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GEOMETRIC PROPERTIES OF NORMALIZED MODIFIED q-SIGMOID FUNCTIONS

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Abstract. We normalize the modified q-sigmoid function and study it within the geometric function theory. Using an Euler product representation, we show that the normalized function is univalent and bounded by 2 in the unit disc for every q between 0 and 1. We establish the coefficient criteria for q-starlikeness and q-convexity of order alpha, and the second and third Hankel determinants are evaluated in closed form as ratios of q-integers. Both are strictly increasing in q, and this gives the sharp bounds 1/12 and 1/8640. We also obtained sharp Fekete–Szegő inequalities for the associated subordination class.

Keywords. q-derivative operator, q-starlike functions, Hankel determinant, modified q-sigmoid function
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1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of functions f that are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = 0$ and $f'(0) = 1$, and let $\mathcal{S} \subset \mathcal{A}$ be the subclass of univalent functions. For $0 \leq \alpha < 1$, the subclasses $\mathcal{S}^*(\alpha)$ of starlike functions of order α and $\mathcal{C}(\alpha)$ of convex functions of order α are defined by $\mathrm{Re}\{zf'(z)/f(z)\} > \alpha$ and $\mathrm{Re}\{1 + zf''(z)/f'(z)\} > \alpha$ in $\mathbb{U}$. A set is starlike with respect to an interior point z_0 if the line segment joining z_0 to any point of the set lies entirely in the set, and it is convex if the segment joining any two of its points lies in the set; an example is any disc or half-plane. We can find the foundational accounts of these classes, and of the coefficient problem in [1] and [16].

For $f \in \mathcal{A}$ with $f(z) = z + \sum_{n\geq 2} a_n z^n$, the Hankel determinant was introduced by [11] and it is represented as $H_{k,n}(f) = \det\left(a_{n+i+j-2}\right)_{i,j=1}^{k}$, where $a_1 = 1$. The second and third order determinants,

$$H_{2,1}(f) = a_3 - a_2^2, \qquad H_{3,1}(f) = a_3a_5 - a_4^2 - a_2^2a_5 + 2a_2a_3a_4 - a_3^3, \tag{1.1}$$

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have attracted sustained attention, and sharp estimates for $|H_{3,1}|$ are available only for a limited number of classes; see [12] for α-starlike functions and [8, 13, 15, 17] for some related families.

The sigmoid function, which is commonly used as an activation function in neural networks, was brought into univalent function theory by [4], who studied the modified sigmoid $2/(1+e^{-z})$; further geometric consequences were developed in [6] and [9]. A q-analogue variant of the function was introduced by [3] in connection with λ-pseudo-starlike functions. The q-calculus framework itself, surveyed by [14], has shown a fertile source of new subclasses of analytic functions, and [7] obtained Fekete–Szegő estimates and second Hankel determinant bounds for a family generated by a q-differential operator. [10] proposed the deformations of the sigmoid function outside complex analysis, notably the piecewise shifted sigmoid function and the κ-operator generalization of [2].

Ezeafulukwe, Darus and Fadipe-Joseph [3] introduced the modified $q-$sigmoid without normalizing it, and neither its coefficient structure nor its Hankel determinants have been considered. On that premise we arrived, together with the subordination class the normalized function generates. All rests on Lemma 2.1, which identifies Φ_q as a Mőbius transform of the Euler product for E_q and so it deals out with the series manipulations used previously. Univalence and the bound $|\Phi_q| < 2$ then hold across the whole parameter range (Theorem 2.4), and the same lemma makes Y_q admissible as a Ma-Minda superodinate function, which is what section 5 requires. The coefficients satisfy a recurrence in which every b_n is positive; Theorems 3.1 and 3.2 use it to evaluate $H_{2,1}(\Phi_q)$ and $H_{3,1}(\Phi_q)$ in closed form and to show that both grow with q. Section 3 stands apart from this process, its criteria for $q-$starlikeness and $q-$convexity of order α applying to any $f \in A$.

