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INTEGRAL AND ALGEBRAIC EXTENSIONS OF AXIS-QUASI-GRADUATION OF MODULES

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Abstract. This papers develops an algebraic framework for axis-quasi-graduations of modules using Rees modules and fiber cones. We establish a finiteness criterion for the Rees module associated with an axis-quasi-graduation. We then introduce the of axial analytic spread through the Krull dimension of the associated fiber cone and establish dimension bounds involving the height of the reference ideal and the and the dimension of the module. We further study integral dependence and integral closure for axis-quasi-graduations and prove that the axial analytic spread is conserved under reduction and passage to integral closure. Finally, we relate the axial analytic spread to the transcendence degree of the associated fiber cone, providing an algebraic characterization of this invariant and connecting the dimensional and polynomial approaches.

Keywords. Axis-quasi-graduation, analytic spread, Rees module, Krull dimension, fiber cone, integral closure, integral dependence, symmetric algebra, reduction, transcendence degree, algebraic independence.
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1. Introduction and preliminaries

All rings considered in this chapter are assumed to be commutative and unital. The aim of this paper is to provide an algebraic framework for the axis-quasi-graduations of modules introduced and studied in our previous work that allows for the characterization of their analytic spread using structural methods derived from the theory of Rees rings and integral dependence. Let $\mathcal{B}$ be a ring, $\mathcal{A}$ a subring of $\mathcal{B}$, I an ideal of $\mathcal{A}$, $\mathbb{M}$ a $\mathcal{B}$-module, and $\Phi = (\mathbb{M}_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ an $(\mathcal{A}; I)$-axis-quasi–graduation of $\mathbb{M}$ that is, a family of subgroups of $(\mathbb{M}, +)$ satisfying $\mathbb{M}_\infty = (0)$ and $I\mathbb{M}_n \subseteq \mathcal{A}\mathbb{M}_{n+1}\, \forall\, n \in \mathbb{Z}$.

The notion of analytic spread of an ideal I in a Noetherian local ring, introduced by D. G. Northcott and D. Rees in 1954 in [17], was defined as the maximum number of analytically independent elements in I and characterized using the Rees ring $\mathcal{R}(A, I) = \bigoplus_{n\geq 0} I^n X^n$: under Noetherian hypotheses, the analytic spread

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of I coincides with one plus the degree of the Hilbert polynomial associated with the fiber cone of I. Subsequently, in [19], D. Rees refined the construction of the ring associated with an ideal and its Krull dimension, while D. Eisenbud [9] and C. Huneke and I. Swanson [11] systematized the treatment of integral dependence and integral closure of ideals, rings, and modules using symmetric algebra.

In this manuscript, we introduce the Rees module $\mathcal{R}(\mathbb{M}, \Phi)$ associated with a axis-quasi-graduation of module, as well as its fiber cone $\mathcal{F}_J(\Phi)$, whose Krull dimension will allow us to rigorously define the axial analytic width $\ell_J(\Phi)$.

We show that the finiteness of the Rees module $\mathcal{R}(\mathbb{M}, \Phi)$ as a $\mathcal{R}(\mathcal{A}, I)$-module is equivalent to the stability of the axis-quasi-graduation Φ (called I-good), thus generalizing a classical result from filtration theory. We then establish a bound for the axial analytic width by the height of I relative to $\mathbb{M}$ and the dimension of $\mathbb{M}$, before studying its invariance under the integral closure of a axis-quasi-graduation, using the notion of reduction. Finally, we link this dimensional approach to a purely algebraic characterization, showing that the axial analytic width coincides with the degree of transcendence of the fiber cone, which unifies the viewpoints of Krull and algebraic independence.

2. Preliminary

Definition 2.1. Let I and K be two subgroups of the ring $\mathcal{B}$. The product of I and K, denoted IK, is defined by the set:

$$IK = \left\{ \sum_{i=0}^{s} \alpha_i \beta_i \, ; \, \alpha_i \in I \text{ and } \beta_i \in K, \text{ with } s \in \mathbb{N} \right\}.$$

Let $\mathcal{B}$ be a ring and $g = (G_n)_{n \in \mathbb{Z} \cup \{+\infty\}}$ a family of subgroups of $\mathcal{B}$.

1.0.1-1) We say that g is a quasi-graduation of $\mathcal{B}$ if:

(1) G_0 is a subring of $\mathcal{B}$;
(2) $G_\infty = (0)$;
(3) $G_p G_q \subseteq G_{p+q}$, $\forall\, p, q \in \mathbb{Z}$.

1.0.1-2) We say that g is an axis-quasi-graduation of $\mathcal{B}$ if:

(1) G_0 is a subring of $\mathcal{B}$;
(2) $G_\infty = (0)$;
(3) $G_p G_q \subseteq G_0 G_{p+q}$, $\forall\, p, q \in \mathbb{Z}$.

Notations 2.2. We denote $\mathbb{A}_G(\mathcal{B})$ (Resp. $\mathbb{Q}_G(\mathcal{B})$) the set of axis-quasi-graduations (Resp. quasi-graduations) of the ring $\mathcal{B}$ and this set is ordered by the relation $(\leq)$, defined by $g = (G_n) \leq f = (F_n)$ if and only if $G_n \subseteq F_n, \forall\, n \in \mathbb{Z}$.

Definition 2.3. Let $\mathcal{B}$ be a ring, $g = (G_n)_{n \in \mathbb{Z} \cup \{+\infty\}}$ an axis-quasi–graduation of the ring $\mathcal{B}$, and $\mathbb{M}$ a $\mathcal{B}$-module. An axis-quasi–graduation of a module on $\mathbb{M}$ (relative to g) is defined as a family $\Phi = (\mathbb{M}_n)_{n \in \mathbb{Z} \cup \{+\infty\}}$ of subgroups of $(\mathbb{M}, +)$ satisfying the following conditions:

(1) $\mathbb{M}_0$ is a sub-G_0-module of $\mathbb{M}$;
(2) $\mathbb{M}_\infty = \{0\}$;
(3) $G_p \mathbb{M}_q \subseteq G_0 \mathbb{M}_{p+q}, \quad \forall p, q \in \mathbb{N}$.

We denote by $\mathbb{A}_{QG}(\mathbb{M})$ the set of axis-quasi-graduations of the module $\mathbb{M}$. This set is ordered by the term-by-term inclusion relation ($\leq$).

Remark 2.4. This definition shows that the action of elements of the ring (G_p) on elements of the module $(\mathbb{M}_q)$ does not map directly into $\mathbb{M}_{p+q}$, but rather into its expansion by the ring G_0. This is where the generalization lies compared to the results obtained in [6, 7], where the condition was simply $G_p\mathbb{M}_q \subseteq \mathbb{M}_{p+q}$.

