( \sum_{u=0}^{1} \sum_{t=0,}^{u} (5-u) + \sum_{u=0}^{1} \sum_{t=1, t\neq 2u}^{4} (2+u) \right) + 8 + \sum_{u=0}^{1} 2(4-u) \right) + 1 && \text{Lemma 5.4} \\ &= 2q_3 \left(58q_2 + 26\right) + 1 \\ &= 2q_3 \left(\frac{58p_2 - 44}{5}\right) + 1 && q_2 = \frac{p_2-3}{5} \\ & && q_3 = \frac{p_3-1}{5p_2} \\ &= \frac{4(p_3-1)(29p_2-22)}{25p_2} + 1 && \text{rearranging} \\ &= N\left(p_3 - 1\right) + 1 \end{aligned}$$

Now, for $r_3 \equiv p_3 - 1$. By Theorem 6.1 in [1], we have $\mathrm{hw}(\Phi_{5p_2p_3}) = Np_3 - (-N + 1) = N(p_3+1) - 1$ □

## 6. Conclusion and Future Work

In this paper we provide a formula for the structure of the cyclotomic polynomials $\Phi_{mp}$ where $p \equiv 1 \mod m$. A previous structure for general cyclotomic

polynomials was given in [1]. However, the one provided in this paper is easier to compute. Then we use the new structure to provide a bound on the density of the ternary cyclotomic polynomial $\Phi_{p_1p_2p_3}$ with $p_3 \equiv 1 \mod p_1p_2$ (Section 3, Theorem 3.5). The computations suggest the following conjectures:

**Conjecture 6.1.** Let $m$ be a square free odd composite integer and $p > m$ be an odd prime, where $p \equiv \pm 1 \mod m$. Then we have

$$d(\Phi_{mp}) = \frac{\mathrm{hw}\,(\Phi_{mp})}{\deg\,(\Phi_{mp})} \leq \frac{\psi(m)}{m}.$$

**Conjecture 6.2.** Let $m$ be a square free odd integer and $p > m$ be an odd prime, where $p \not\equiv \pm 1 \mod m$. Then we have

$$d(\Phi_{mp}) = \frac{\mathrm{hw}\,(\Phi_{mp})}{\deg\,(\Phi_{mp})} \geq \frac{\psi(m)}{m}.$$

For the number of terms we have the following conjectures:

**Conjecture 6.3.** Let $7 < p_2 < p_3$ be odd prime numbers such that $p_2 \equiv -2 \mod 7$. Then

$$\mathrm{hw}(\Phi_{7p_2p_3}) = N(p_3 - 1) + 1,$$

where $N = \frac{2}{49p_2}(150p_2 - 190)$ and $p_3 \equiv 1 \mod 7p_2$.

**Conjecture 6.4.** Let $p_1 < p_2 < p_3$ be odd prime numbers such that $p_2 \equiv 2 \mod p_1$. Then

$$\mathrm{hw}(\Phi_{p_1p_2p_3}) = \begin{cases} N(p_3 - 1) + 1 & r_3 = 1 \\ N(p_3 + 1) - 1 & r_3 = p_1p_2 - 1 \end{cases},$$

where $N = \frac{(p_1-1)}{6p_1p_2}(4p_1p_2 - 5p_1 + 16p_2 - 17)$.

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