All through this study, $0 < q < 1$ is fixed. For $n \in \mathbb{N}_0$ the q-integer and q-factorial are

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}, \qquad [0]_q = 0, \qquad [n]_q! = \prod_{k=1}^{n} [k]_q, \quad [0]_q! = 1. \tag{1.2}$$

The Jackson q-derivative of f is

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \quad (z \neq 0), \qquad D_q f(0) = f'(0), \tag{1.3}$$

so that $D_q z^n = [n]_q z^{n-1}$ and $D_q f \to f'$ as $q \to 1^-$. For u, v analytic, $D_q(uv)(z) = u(qz)D_q v(z) + v(z)D_q u(z)$; in particular

$$\frac{zD_q\big(zD_q f(z)\big)}{zD_q f(z)} = 1 + \frac{qzD_q^2 f(z)}{D_q f(z)}. \tag{1.4}$$

A function $f \in \mathcal{A}$ belongs to $\mathcal{S}_q^*(\alpha)$ if $\mathrm{Re}\{zD_q f(z)/f(z)\} > \alpha$ in $\mathbb{U}$, and to $\mathcal{C}_q(\alpha)$ if $\mathrm{Re}\{1 + qzD_q^2 f(z)/D_q f(z)\} > \alpha$ in $\mathbb{U}$, where $0 \leq \alpha < 1$. Identity (1.4) gives the q-Alexander theorem: $f \in \mathcal{C}_q(\alpha)$ if and only if $zD_q f \in \mathcal{S}_q^*(\alpha)$.

We write $f \prec g$ if $f = g \circ \omega$ for some Schwarz function ω, that is, ω analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$. If g is univalent then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. We shall use the classical estimate of [5]: if $\omega(z) = w_1 z + w_2 z^2 + \dots$ is a Schwarz function and $\nu \in \mathbb{C}$, then

$$|w_2 - \nu w_1^2| \le \max\{1, |\nu|\}, \tag{1.5}$$

with equality for $\omega(z) = z^2$ when $|\nu| \le 1$ and for $\omega(z) = z$ when $|\nu| \ge 1$.

Two q-exponentials occur below,

$$e_q(z) = \sum_{n\ge 0} \frac{z^n}{[n]_q!} \quad \Big(|z| < \tfrac{1}{1-q}\Big), \qquad \mathcal{E}_q(z) = \sum_{n\ge 0} \frac{q^{n(n-1)/2} z^n}{[n]_q!} \quad (z \in \mathbb{C}). \tag{1.6}$$

The modified q-sigmoid function of [3] is $Y_q(z) = 2/\big(1 + e_q(-z)\big)$; note that $e_q(-z) = \sum_{n\ge 0} (-1)^n z^n/[n]_q!$, so the sign pattern in the expansion alternates.

2. The normalized function and its product representation

Since $Y_q(0) = 1$ and $Y_q'(0) = \frac{1}{2}$, the normalization of Y_q into $\mathcal{A}$ is

$$\Phi_q(z) = \frac{Y_q(z) - Y_q(0)}{Y_q'(0)} = 2\big(Y_q(z) - 1\big) = 2\,\frac{1 - e_q(-z)}{1 + e_q(-z)} = \sum_{n\ge 1} a_n z^n, \qquad a_1 = 1. \tag{2.1}$$

Lemma 2.1. *For $0 < q < 1$ put $c_k = (1-q)q^k$, $k \ge 0$, so that $\sum_{k\ge 0} c_k = 1$. Then*

$$\mathcal{E}_q(z) = \prod_{k=0}^{\infty} \big(1 + c_k z\big), \qquad e_q(-z)\,\mathcal{E}_q(z) = 1, \tag{2.2}$$

and consequently

$$\Phi_q(z) = 2\,\frac{\mathcal{E}_q(z) - 1}{\mathcal{E}_q(z) + 1}, \qquad Y_q(z) = \frac{2\mathcal{E}_q(z)}{1 + \mathcal{E}_q(z)}. \tag{2.3}$$

The Taylor coefficients of Φ_q satisfy the recurrence

$$a_n = b_n - \frac{1}{2}\sum_{k=1}^{n-1} a_k b_{n-k}, \qquad b_n = \frac{q^{n(n-1)/2}}{[n]_q!} \quad (n \ge 1), \tag{2.4}$$

in which every b_n is positive.

Proof. Euler's identity $\prod_{k\ge 0}(1+xq^k) = \sum_{n\ge 0} q^{n(n-1)/2}x^n/(q;q)_n$ with $x = (1-q)z$ and $[n]_q! = (q;q)_n/(1-q)^n$ gives the first equality in (2.2); the second is the classical relation $e_q(z)\mathcal{E}_q(-z) = 1$. Substituting $e_q(-z) = 1/\mathcal{E}_q(z)$ in (2.1) gives (2.3). Finally, writing $\mathcal{E}_q(z) = \sum_{n\ge 0} b_n z^n$ with $b_0 = 1$ and comparing coefficients of z^n in $\Phi_q(z)\big(\mathcal{E}_q(z) + 1\big) = 2\big(\mathcal{E}_q(z) - 1\big)$ yields $2a_n + \sum_{k=1}^{n-1} a_k b_{n-k} = 2b_n$, which is (2.4). □

Remark 2.2. Recurrence (2.4) is preferable to the one obtained from $e_q(-z)$ directly, because every b_n is positive and so the coefficients of $\mathcal{E}_q$ are them computed without subtraction. $\mathcal{E}_q$ is also entire, so (2.3) presents Φ_q as a meromorphic function on $\mathbb{C}$ rather than as a series with a finite radius of convergence.