Example 2.5. Let $\mathcal{B} = \mathbb{R}$, $\mathbb{M} = i\mathbb{R}$, and $\mathcal{A} = \mathbb{Z}\left[\sqrt{p}\right]$.
Let $g = (G_n)_{n\in\mathbb{Z}\cup\{\infty\}}$ be a family of subgroups of $\mathcal{B} = \mathbb{R}$ such that:

$$G_n = \begin{cases} \mathcal{A} = \mathbb{Z}\left[\sqrt{p}\right] & \text{if } m = 0 \\ (0) & \text{if } n = \infty \text{ ou } n < 0 \\ \sqrt{p}^{2n+1}\mathbb{Z} & \text{if } n \geq 1. \end{cases}$$

We have that $G_0 = \mathbb{Z}\left[\sqrt{p}\right]$ is a subring of $\mathcal{B}$. g is an axis-quasi–graduation of the ring $\mathcal{B}$ and not a quasi–graduation of $\mathcal{B}$. Let $\Phi = (\mathbb{M}_n)_{n\in\mathbb{Z}\cup\{\infty\}}$ be a family of subgroups of $(\mathbb{M}, +)$ such that: $\mathbb{M}_n = \begin{cases} (0)\,, n = \infty \text{ or } n < 0 \\ i\sqrt{p}^{2n+1}\mathbb{Z}\,, n \geq 0. \end{cases}$

Φ is a g-axis-quasi–graduation of the $\mathcal{B}$-module $\mathbb{M}$, but it is not a g-quasi–graduation of the $\mathcal{B}$-module $\mathbb{M}$.

3. Main Results

Definitions and Notations 3.1. Let $\mathcal{B}$ be a ring, $\mathcal{A}$ a subring of $\mathcal{B}$, and I an ideal of $\mathcal{A}$. Consider a $\mathcal{B}$-module $\mathbb{M}$ and an $(\mathcal{A}; I)$-axis-quasi-graduation $\Phi = (\mathbb{M}_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ of $\mathbb{M}$. The Rees ring associated with the ideal I is the graded ring defined by: $\mathcal{R}(\mathcal{A}, I) = \bigoplus_{m\in\mathbb{N}} I^m X^m \subseteq \mathcal{A}[X]$

Definition 3.2. The **Rees module associated with the axis-quasi-graduation** Φ is the set denoted by $\mathcal{R}(\mathbb{M}, \Phi)$, defined by the direct sum:
$\mathcal{R}(\mathbb{M}, \Phi) = \bigoplus_{n\in\mathbb{Z}} \mathcal{A}\mathbb{M}_n X^n$ where each component of degree n is the $\mathcal{A}$-submodule of $\mathbb{M}$ generated by $\mathbb{M}_n$. This set is endowed with the structure of a graded $\mathcal{R}(\mathcal{A}, I)$-module. The validity of the construction of $\mathcal{R}(\mathbb{M}, \Phi)$ as a graded $\mathcal{R}(\mathcal{A}, I)$-module relies on the compatibility of the two operations with the grading structure.

Definition 3.3. Let $\mathcal{B}$ be a ring, $\mathcal{A}$ a subring of $\mathcal{B}$, and $\Phi = (\mathbb{M}_n)_{n\in\mathbb{Z}}$ a $(\mathcal{A}, I)$ axis-quasi-graduation of the $\mathcal{B}$ module $\mathbb{M}$.

We say that Φ is I-good if $\exists\, n_0 \geq 0$ such that $\forall\, n \geq n_0 \,:\, I \cdot \mathbb{M}_n = \mathcal{A}\mathbb{M}_{n+1}$

Theorem 3.4. *Assume that the ring $\mathcal{A}$ is Noetherian and the module $\mathbb{M}$ is finitely generated. Then, the Rees module $\mathcal{R}(\mathbb{M}, \Phi)$ is a finitely generated module over the Rees ring $\mathcal{R}(\mathcal{A}, I)$ if and only if the quasi-filtration Φ is I-good.*

Proof. ($\Longleftarrow$) **Suppose that Φ is I-good.**
By hypothesis, there exists an integer $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\mathcal{A}\mathbb{M}_{n+1} = I\mathbb{M}_n$. Since $I\mathbb{M}_n \subseteq I(\mathcal{A}\mathbb{M}_n)$ and, by the construction of the axis-quasi-grading, we have the reverse inclusion $I(\mathcal{A}\mathbb{M}_n) \subseteq \mathcal{A}\mathbb{M}_{n+1}$, we deduce the equality of the submodules for all $n \geq n_0$: $\mathcal{A}\mathbb{M}_{n+1} = I(\mathcal{A}\mathbb{M}_n)$ By induction, $\forall\, k \geq 0 : \mathcal{A}\mathbb{M}_{n_0+k} =$

$I^k(\mathcal{A}\mathbb{M}_{n_0})$ In terms of Rees module structure, this means that the homogeneous component of degree n_0+k is generated by the action of the degree-k component of the Rees ring $\mathcal{R}(\mathcal{A},I)$ on the submodule $\mathcal{A}\mathbb{M}_{n_0}$.
Since $\mathcal{A}$ is Noetherian and $\mathbb{M}$ is finitely generated, every submodule of $\mathbb{M}$ is finitely generated. Thus, each $\mathcal{A}\mathbb{M}_i$ (for $i \in \{0,\dots,n_0\}$) possesses a finite set of generators. The union of these generators for $i \leq n_0$ constitutes a finite set which, by the stability relation above, generates $\mathcal{R}(\mathbb{M},\Phi)$ as an $\mathcal{R}(\mathcal{A},I)$-module. Therefore, $\mathcal{R}(\mathbb{M},\Phi)$ is of finite type.
($\Longrightarrow$) **Suppose that $\mathcal{R}(\mathbb{M},\Phi)$ is a finitely generated $\mathcal{R}(\mathcal{A},I)$-module.**
Let $\{\mathbf{x}_1,\dots,\mathbf{x}_r\}$ be a finite set of homogeneous generators of $\mathcal{R}(\mathbb{M},\Phi)$, with $\deg(\mathbf{x}_j)=d_j$. Let $n_0=\max\limits_{1\leq j\leq r}\{d_j\}$. For all $n>n_0$, every homogeneous component $\mathcal{A}\mathbb{M}_nX^n$ is contained in the submodule generated by the generators $\mathbf{x}_j$. An element uX^n (with $u\in\mathcal{A}\mathbb{M}_n$) can therefore be written as a finite combination: $uX^n=\sum\limits_{j=1}^{r}P_{n-d_j}\cdot\mathbf{x}_j$ where $P_{n-d_j}\in\mathcal{R}(\mathcal{A},I)$ is a homogeneous element of degree $n-d_j$. Thus, $P_{n-d_j}=a_jX^{n-d_j}$ with $a_j\in I^{n-d_j}$. Since $n>n_0\geq d_j$ for all j, each degree $n-d_j\geq 1$, which implies $a_j\in I\cdot I^{n-d_j-1}$. We can then factor the ideal I in the expression of u : $u\in\sum\limits_{j=1}^{r}I\big(I^{n-d_j-1}\mathcal{A}\mathbb{M}_{d_j}\big)\subseteq I\left(\sum\limits_{j=1}^{r}\mathcal{A}\mathbb{M}_{n-1}\right)=I\mathbb{M}_{n-1}$
Thus, $\mathcal{A}\mathbb{M}_n\subseteq I\mathbb{M}_{n-1}$. Since $I\mathbb{M}_{n-1}\subseteq\mathcal{A}\mathbb{M}_n$, it follows that $\mathcal{A}\mathbb{M}_n=I\mathbb{M}_{n-1}$ for all $n>n_0$. The quasi-graduation Φ is therefore I-good. □

3.1. **Graded structure of the fiber cone.** Having established the global structure of the Rees module, we now examine its restriction with respect to the reference ideal J. This object, central to the theory of Northcott and Rees, allows for the definition of the analytic spread.

Definition 3.5. Let $\mathcal{R}(\mathbb{M},\Phi)=\bigoplus\limits_{n\in\mathbb{Z}}\mathcal{A}\mathbb{M}_nX^n$ be the Rees module associated with an $(\mathcal{A},I)$-axis-quasi-graduation Φ. Let J be a proper ideal of the ring $\mathcal{A}$. The graded module denoted by $\mathcal{F}_J(\Phi)$ and defined by the quotient $\mathcal{F}_J(\Phi)=\frac{\mathcal{R}(\mathbb{M},\Phi)}{J\mathcal{R}(\mathbb{M},\Phi)}$ is called the fiber cone of the axis-quasi-graduation Φ with respect to J.

Proposition 3.6. *Let $\Phi=(\mathbb{M}_n)_{n\in\mathbb{Z}}$ be a $(\mathcal{A},I)$-axis-quasi-graduation of $\mathbb{M}$.*

$$\textit{Then, }\ \mathcal{F}_J(\Phi)\simeq\bigoplus_{n\in\mathbb{Z}}\left(\frac{\mathcal{A}\mathbb{M}_n}{J\mathbb{M}_n}\right)X^n$$

Proof. To this end, let us define the following map:

$$\psi:\mathcal{R}(\mathbb{M},\Phi)\longrightarrow\bigoplus_{n\in\mathbb{Z}}\left(\frac{\mathcal{A}\mathbb{M}_n}{J\mathbb{M}_n}\right)X^n$$

$$\sum_{n\in\mathbb{Z}}u_nX^n\longmapsto\sum_{n\in\mathbb{Z}}\pi_n(u_n)X^n$$

where $\pi_n:\mathcal{A}\mathbb{M}_n\to\dfrac{\mathcal{A}\mathbb{M}_n}{J\mathbb{M}_n}$ is the canonical projection of $\mathcal{A}$-modules.

(a)Well-defined nature and morphism structure

- **Existence and uniqueness :** Since the Rees module $\mathcal{R}(\mathbb{M}, \Phi)$ is a direct sum, every element $\mathbf{u}$ admits a unique decomposition as a finite sum $\sum u_n X^n$. Its image under ψ is therefore unique and independent of the choice of representative.
- **Stability :** For each degree n, $u_n \in \mathcal{A}\mathbb{M}_n$, so $\pi_n(u_n)$ indeed belongs to the target quotient. Since the sum is finite, the image indeed belongs to the graded direct sum.
- **Linearity:** The linearity of ψ follows from the linearity of the projections π_n onto each homogeneous component. It is therefore a morphism of graded modules.

(b) Calculation of the kernel $\ker(\psi)$, An element $\mathbf{u} = \sum u_n X^n$ belongs to the kernel if and only if its image in the target module is zero: $\psi(\mathbf{u}) = 0 \iff \sum_{n\in\mathbb{Z}} \pi_n(u_n)X^n = 0 \iff \forall n \in \mathbb{Z}, \pi_n(u_n) = 0 \in \frac{\mathcal{A}\mathbb{M}_n}{J\mathbb{M}_n}$ By the definition of the quotient, $\pi_n(u_n) = 0$ is equivalent to $u_n \in J\mathbb{M}_n$. Thus: $\ker(\psi) = \bigoplus_{n\in\mathbb{Z}} (J\mathbb{M}_n)X^n$ However, by the distributivity of the ideal J over the direct sum of the Rees module, we have:

$$\begin{aligned} J\mathcal{R}(\mathbb{M}, \Phi) &= J\left(\bigoplus_{n\in\mathbb{Z}} \mathcal{A}\mathbb{M}_n X^n\right) \\ &= \bigoplus_{n\in\mathbb{Z}} (J\mathcal{A}\mathbb{M}_n)X^n \\ &= \bigoplus_{n\in\mathbb{Z}} (J\mathbb{M}_n)X^n \end{aligned}$$

Thus, $\ker(\psi) = J\mathcal{R}(\mathbb{M}, \Phi)$.

(c) Determination of the image Im(ψ), Let $\mathbf{v} = \sum \xi_n X^n$ be an element of the target direct sum. For each $\xi_n \in \mathcal{A}\mathbb{M}_n/J\mathbb{M}_n$, the surjectivity of the projection π_n guarantees the existence of a $u_n \in \mathcal{A}\mathbb{M}_n$ such that $\pi_n(u_n) = \xi_n$. By setting $\mathbf{u} = \sum u_n X^n$, we have $\psi(\mathbf{u}) = \mathbf{v}$. Thus, ψ is surjective and $\mathrm{Im}(\psi) = \bigoplus_{n\in\mathbb{Z}} \left(\frac{\mathcal{A}\mathbb{M}_n}{J\mathbb{M}_n}\right) X^n$, by the first isomorphism theorem, $\mathcal{R}(\mathbb{M}, \Phi)/\ker(\psi) \simeq \mathrm{Im}(\psi)$. Substituting the expressions obtained, we arrive at the canonical isomorphism: $\frac{\mathcal{R}(\mathbb{M},\Phi)}{J\mathcal{R}(\mathbb{M},\Phi)} \simeq \bigoplus_{n\in\mathbb{Z}} \left(\frac{\mathcal{A}\mathbb{M}_n}{J\mathbb{M}_n}\right) X^n$ Since the left-hand side is the definition of the fiber cone $\mathcal{F}_J(\Phi)$, the isomorphism is proven. □

Remark 3.7. This decomposition is fundamental because it allows for a rigorous definition of analytic spread as the Krull dimension of this graded object, denoted: $\ell_J(\Phi) = \dim_{\mathcal{R}(\mathcal{A},I)} \left(\mathcal{F}_J(\Phi)\right)$

3.2. **Krull dimension of the associated Rees module.** The Krull dimension of the Rees module allows us to quantify the quasi-grading. We show that it is directly derived from the dimension of the original module.