Lemma 2.3. *For $0 < q < 1$ we have $\mathrm{Re}\{\mathcal{E}_q(z)\} > 0$ and $|\arg \mathcal{E}_q(z)| < \pi/2$ for all $|z| \leq 1$.*

Proof. For $|z| \leq 1$ the point $1 + c_k z$ lies in the closed disc of centre 1 and radius $c_k < 1$, whence $|\arg(1+c_k z)| \leq \arcsin c_k$. Since arcsin is convex on $[0,1]$, we have $\arcsin x \leq \frac{\pi}{2}x$ there, with strict inequality for $x \in (0,1)$. Therefore

$$\sum_{k\geq 0} \big|\arg(1 + c_k z)\big| \leq \sum_{k\geq 0} \arcsin c_k < \frac{\pi}{2} \sum_{k\geq 0} c_k = \frac{\pi}{2}.$$

As the total variation of the argument is smaller than $\pi/2$, no branch ambiguity arises and $|\arg \mathcal{E}_q(z)| < \pi/2$, that is, $\mathrm{Re}\{\mathcal{E}_q(z)\} > 0$. □

Theorem 2.4. *For every $q \in (0,1)$ the function Φ_q is analytic and univalent in $\mathbb{U}$, $\Phi_q(z) \neq 0$ for $0 < |z| < 1$, and $|\Phi_q(z)| < 2$ in $\mathbb{U}$. Moreover $Y_q = 1 + \frac{1}{2}\Phi_q$ is univalent in $\mathbb{U}$ and satisfies $|Y_q(z) - 1| < 1$; in particular Y_q is a Carathéodory function with $Y_q(0) = 1$ and $Y_q'(0) = \frac{1}{2} > 0$.*

Proof. By Lemma 2.3, $\mathcal{E}_q(z) + 1 \neq 0$ on $|z| \leq 1$, so Φ_q is analytic on a neighbourhood of $\overline{\mathbb{U}}$.

Put $h(z) = \log \mathcal{E}_q(z) = \sum_{k\geq 0} \log(1 + c_k z)$, the branch being fixed by $h(0) = 0$; this is legitimate by Lemma 2.3. For $|z| < 1$ the point $1 + c_k z$ lies in the disc $D(1, c_k)$, whose image under inversion is the disc $D\big((1 - c_k^2)^{-1}, c_k(1 - c_k^2)^{-1}\big)$, contained in the half-plane $\mathrm{Re}\, w > 1/(1 + c_k) > 0$. Hence

$$\mathrm{Re}\{h'(z)\} = \sum_{k\geq 0} \mathrm{Re}\, \frac{c_k}{1 + c_k z} > \sum_{k\geq 0} \frac{c_k}{1 + c_k} > 0,$$

so by the Noshiro–Warschawski criterion h is univalent in the convex domain $\mathbb{U}$. By Lemma 2.3, $\mathrm{Im}\, h(\mathbb{U}) \subset (-\pi/2, \pi/2)$, an interval of length $\pi < 2\pi$, so the exponential is injective on $h(\mathbb{U})$. Therefore $\mathcal{E}_q = \exp \circ h$ is univalent in $\mathbb{U}$, and so is its Möbius transform $\Phi_q = 2(\mathcal{E}_q - 1)/(\mathcal{E}_q + 1)$.

Since $\mathcal{E}_q$ is univalent with $\mathcal{E}_q(0) = 1$, we get $\mathcal{E}_q(z) \neq 1$ for $z \neq 0$, that is, $\Phi_q(z) \neq 0$ for $0 < |z| < 1$. Finally $w \mapsto 2(w-1)/(w+1)$ maps the right half-plane onto $|\zeta| < 2$ and $w \mapsto 2w/(1+w)$ maps it onto $|\zeta - 1| < 1$; combining with Lemma 2.3 gives $|\Phi_q| < 2$ and $|Y_q - 1| < 1$. □

Remark 2.5. Theorem 2.4 settles univalence for the whole range $q \in (0,1)$. The familiar coefficient test $\sum_{n\geq 2} n|a_n| \leq 1$ is not adequate here: numerical evaluation of (2.4) shows that $\sum_{n\geq 2} n|a_n| > 1$ for $q < q_0 \approx 0.2053$, so a proof of univalence resting on that test would leave a whole subinterval of the parameter range uncovered.