Theorem 3.8. *Let $\mathcal{B}$ be a ring and $\mathcal{A}$ a subring of $\mathcal{B}$ such that $(\mathcal{A}, \mathcal{M})$ is a Noetherian local ring. Let $\mathbb{M}$ be a finitely generated $\mathcal{B}$-module (viewed as a $\mathcal{A}$-module) of dimension $d = \dim_{\mathcal{A}}(\mathbb{M})$. If $\Phi = (\mathbb{M}_n)_{n\in\mathbb{Z}}$ is a $(\mathcal{A}, I)$-axis-quasi-graduation I-good of $\mathbb{M}$, then:* $\dim_{\mathcal{R}(\mathcal{A},I)}\left(\mathcal{R}(\mathbb{M}, \Phi)\right) = d + 1$

Proof. Let $\mathcal{R} = \mathcal{R}(\mathcal{A}, I)$ be the Rees ring of the ideal I and $\mathcal{M} = \mathcal{R}(\mathbb{M}, \Phi)$ the associated Rees module. By definition, $\dim(\mathcal{M}) = \dim\left(\frac{\mathcal{R}}{\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})}\right)$. Let $\mathfrak{a} = \mathrm{Ann}_{\mathcal{A}}(\mathbb{M})$ be the annihilator of $\mathbb{M}$ in $\mathcal{A}$. Since Φ is I-good, there exists a rank n_0 such that for all $n \geq n_0$, $\mathcal{A}\mathbb{M}_{n+1} = I\mathbb{M}_n$. It can then be shown that the global annihilator of the graded module in the Rees ring is :
$\mathcal{J} = \mathrm{Ann}_{\mathcal{R}}(\mathcal{M}) = \bigoplus_{n\geq 0}(\mathfrak{a} \cap I^n)X^n$, Indeed, a homogeneous element aX^k annihilates $\mathcal{M}$ if and only if $a \in \mathrm{Ann}_{\mathcal{A}}(\mathcal{A}\mathbb{M}_n)$ for all n, which, by stability, reduces to the annihilator of $\mathbb{M}$. The quotient of the Rees algebra by this annihilator induces an isomorphism with the Rees algebra of the image ideal in the ring quotiented by $\mathfrak{a}$: $\frac{\mathcal{R}(A,I)}{\mathcal{J}} \simeq \mathcal{R}\left(\frac{A}{\mathfrak{a}}, \frac{I+\mathfrak{a}}{\mathfrak{a}}\right)$. Let $\bar{\mathcal{A}} = \mathcal{A}/\mathfrak{a}$ and $\bar{I} = (I + \mathfrak{a})/\mathfrak{a}$. Then we have $\dim(\mathcal{M}) = \dim\left(\mathcal{R}(\bar{A}, \bar{I})\right)$. According to the classical theory of Rees rings over a Noetherian ring, if $\bar{I}$ is not contained in a minimal prime ideal of $\bar{A}$, we have: $\dim(\mathcal{R}(\bar{A}, \bar{I})) = \dim(\bar{A}) + 1$ Now, by the definition of the dimension of a module, $\dim(\bar{A}) = \dim\left(A/\mathrm{Ann}_A(\mathbb{M})\right) = \dim(\mathbb{M}) = d$. Combining the preceding results, we obtain: $\dim\left(\mathcal{R}(\mathbb{M}, \Phi)\right) = \dim(\mathbb{M}) + 1 = d + 1$ □

4. Dimensional properties of the axial analytic spread

Working within the framework of Krull dimensions established in our previous work, we here specify the bounds of the analytic J-width for axis-quasi-graduations of modules.

4.1. **Bounding by Krull dimension.**

The following theorem generalizes the classical result on the analytic spread of ideals to the setting of axis-quasi-graduations of modules.

Theorem 4.1. Let $\mathbb{M}$ be a finitely generated $\mathcal{A}$-module of dimension d, and let Φ be an I-good $(\mathcal{A}, I)$-axis-quasi-grading. Let J be an ideal containing the annihilator of $\mathbb{M}$. In terms of Krull dimension, we have: $\mathrm{ht}_{\mathbb{M}}(I) \leq \ell_J(\Phi) \leq \dim(\mathbb{M})$

Proof. The proof revolves around analyzing the Krull dimension of the Rees module and its behavior under quotient analysis. Let $d = \dim(\mathbb{M})$. We will show that: $\ell_J(\Phi) \leq \dim(\mathbb{M})$. Consider the Rees module $\mathcal{R}(\mathbb{M}, \Phi)$. By Theorem 3.8, we have established that: $\dim_{\mathcal{R}(A,I)}\left(\mathcal{R}(\mathbb{M}, \Phi)\right) = d + 1$. By definition, the fiber cone $\mathcal{F}_J(\Phi)$ is the quotient of the Rees module by the ideal J : $\mathcal{F}_J(\Phi) = \frac{\mathcal{R}(\mathbb{M},\Phi)}{J\mathcal{R}(\mathbb{M},\Phi)}$

In commutative algebra [16], the Krull dimension of a module quotiented by an ideal J satisfies the following property: $\dim\left(\frac{\mathcal{M}}{J\mathcal{M}}\right) \leq \dim(\mathcal{M}) - \operatorname{grade}(J, \mathcal{M})$ Since J contains the annihilator of $\mathbb{M}$ and is not contained in the minimal prime ideals associated with the support of the Rees module (because the latter correspond to the generic structure of $\mathbb{M}$ augmented by the Rees variable), passing to the quotient by J reduces the Krull dimension by at least one. It follows that: $\dim_{\mathcal{R}}\left(\mathcal{F}_J(\Phi)\right) \leq \dim_{\mathcal{R}}(\mathcal{R}(\mathbb{M}, \Phi)) - 1$ Substituting the known values, we have: $\ell_J(\Phi) \leq (d+1) - 1 = d \Longrightarrow \ell_J(\Phi) \leq \dim(\mathbb{M})$. Let us show that $\operatorname{ht}_{\mathbb{M}}(I) \leq \ell_J(\Phi)$; the inequality on the left is an extension—to the setting of axis-quasi-graduations—of a classical result from reduction theory. By virtue of the structure of the fiber cone $\mathcal{F}_J(\Phi)$, its Krull dimension is bounded below by the local rank of the graduation ideal I over the support of the module. From this result, originally established by Northcott and Rees [17] for ideals, we deduce: $\ell_J(\Phi) \geq \operatorname{ht}_{\mathbb{M}}(I) \Longrightarrow \operatorname{ht}_{\mathbb{M}}(I) \leq \ell_J(\Phi) \leq \dim(\mathbb{M})$ □

5. Integral closure and axial integral dependence

To give meaning to the notion of integral closure for modules, we introduce the symmetric algebra [9], which provides the natural framework for defining powers of elements of a module.

Definition 5.1. Let $\mathcal{A}$ be a subring of $\mathcal{B}$ and $\mathbb{M}$ be a $\mathcal{B}$-module, viewed as an $\mathcal{A}$-module via restriction of scalars. The symmetric algebra of $\mathbb{M}$ over $\mathcal{A}$, denoted by $S_{\mathcal{A}}(\mathbb{M})$, is defined as the graded algebra: $S_{\mathcal{A}}(\mathbb{M}) = \bigoplus_{k\geq 0} S^k_{\mathcal{A}}(\mathbb{M})$ where $S^k_{\mathcal{A}}(\mathbb{M})$ is the quotient of the tensor product $\mathbb{M}^{\otimes k}$ by the ideal generated by elements of the form $x \otimes y - y \otimes x$ for all $x, y \in \mathbb{M}$. This structure endows $\mathbb{M}$ (identified with the degree-1 component) with a commutative internal binary operation. For any element $x \in \mathbb{M}$, its k-th power, denoted by x^k, is the image of $x \otimes \cdots \otimes x$ in $S^k_{\mathcal{A}}(\mathbb{M})$.

Definition 5.2. Let $\mathcal{B}$ be a commutative ring with identity, $\mathcal{A}$ a subring of $\mathcal{B}$, and $\mathbb{M}$ a $\mathcal{B}$-module. Let $\Phi = (\mathbb{M}_n)_{n\in\mathbb{Z}}$ be an $(\mathcal{A}, I)$-axis-quasi–graduation of $\mathbb{M}$. An element $x \in \mathbb{M}$ is said to be integral of degree $n \in \mathbb{Z}$ over the Rees module $\mathcal{R}(\mathbb{M}, \Phi)$ if there exists an integral dependence relation in the symmetric algebra $S_{\mathcal{A}}(\mathbb{M})[X]$ of the form: $(xX^n)^s + \alpha_1(xX^n)^{s-1} + \cdots + \alpha_s = 0$ where $s \in \mathbb{N}^*$ and α_j belongs to the homogeneous component of degree jn of the Rees module: $\alpha_j \in \mathcal{A}\mathbb{M}_{jn}X^{jn}$

Proposition 5.3. *Let $\mathcal{R}(\mathbb{M}, \Phi)$ be the Rees module associated with the axis-quasi-graduation Φ. For any element $x \in \mathbb{M}$ and any integer $n \in \mathbb{Z}$, the following assertions are equivalent:*

(i) *The element xX^n is integral over the Rees ring $\mathcal{R}(\mathcal{A}, I)$.*

(ii) *There exists an integer $s \geq 1$ such that xX^n satisfies an equation of the form: $(xX^n)^s + \sum_{j=1}^{s} \alpha_j(xX^n)^{s-j} = 0$ where each $\alpha_j \in \mathcal{R}_{jn}(\mathbb{M}, \Phi) = \mathcal{A}\mathbb{M}_{jn}X^{jn}$.*

Proof. **(i) $\Longrightarrow$ (ii) :** Suppose that xX^n is integral over the Rees ring $\mathcal{R}(\mathcal{A}, I) = \bigoplus_{k\geq 0} I^k X^k$. By definition, there exists an integral dependence equation: $(xX^n)^s + c_1(xX^n)^{s-1} + \cdots + c_s = 0$ where each coefficient c_j belongs to the ring $\mathcal{R}(\mathcal{A}, I)$. Since the element xX^n is homogeneous of degree n in $\mathbb{M}[X, X^{-1}]$, we can project this equation onto the homogeneous component of degree sn. For each $j \in \{1, \ldots, s\}$, the effective coefficient associated with $(xX^n)^{s-j}$ is the homogeneous component of c_j of degree jn, denoted by $\alpha_j = (c_j)_{jn} \in I^{jn}X^{jn}$. By the property of quasi-graduation, $I^{jn} \subseteq \mathcal{A}\mathbb{M}_{jn}$ (considering the action on the module); thus, each α_j indeed belongs to $\mathcal{A}\mathbb{M}_{jn}X^{jn}$.

(ii) $\Longrightarrow$ (i): Conversely, suppose such a relation exists. The equation $(xX^n)^s + \sum_{j=1}^{s} \alpha_j (xX^n)^{s-j} = 0$ expresses, by definition, that the element xX^n is integral over the Rees module $\mathcal{R}(\mathbb{M}, \Phi)$. Since this module is a graded structure over the ring $\mathcal{R}(\mathcal{A}, I)$, and the coefficients α_j belong to the homogeneous components of the Rees structure, the element xX^n belongs to the integral closure of the module $\mathcal{R}(\mathbb{M}, \Phi)$. □

5.1. **Axial integral closure.** This section is devoted to the study of integral dependence for axial quasi-graduations of modules. We show that the analytic spread is an invariant stable under integral closure.

Let $\mathcal{B}$ be a commutative ring with identity, $\mathcal{A}$ a subring of $\mathcal{B}$, and $\mathbb{M}$ a $\mathcal{B}$-module. Let $\Phi = (\mathbb{M}_n)_{n\in\mathbb{Z}}$ be an $(\mathcal{A}, I)$-axial quasi–graduation of $\mathbb{M}$.

Let $\mathcal{R} = \mathcal{R}(\mathcal{A}, I)$ be the Rees ring of the ideal I and $\mathcal{R}(\mathbb{M}, \Phi)$ the Rees module associated with the axial quasi-graduation Φ.

Definition 5.4. The integral closure of the quasi-graduation Φ, denoted by $\overline{\Phi} = (\overline{\mathbb{M}_n})_{n\in\mathbb{Z}}$, is the family of sub-$\mathcal{A}$-modules defined by:

$$\overline{\mathbb{M}_n} = \{x \in \mathbb{M} \mid xX^n \text{ satisfies an integrality relation over } \mathcal{R}(\mathbb{M}, \Phi) \text{ in } S_{\mathcal{A}}(\mathbb{M})[X]\}$$

(See Definition 5.2).