3. Coefficient criteria for q-starlikeness and q-convexity

Theorem 3.1. *Let $f(z) = z + \sum_{n\geq 2} a_n z^n \in \mathcal{A}$ and $0 \leq \alpha < 1$. If*

$$\sum_{n=2}^{\infty} \big([n]_q - \alpha\big)|a_n| \leq 1 - \alpha, \tag{3.1}$$

then $f \in \mathcal{S}_q^*(\alpha)$.

Proof. Since $[n]_q - \alpha \geq [2]_q - \alpha = 1 + q - \alpha > 1 - \alpha$ for $n \geq 2$, condition (3.1) forces $\sum_{n\geq 2} |a_n| < 1$, so $f(z)/z$ does not vanish in $\mathbb{U}$ and the quotient below is analytic. For $|z| < 1$,

$$\left| \frac{zD_q f(z)}{f(z)} - 1 \right| = \left| \frac{\sum_{n\geq 2} \big([n]_q - 1\big) a_n z^{n-1}}{1 + \sum_{n\geq 2} a_n z^{n-1}} \right| \leq \frac{\sum_{n\geq 2} \big([n]_q - 1\big)|a_n|}{1 - \sum_{n\geq 2} |a_n|}.$$

The last quantity is at most $1 - \alpha$ if and only if

$$\sum_{n\geq 2} \big([n]_q - 1\big)|a_n| \leq (1-\alpha)\Big(1 - \sum_{n\geq 2} |a_n|\Big) = 1 - \alpha - (1-\alpha) \sum_{n\geq 2} |a_n|,$$

that is, after transposing the last sum,

$$\sum_{n\geq 2} \big([n]_q - 1 + 1 - \alpha\big)|a_n| = \sum_{n\geq 2} \big([n]_q - \alpha\big)|a_n| \leq 1 - \alpha,$$

which is exactly (3.1). Hence $|zD_q f/f - 1| < 1-\alpha$, and therefore $\operatorname{Re}\{zD_q f(z)/f(z)\} > \alpha$ in $\mathbb{U}$. □

Theorem 3.2. *Let $f(z) = z + \sum_{n\geq 2} a_n z^n \in \mathcal{A}$ and $0 \leq \alpha < 1$. If*

$$\sum_{n=2}^{\infty} [n]_q \big([n]_q - \alpha\big)|a_n| \leq 1 - \alpha, \tag{3.2}$$

then $f \in \mathcal{C}_q(\alpha)$.

Proof. By the q-Alexander theorem, $f \in \mathcal{C}_q(\alpha)$ if and only if $g = zD_q f \in \mathcal{S}_q^*(\alpha)$. Now $g(z) = z + \sum_{n\geq 2} [n]_q a_n z^n$ has coefficients $A_n = [n]_q a_n$, and Theorem 3.1 applied to g requires $\sum_{n\geq 2} ([n]_q - \alpha)|b_n| = \sum_{n\geq 2} [n]_q ([n]_q - \alpha)|a_n| \leq 1 - \alpha$, which is (3.2). □

Remark 3.3. In (3.1) and (3.2), we have the correct rearrangements. Forms such as $\sum([n]_q \pm 1)|a_n| \leq 1 - \alpha$ and $\sum [n]_q([n]_q - 1)|a_n| \leq 1 - \alpha$ do not follow from the displayed inequality: moving $(1-\alpha)\sum |a_n|$ to the left-hand side contributes $+(1-\alpha)$, not $+1$ or -1, to each bracket. Setting $\alpha = 0$ recovers $\sum_{n\geq 2} [n]_q |a_n| \leq 1$, and letting $q \to 1^-$ recovers the classical conditions $\sum (n - \alpha)|a_n| \leq 1 - \alpha$ and $\sum n(n - \alpha)|a_n| \leq 1 - \alpha$.