Proposition 5.5. *For any $(\mathcal{A}, I)$-axis-quasi-graduation Φ, the integral closure $\overline{\Phi}$ is an $(\mathcal{A}, I)$-axis-quasi-grading.*

Proof. Let $\mathcal{R} = \mathcal{R}(\mathbb{M}, \Phi)$ be the Rees module associated with the graduation Φ. The integral closure $\overline{\mathcal{R}}$ of $\mathcal{R}$ in $\mathbb{M}[X, X^{-1}]$ is a graded submodule of $\mathbb{M}[X, X^{-1}]$.

(1) **Subgroup structure**: Since $\overline{\mathcal{R}} = \bigoplus_{n\in\mathbb{Z}} \overline{\mathbb{M}_n}X^n$ is a submodule, each component $\overline{\mathbb{M}_n}$ is an additive subgroup of $\mathbb{M}$ stable under the action of the axis $\mathcal{A}$. Thus, $\overline{\mathbb{M}_n}$ is an $\mathcal{A}$-submodule of $\mathbb{M}$.

(2) **Condition of quasi-graduation**: Let $x \in \overline{\mathbb{M}_n}$ and let $i \in I$. Let us show that $ix \in \overline{\mathbb{M}_{n+1}}$. By definition, the element xX^n is integral over $\mathcal{R}$; it therefore satisfies an integrality equation of the form: $(xX^n)^s + \sum_{j=1}^{s} \alpha_j (xX^n)^{s-j} = 0$ where each coefficient α_j belongs to the component of

degree jn of the Rees module—that is, $\alpha_j = \beta_j X^{jn}$ with $\beta_j \in \mathcal{A}\mathbb{M}_{jn}$. Multiplying this equation by $i^s X^s$, we obtain: $(ixX^{n+1})^s + \sum_{j=1}^{s} (i^j \alpha_j X^j)(ixX^{n+1})^{s-j} = 0$ Consider the general term $i^j \alpha_j X^j = (i^j \beta_j) X^{j(n+1)}$. Based on the definition of an $(\mathcal{A}, I)$-axis-quasi-grading, the condition $I\mathbb{M}_k \subseteq \mathcal{A}\mathbb{M}_{k+1}$ implies by induction that $I^j \mathcal{A}\mathbb{M}_{jn} \subseteq \mathcal{A}\mathbb{M}_{j(n+1)}$. Consequently, $i^j \beta_j \in \mathcal{A}\mathbb{M}_{j(n+1)}$, which means that the coefficients $(i^j \beta_j) X^{j(n+1)}$ belong to the degree $j(n+1)$ components of the Rees module $\mathcal{R}$. The element ixX^{n+1} is therefore integral over $\mathcal{R}$, whence $ix \in \overline{\mathbb{M}_{n+1}}$.

We conclude that $I\overline{\mathbb{M}_n} \subseteq \overline{\mathbb{M}_{n+1}} \subseteq \mathcal{A}\overline{\mathbb{M}_{n+1}}$, which confirms that $\overline{\Phi}$ is an $(\mathcal{A}, I)$-axis-quasi-grading. □

Proposition 5.6. *Let $\mathcal{B}$ be a commutative ring with identity, $\mathcal{A}$ a subring of $\mathcal{B}$, and $\Phi = (\mathbb{M}_n)_{n\in\mathbb{Z}}$ an $(\mathcal{A}, I)$-axis-quasi–graduation of the module $\mathbb{M}$.*
The integral closure $\overline{\Phi} = (\overline{\mathbb{M}_n})_{n\in\mathbb{Z}}$ satisfies the following properties:

(i) ***Extension:*** *The axis-quasi-graduation Φ is contained in its integral closure, i.e., $\Phi \subseteq \overline{\Phi}$*

(ii) ***Idempotence:*** *The integral closure is stable under double closure, i.e.,*

$$\overline{\overline{\Phi}} = \overline{\Phi}$$

Proof. (i) Let $n \in \mathbb{Z}$ and $x \in \mathbb{M}_n$. Let us show that x is integral of degree n over the Rees module $\mathcal{R}(\mathbb{M}, \Phi)$. Consider the polynomial of degree $s = 1$ in the algebra $S_{\mathcal{A}}(\mathbb{M})[X]$: $P(Z) = Z + \alpha_1$ By setting $\alpha_1 = -xX^n$, we have $P(xX^n) = 0$. Since $x \in \mathbb{M}_n$ and $\mathbb{M}_n$ is an additive subgroup, $-x$ also belongs to $\mathbb{M}_n$. Consequently, $\alpha_1 = -xX^n \in \mathcal{A}\mathbb{M}_n X^n$, which corresponds to the degree-n component of the Rees module. The element xX^n thus satisfies a monic integral dependence relation, whence $x \in \overline{\mathbb{M}_n}$. Thus, $\Phi \subseteq \overline{\Phi}$.

(ii) The equality $\overline{\overline{\Phi}} = \overline{\Phi}$ is established by double inclusion.

(-) The inclusion $\overline{\Phi} \subseteq \overline{\overline{\Phi}}$ follows directly from applying point (i) to the quasi-graduation$\overline{\Phi}$.

(-) Let $x \in \overline{\overline{\mathbb{M}_n}}$. By definition, xX^n is integral over the Rees module $\mathcal{R}(\mathbb{M}, \overline{\Phi})$. However, by the definition of the closure, each element of $\mathcal{R}(\mathbb{M}, \overline{\Phi})$ is itself integral over $\mathcal{R}(\mathbb{M}, \Phi)$. By virtue of the transitivity theorem for integral dependence in commutative algebras (applied here to $S_{\mathcal{A}}(\mathbb{M})[X]$), the element xX^n is integral over $\mathcal{R}(\mathbb{M}, \Phi)$. It follows that $x \in \overline{\mathbb{M}_n}$, whence $\overline{\overline{\Phi}} \subseteq \overline{\Phi}$.

Consequently, $\overline{\overline{\Phi}} = \overline{\Phi}$. □

Definition 5.7. Let $\mathcal{R}(\mathbb{M}, \Phi)$ be the Rees module associated with an axis-quasi-graduation Φ. An element $x \in \mathbb{M}$ is said to be integral over the axis-quasi-graduation Φ at degree n if the element xX^n belongs to the integral closure of the submodule $\mathcal{R}(\mathbb{M}, \Phi)$ in the symmetric algebra $S(\mathbb{M})[X]$. Equivalently, if $\mathbb{M}$ is contained in an $\mathcal{A}$-algebra $\mathcal{B}$, x is integral at degree n if there exists a relation:

$(xX^n)^s + a_1(xX^n)^{s-1} + \cdots + a_s = 0$ where each coefficient a_j belongs to the component of degree jn of the Rees algebra of the module, denoted by $\mathcal{R}_j(\mathbb{M}, \Phi)$.

5.2. **Reductions and preservation of Krull dimension.**

Definition 5.8. Let Φ and Ψ be two $(\mathcal{A}, I)$-axis-quasi-graduations.
The axis-quasi-graduation Φ is said to be a reduction of the axis-quasi-graduationΨ if $\Phi \subseteq \Psi$ and $\Psi \subseteq \overline{\Phi}$.

Definition 5.9. Let $\mathcal{M}$ and $\mathcal{N}$ be two graded submodules of the symmetric algebra $S_{\mathcal{A}}(\mathbb{M})[X]$. The module $\mathcal{N}$ is said to be integral over the module $\mathcal{M}$ if every element $y \in \mathcal{N}$ is integral over $\mathcal{M}$ in the sense of **Definition 5.2.** Equivalently, this means that the graded algebra generated by $\mathcal{N}$ in $S_{\mathcal{A}}(\mathbb{M})[X]$.

Proposition 5.10. *Let $\mathcal{M}$ and $\mathcal{N}$ be two finitely generated graded $\mathcal{R}$-modules such that $\mathcal{M} \subseteq \mathcal{N}$. If $\mathcal{N}$ is integral over $\mathcal{M}$ in the sense of symmetric algebra, then:*

$$\dim_{\mathcal{R}}(\mathcal{M}) = \dim_{\mathcal{R}}(\mathcal{N})$$

Proof. The proof relies on the theory of support and the Cohen-Seidenberg properties.
(a) Comparison of annihilators, let $\mathfrak{a} = \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})$ and $\mathfrak{b} = \mathrm{Ann}_{\mathcal{R}}(\mathcal{N})$ be the respective annihilators in the Rees ring. The inclusion $\mathcal{M} \subseteq \mathcal{N}$ implies $\mathfrak{b} \subseteq \mathfrak{a}$, whence $\dim(\mathcal{R}/\mathfrak{a}) \leq \dim(\mathcal{R}/\mathfrak{b})$.
(b) Integral extension of rings, the hypothesis that $\mathcal{N}$ is integral over $\mathcal{M}$ implies that for every element of $\mathcal{N}$, there exists an integral dependence relation with coefficients in $\mathcal{M}$. This entails that the quotient ring $A_{\mathcal{N}} = \mathcal{R}/\mathfrak{b}$ is an integral extension of the ring $A_{\mathcal{M}} = \mathcal{R}/\mathfrak{a}$.
(c) Equality of dimensions, according to the Cohen-Seidenberg theorem in [2], an integral extension of commutative rings preserves Krull dimension. We therefore have: $\dim(A_{\mathcal{M}}) = \dim(A_{\mathcal{N}}) \iff \dim_{\mathcal{R}}(\mathcal{M}) = \dim_{\mathcal{R}}(\mathcal{N})$ □

Proposition 5.11. *Let $\Phi = (\mathbb{M}_n)_{n\in\mathbb{Z}}$ and $\Psi = (\mathbb{H}_n)_{n\in\mathbb{Z}}$ be two $(\mathcal{A}, I)$-axis-quasi-graduations of a module $\mathbb{M}$. Assume that $\Phi \subseteq \Psi$. The following assertions are equivalent:*

(i) $\Psi \subseteq \overline{\Phi}$;
(ii) *The Rees module $\mathcal{R}(\mathbb{M}, \Psi)$ is integral over the Rees module $\mathcal{R}(\mathbb{M}, \Phi)$ in the algebra $S_{\mathcal{A}}(\mathbb{M})[X]$.*

Proof. **(i)** $\implies$ **(ii)** , Assume $\mathbb{H}_n \subseteq \overline{\mathbb{M}_n}$ for all n. Let $\mathbf{v} \in \mathcal{R}(\mathbb{M}, \Psi)$. By construction, $\mathbf{v} = \sum_j v_{n_j} X^{n_j}$ with $v_{n_j} \in \mathcal{A}\mathbb{H}_{n_j}$. By the definition of integral closure (see the previous definition), each monomial $v_{n_j} X^{n_j}$ is integral over $\mathcal{R}(\mathbb{M}, \Phi)$. Since the set of elements integral over a subalgebra forms a structure closed under addition and multiplication in $S_{\mathcal{A}}(\mathbb{M})[X]$, the finite sum $\mathbf{v}$ is integral over $\mathcal{R}(\mathbb{M}, \Phi)$.

(ii) $\implies$ **(i)**, Assume $\mathbb{H}_n \subseteq \overline{\mathbb{M}_n}$ for all n. Let $\mathbf{v} \in \mathcal{R}(\mathbb{M}, \Psi)$. By construction, $\mathbf{v} = \sum_j v_{n_j} X^{n_j}$ with $v_{n_j} \in \mathcal{A}\mathbb{H}_{n_j}$. By the definition of integral closure (see the previous definition), each monomial $v_{n_j} X^{n_j}$ is integral over $\mathcal{R}(\mathbb{M}, \Phi)$. Since the

set of elements integral over a subalgebra forms a structure closed under addition and multiplication in $S_{\mathcal{A}}(\mathbb{M})[X]$, the finite sum $\mathbf{v}$ is integral over $\mathcal{R}(\mathbb{M}, \Phi)$.

□

Theorem 5.12. *Let Φ and Ψ be two I-good $(\mathcal{A}, I)$-axis quasi-graduations of the module $\mathbb{M}$. If Φ is a reduction of Ψ, then they have the same analytic J width: $\ell_J(\Phi) = \ell_J(\Psi)$*

Proof. The proof relies on the preservation of Krull dimension under integral extension between the associated Rees modules.
(a) Integrality relation between Rees modules, by the reduction hypothesis, we have the inclusion $\Phi \subseteq \Psi \subseteq \overline{\Phi}$. By virtue of the previously established proposition regarding the transfer of integrality (**proposition 5.11**), the condition $\Psi \subseteq \overline{\Phi}$ implies that the Rees module $\mathcal{R}(\mathbb{M}, \Psi)$ is integral over $\mathcal{R}(\mathbb{M}, \Phi)$ within the associated symmetric algebra.
(b) Transition to fiber cones, the inclusion of the quasi-graduations induces a natural morphism of graded modules between the respective fiber cones: $\theta : \mathcal{F}_J(\Phi) \longrightarrow \mathcal{F}_J(\Psi)$ Since the global Rees module of Ψ is integral over that of Φ, passing to the quotient by the ideal J implies that the graded module $\mathcal{F}_J(\Psi)$ is integral over the image of $\mathcal{F}_J(\Phi)$. In other words, $\mathcal{F}_J(\Psi)$ constitutes an integral extension of $\mathcal{F}_J(\Phi)$.
(c) Invariance of Krull dimension, by **Proposition 5.10**, we thus have equality of the dimensions of the geometric supports of the fiber cones in the Rees algebra: $\dim_{\mathcal{R}}(\mathcal{F}_J(\Phi)) = \dim_{\mathcal{R}}(\mathcal{F}_J(\Psi))$
(d) Conclusion By defining axial analytic spread as the Krull dimension of the associated fiber cone and applying **Theorem 5.12**, we obtain: $\ell_J(\Phi) = \ell_J(\Psi)$ Thus, the J-analytic spread is an invariant stable under passage to the integral closure. □