Corollary 3.4. *Write $\Sigma_1 = \sum_{n\geq 2} |a_n|$, $\Sigma_2 = \sum_{n\geq 2} [n]_q |a_n|$ and $\Sigma_3 = \sum_{n\geq 2} [n]_q^2 |a_n|$ for the coefficients of Φ_q. If $\Sigma_1 < 1$ then $\Phi_q \in \mathcal{S}_q^*(\alpha)$ for every $\alpha \leq \alpha_S^\sharp(q) = (1 - \Sigma_2)/(1 - \Sigma_1)$; if $\Sigma_2 < 1$ then $\Phi_q \in \mathcal{C}_q(\alpha)$ for every $\alpha \leq \alpha_C^\sharp(q) = (1 - \Sigma_3)/(1 - \Sigma_2)$.*

Numerical values of $\alpha_S^\sharp$ and $\alpha_C^\sharp$ are listed in Table 1 alongside the exact orders obtained in Section 6; as is to be expected of criteria derived from the triangle inequality, they are conservative, markedly so for convexity when q is close to 1.

4. Hankel determinants

Theorem 4.1. *For $0 < q < 1$ the first five Taylor coefficients of Φ_q are*

$$a_1 = 1, \qquad a_2 = -\frac{1-q}{2[2]_q}, \qquad a_3 = \frac{1-3q+q^2}{4[3]_q},$$
$$a_4 = -\frac{(1-q)(1-4q+q^2)}{8[4]_q}, \qquad a_5 = \frac{\sum_{j=0}^{8} d_j q^j}{16\,[2]_q^2[3]_q[5]_q}, \tag{4.1}$$

where $(d_0, d_1, \ldots, d_8) = (1, -4, -4, 7, 8, 7, -4, -4, 1)$, and the Hankel determinants defined in (1.1) are

$$H_{2,1}(\Phi_q) = a_3 - a_2^2 = -\frac{q^2}{[2]_q^2\,[3]_q}, \tag{4.2}$$

$$H_{3,1}(\Phi_q) = -\frac{q^9}{[2]_q^2\,[3]_q^3\,[4]_q^2\,[5]_q} = -\frac{q^9(1-q)^8}{(1-q^2)^2(1-q^3)^3(1-q^4)^2(1-q^5)}. \tag{4.3}$$

Proof. Formulae (4.1) follow from (2.4) on substituting $[2]_q! = [2]_q$, $[3]_q! = [2]_q[3]_q$, $[4]_q! = [2]_q[3]_q[4]_q$ and $[5]_q! = [2]_q[3]_q[4]_q[5]_q$ and clearing denominators; the factorizations $[4]_q = (1+q)(1+q^2)$ and $1 + q + q^2 + q^3 + q^4 = [5]_q$ are used to reduce a_4 and a_5. Substituting these rational functions into (1.1) and simplifying gives, after cancellation of a factor q^2 in the second determinant and q^9 in the third, the expressions (4.2) and (4.3). The alternative form in (4.3) comes from $[n]_q = (1-q^n)/(1-q)$ together with $2+3+2+1 = 8$. □

Theorem 4.2. *The functions $q \mapsto |H_{2,1}(\Phi_q)|$ and $q \mapsto |H_{3,1}(\Phi_q)|$ are strictly increasing on $(0,1)$. Consequently, for all $q \in (0,1)$,*

$$\left|H_{2,1}(\Phi_q)\right| < \frac{1}{12}, \qquad \left|H_{3,1}(\Phi_q)\right| < \frac{1}{8640} \approx 1.1574 \times 10^{-4}, \tag{4.4}$$

and both bounds are sharp, being attained in the limit $q \to 1^-$, where $\Phi_1(z) = 2\tanh(z/2)$.

Proof. For $n \geq 2$ set $\mu_n(q) = q\,\dfrac{d}{dq}\log[n]_q = \dfrac{\sum_{k=0}^{n-1} kq^k}{\sum_{k=0}^{n-1} q^k}$, the mean of $\{0, 1, \ldots, n-1\}$ with respect to the weights q^k. For $0 < q < 1$ these weights are strictly decreasing in k, so the weighted mean is strictly smaller than the unweighted mean:

$$\mu_n(q) < \frac{n-1}{2} \qquad (0 < q < 1,\ n \geq 2). \tag{4.5}$$

From (4.3), $\log|H_{3,1}(\Phi_q)| = 9\log q - 2\log[2]_q - 3\log[3]_q - 2\log[4]_q - \log[5]_q$, hence

$$q\frac{d}{dq}\log\left|H_{3,1}(\Phi_q)\right| = 9 - \Big(2\mu_2 + 3\mu_3 + 2\mu_4 + \mu_5\Big) > 9 - \Big(2\cdot\tfrac{1}{2} + 3\cdot 1 + 2\cdot\tfrac{3}{2} + 2\Big) = 9 - 9 = 0,$$

by (4.5). Thus $|H_{3,1}(\Phi_q)|$ is strictly increasing. The same argument applied to $\log|H_{2,1}(\Phi_q)| = 2\log q - 2\log[2]_q - \log[3]_q$ gives $q\,(\log|H_{2,1}|)' = 2 - (2\mu_2 + \mu_3) > 2 - (1+1) = 0$.

Letting $q \to 1^-$ in (4.2) and (4.3) with $[n]_1 = n$ gives $1/(4\cdot 3) = 1/12$ and $1/(2^2 3^3 4^2 5) = 1/8640$; letting $q \to 0^+$ gives 0 in both cases. Since the functions

are strictly increasing, (4.4) holds with strict inequality and the constants cannot be lowered. The limiting function is $\Phi_1(z) = 2\tanh(z/2) = z - \frac{1}{12}z^3 + \frac{1}{120}z^5 - \dots$, for which $a_2 = a_4 = 0$, $a_3 = -1/12$, $a_5 = 1/120$ and $H_{3,1} = a_3a_5 - a_3^3 = -1/8640$. □

Remark 4.3. The exponent 9 in (4.3) is exactly $2\cdot\frac{1}{2}+3\cdot1+2\cdot\frac{3}{2}+1\cdot2$, the weighted sum of the classical means appearing in the proof. This is why the monotonicity argument is borderline: the inequality is strict for every $q \in (0,1)$ but degenerates as $q \to 1^-$, which is precisely where the extremal value is attained. We emphasize that (4.3) is an exact evaluation for a single function; the corresponding sharp bound over an entire class, as obtained for α-starlike functions by [12], remains open here and is recorded in Section 7.

5. The subordination class generated by Y_q

By Theorem 2.4, Y_q is univalent in $\mathbb{U}$ with $Y_q(0) = 1$, $Y_q'(0) = 1/2 > 0$ and $\operatorname{Re} Y_q > 0$, so it is admissible as a Ma–Minda superordinate function. Define

$$\mathcal{S}_q^*(Y_q) = \left\{ f \in \mathcal{A} : \frac{zD_qf(z)}{f(z)} \prec Y_q(z) \right\}.$$

Writing $Y_q(z) = 1 + \sum_{n\ge1} B_n z^n$ we have, from $Y_q = 1 + \frac{1}{2}\Phi_q$,

$$B_1 = \tfrac{1}{2}, \qquad B_2 = \tfrac{1}{2}a_2 = -\frac{1-q}{4[2]_q}. \tag{5.1}$$

The theorem below is the analogue, for the modified q-sigmoid, of the estimates obtained by [7] for a family defined through a q-differential operator.

Theorem 5.1. *Let $f(z) = z + \sum_{n\ge2} a_n z^n \in \mathcal{S}_q^*(Y_q)$ and $\mu \in \mathbb{C}$. Then*

$$|a_2| \le \frac{1}{2q}, \tag{5.2}$$

and

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{2q[2]_q} \max\left\{1, \frac{\left|\mu[2]_q^2 - (1+q^2)\right|}{2q[2]_q}\right\}. \tag{5.3}$$

Both estimates are sharp. In particular

$$\left|H_{2,1}(f)\right| = \left|a_3 - a_2^2\right| \le \frac{1}{2q[2]_q}, \tag{5.4}$$

and

$$|a_3| \le \begin{cases} \dfrac{1+q^2}{4q^2[2]_q^2}, & 0 < q \le \sqrt{2}-1, \\[2mm] \dfrac{1}{2q[2]_q}, & \sqrt{2}-1 \le q < 1. \end{cases}$$

Proof. Let $p(z) = zD_qf(z)/f(z) = 1 + p_1z + p_2z^2 + \dots$. Comparing coefficients in $zD_qf(z) = f(z)p(z)$ gives $[2]_qa_2 = a_2 + p_1$ and $[3]_qa_3 = a_3 + a_2p_1 + p_2$, so that, since $[2]_q - 1 = q$ and $[3]_q - 1 = q[2]_q$,

$$a_2 = \frac{p_1}{q}, \qquad a_3 = \frac{1}{q[2]_q}\left(\frac{p_1^2}{q} + p_2\right). \tag{5.5}$$

Subordination means $p = Y_q \circ \omega$ for a Schwarz function $\omega(z) = w_1 z + w_2 z^2 + \dots$, whence $p_1 = B_1 w_1$ and $p_2 = B_1 w_2 + B_2 w_1^2$. Then $|a_2| = |B_1||w_1|/q \le 1/(2q)$, which is (5.2), with equality for $\omega(z) = z$.

Using (5.5),

$$a_3 - \mu a_2^2 = \frac{B_1}{q[2]_q}\left[w_2 - \nu w_1^2\right], \qquad \nu = \frac{\mu[2]_q^2 - (1+q^2)}{2q[2]_q},$$

where ν is obtained by collecting the w_1^2 terms and using $B_2/B_1 = -(1-q)/(2[2]_q)$ together with $[2]_q + q(q-1) = 1+q^2$. Estimate (1.5) now gives (5.3), and sharpness follows from the sharpness in (1.5): take $\omega(z) = z^2$ if $|\nu| \le 1$ and $\omega(z) = z$ if $|\nu| \ge 1$.

For $\mu = 1$ we get $\mu[2]_q^2 - (1+q^2) = (1+q)^2 - (1+q^2) = 2q$, so $|\nu| = 1/[2]_q \le 1$ and (5.3) reduces to (5.4). For $\mu = 0$ we get $|\nu| = (1+q^2)/(2q[2]_q)$, and $|\nu| \ge 1$ precisely when $1 - 2q - q^2 \ge 0$, that is when $q \le \sqrt{2} - 1$; the two cases of the stated bound for $|a_3|$ follow. □

Remark 5.2. As $q \to 1^-$, (5.2)–(5.4) become $|a_2| \le \frac{1}{2}$, $|a_3 - \mu a_2^2| \le \frac{1}{4}\max\{1, |2\mu - 1|/2\}$ and $|a_3 - a_2^2| \le \frac{1}{4}$, the Ma–Minda estimates for the class generated by the classical modified sigmoid $2/(1+e^{-z})$ studied in [4] and [6], for which $B_1 = \frac{1}{2}$ and $B_2 = 0$.

6. Numerical illustration and applied interpretation

Recurrence (2.4) together with $D_q\Phi_q(z) = \sum_{n\ge1}[n]_q a_n z^{n-1}$ and $D_q^2\Phi_q(z) = \sum_{n\ge2}[n]_q[n-1]_q a_n z^{n-2}$ allows the q-geometric quantities

$$S_q(z) = \frac{z D_q\Phi_q(z)}{\Phi_q(z)}, \qquad C_q(z) = 1 + \frac{q z D_q^2\Phi_q(z)}{D_q\Phi_q(z)} \tag{6.1}$$

to be evaluated. By Theorem 2.4 both are analytic on a neighbourhood of $\overline{\mathbb{U}}$, the zero of Φ_q at the origin being cancelled by that of $zD_q\Phi_q$, and $D_q\Phi_q$ not vanishing on $\overline{\mathbb{U}}$. Since $\operatorname{Re} S_q$ and $\operatorname{Re} C_q$ are harmonic, their minima over $|z| \le r$ are attained on $|z| = r$, so it suffices to sample the circle; the exact orders

$$\alpha_S(q) = \min_{|z|\le1} \operatorname{Re} S_q(z), \qquad \alpha_C(q) = \min_{|z|\le1} \operatorname{Re} C_q(z)$$

are obtained as $r \to 1^-$. Table 1 reports these together with the analytic lower bounds of Corollary 3.4.

Both α_S and α_C fall strictly as q rises, and $\alpha_C(q) < \alpha_S(q)$ at every value, so q-convexity is the harder condition to meet. What Table 1 shows beyond that is how far the two separate: at $q = 0.25$ they are less than 0.03 apart, at $q = 0.90$ more than 0.24. A small deformation parameter therefore leaves room in both classes, whereas $q \to 1^-$ returns the classical $2\tanh(z/2)$ and spends most of the convexity margin doing so.

Figure 1 displays $\operatorname{Re} S_q$ and $\operatorname{Re} C_q$ along the real segment together with the exact curve $q \mapsto |H_{3,1}(\Phi_q)|$ of Theorem 4.1, whose values at $q = 0.25, 0.50, 0.75, 0.90$ reproduce the last column of Table 2 to machine precision and confirm the monotone increase towards $1/8640$.

TABLE 1. Margins for Φ_q. Columns 2–3: minima over $|z| \le 0.90$; columns 4–5: exact orders over $\mathbb{U}$; columns 6–7: analytic lower bounds from Corollary 3.4.

q	$\alpha_S(q, 0.90)$	$\alpha_C(q, 0.90)$	$\alpha_S(q)$	$\alpha_C(q)$	$\alpha_S^\sharp(q)$	$\alpha_C^\sharp(q)$
0.25	0.9305	0.9078	0.9227	0.8967	0.8473	0.7722
0.50	0.9001	0.8326	0.8869	0.8082	0.8204	0.6439
0.75	0.8845	0.7462	0.8656	0.7002	0.7632	0.2884
0.90	0.8791	0.6775	0.8562	0.6117	0.7572	0.0627

TABLE 2. Coefficients of Φ_q and the exact Hankel determinants of Theorem 4.1.

q	a_2	a_3	a_4	a_5	$H_{2,1}(\Phi_q)$	$H_{3,1}(\Phi_q)$
0.25	-0.30000	0.05952	-0.00441	-0.00237	-0.030476	-4.5957×10^{-7}
0.50	-0.16667	-0.03571	0.02500	-0.00406	-0.063492	-2.3779×10^{-5}
0.75	-0.07143	-0.07432	0.01643	0.00506	-0.079426	-8.6917×10^{-5}
0.90	-0.02632	-0.08210	0.00651	0.00786	-0.082796	-1.1134×10^{-4}
$q \to 1^-$	0	$-1/12$	0	$1/120$	$-1/12$	$-1/8640$

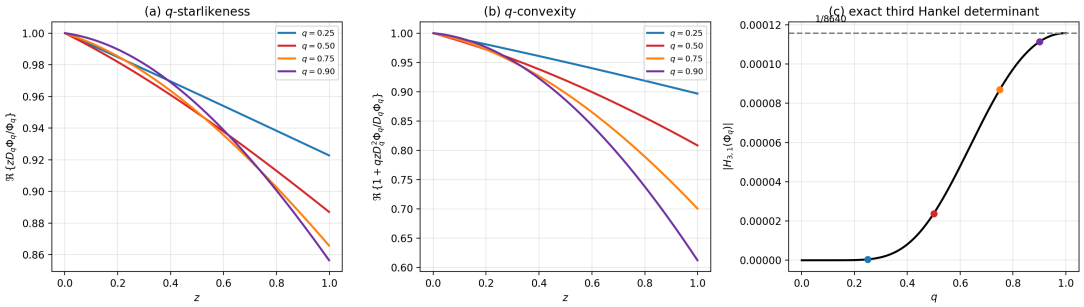


FIGURE 1. Geometric behaviour of Φ_q. (a) q-starlikeness quantity; (b) q-convexity quantity, including the factor q required by the definition of $\mathcal{C}_q(\alpha)$; (c) the exact third Hankel determinant $|H_{3,1}(\Phi_q)|$ with its supremum $1/8640$.

Restricting Φ_q to the real line yields $A_q(x) = 2\big(\mathcal{E}_q(x) - 1\big)/\big(\mathcal{E}_q(x) + 1\big)$, which reduces to $A_1(x) = 2\tanh(x/2)$ as $q \to 1^-$. From (4.1),

$$A_q(x) = x - \frac{1-q}{2(1+q)}x^2 + O(x^3), \qquad A_q(0) = 0, \quad A_q'(0) = 1,$$

so q controls the local asymmetry while the normalization is preserved, and $|A_q| < 2$ on $(-1, 1)$ by Theorem 2.4. The associated q-sensitivity is $D_q A_q(x) = \big(A_q(qx) - A_q(x)\big)/\big((q-1)x\big)$. This gives a one-parameter family of normalized, bounded,

centred activation-type functions with tunable curvature, in the spirit of the deformations studied by [10] and [2]; we make no claim about performance in learning tasks, as no empirical evaluation is carried out here.

7. Conclusion

The Euler product representation (2.2) is what makes the rest of the paper possible. It gives univalence and the bound 2 on $\mathbb{U}$ for every $q \in (0,1)$ with nothing assumed about the coefficients, and it yields a recurrence in which every b_n is positive. From that recurrence come the closed forms $-q^2/([2]_q^2[3]_q)$ and $-q^9/([2]_q^2[3]_q^3[4]_q^2[5]_q)$, each increasing strictly in q towards $1/12$ and $1/8640$.

The next clear problem is the class-level one. Our $H_{3,1}$ is an exact value for a single function; a sharp bound for $|H_{3,1}(f)|$ over the whole of $\mathcal{S}_q^*(Y_q)$ is open, and we expect it to be much harder. Closed forms for $\alpha_S(q)$ and $\alpha_C(q)$ would also be welcome, since Table 1 gives only numerical values, as would uniform convexity and close-to-convexity for Φ_q. Using A_q as an activation function is an open problem which can be subjected to test using the right apparatus.

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