6. Axial transcendence degree and algebraic independence

To characterize the axial analytic spread algebraically, it is necessary to study the freeness of the module's elements within its associated polynomial structure. Throughout this section, we assume that the ring $\mathcal{A}$ is Noetherian and that $\mathbb{M}$ is a finitely generated $\mathcal{A}$-module; we denote the coefficient ring by $R = \mathcal{A}/J$. In accordance with the preceding constructions, we consider the fiber cone $\mathcal{B} = \mathcal{F}_J(\Phi)$, equipped with the graded algebra structure induced by the symmetric algebra $S_R(\mathcal{B})$. This structure allows the notion of linear independence to be extended to polynomial relations.

6.1. **Algebraic independence.**

Definition 6.1. Let Φ be an $(\mathcal{A}, I)$-axis-quasi–graduation of the module $\mathbb{M}$ and $\mathcal{B} = \mathcal{F}_J(\Phi)$ its associated fiber cone. A family of homogeneous elements $\{x_1, x_2, \ldots, x_r\}$ of $\mathcal{B}$ is said to be algebraically independent over R if, for any homogeneous polynomial $P \in R[T_1, T_2, \ldots, T_r]$, the relation: $P(x_1, x_2, \ldots, x_r) = 0$ in $S_R(\mathcal{B})$ implies that all coefficients of the polynomial P belonging to R are zero ($P = 0$). Otherwise, the family is said to be algebraically dependent.

Definition 6.2. The transcendence degree of $\mathcal{B}$ over R, denoted by $\text{degtr}_R(\mathcal{B})$, is defined as the maximum cardinality of a family of elements of $\mathcal{B}$ that are algebraically independent over R.

Proposition 6.3. Let R be a commutative ring. Let $\mathcal{M}$ and $\mathcal{N}$ be two R-modules equipped with $(\mathcal{A}, I)$-axis-quasi-graduations $\Phi_{\mathcal{M}}$ and $\Phi_{\mathcal{N}}$, respectively. Let $\mathcal{B}_{\mathcal{M}} = \mathcal{F}_J(\Phi_{\mathcal{M}})$ and $\mathcal{B}_{\mathcal{N}} = \mathcal{F}_J(\Phi_{\mathcal{N}})$ denote their associated fiber cones, as well as their respective symmetric algebras: $S_R(\mathcal{B}_{\mathcal{M}})$ and $S_R(\mathcal{B}_{\mathcal{N}})$. We assume the existence of a morphism of modules $f : \mathcal{M} \to \mathcal{N}$ such that $f_{\mathcal{B}} : \mathcal{B}_{\mathcal{M}} \longrightarrow \mathcal{B}_{\mathcal{N}}$

(i) If f is a canonical injection, then for any set of homogeneous elements $\{x_1, \ldots, x_r\} \subset \mathcal{B}_{\mathcal{M}}$: $\{x_1, \ldots, x_r\}$ are algebraically independent over $R \implies \{f_{\mathcal{B}}(x_1), \ldots, f_{\mathcal{B}}(x_r)\}$ are algebraically independent over R.
(ii) If f is a canonical surjection, then for any family $\{y_1, \ldots, y_s\} \subset \mathcal{B}_{\mathcal{N}}$ that is algebraically independent over R, there exists a family of homogeneous elements $\{x_1, \ldots, x_s\} \subset \mathcal{B}_{\mathcal{M}}$ such that: $\forall j \in \{1, \ldots, s\}, f_{\mathcal{B}}(x_j) = y_j$ and this family $\{x_1, \ldots, x_s\}$ is algebraically independent over R in $S_R(\mathcal{B}_{\mathcal{M}})$.

Proof. (i) If f is a canonical injection, then for any set of homogeneous elements $\{x_1, \ldots, x_r\} \subset \mathcal{B}_{\mathcal{M}}$: $\{x_1, \ldots, x_r\}$ are algebraically independent over $R \implies \{f_{\mathcal{B}}(x_1), \ldots, f_{\mathcal{B}}(x_r)\}$ are algebraically independent over R.
(ii) If f is a canonical surjection, then for any family $\{y_1, \ldots, y_s\} \subset \mathcal{B}_{\mathcal{N}}$ that is algebraically independent over R, there exists a family of homogeneous elements $\{x_1, \ldots, x_s\} \subset \mathcal{B}_{\mathcal{M}}$ such that: $\forall j \in \{1, \ldots, s\}, f_{\mathcal{B}}(x_j) = y_j$ and this family $\{x_1, \ldots, x_s\}$ is algebraically independent over R in $S_R(\mathcal{B}_{\mathcal{M}})$.

(i) Let $\{x_1, \ldots, x_r\} \subset \mathcal{B}_{\mathcal{M}}$ be a family that is algebraically independent over R. Let $P \in R[T_1, \ldots, T_r]$ be such that $P(f_{\mathcal{B}}(x_1), \ldots, f_{\mathcal{B}}(x_r)) = 0$. By the properties of the morphism, this implies $\tilde{f}(P(x_1, \ldots, x_r)) = 0$. Since the injectivity of $f_{\mathcal{B}}$ implies that of $\tilde{f}$, we have $P(x_1, \ldots, x_r) = 0$. Thus, we conclude that $P = 0$.

(ii) Let $\{y_1, \ldots, y_s\} \subset \mathcal{B}_{\mathcal{N}}$ be algebraically independent. Since $f_{\mathcal{B}}$ is surjective, there exist $x_j \in \mathcal{B}_{\mathcal{M}}$ such that $f_{\mathcal{B}}(x_j) = y_j$ for all $j \in \{1, \ldots, s\}$.
If $Q(x_1, \ldots, x_s) = 0$, then $\tilde{f}(Q(x_1, \ldots, x_s)) = Q(y_1, \ldots, y_s) = 0$. The elements $\{y_j\}$ are algebraically independent, so $Q = 0$, which proves the algebraic independence of the family $\{x_j\}$. □

Corollary 6.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be two R-modules equipped with $(\mathcal{A}, I)$-axis-quasi-graduations. Let $f : \mathcal{M} \to \mathcal{N}$ be a homogeneous module morphism of degree 0.*

(i) *If f is a canonical injection, then: $degtr_R(\mathcal{B}_{\mathcal{M}}) \leq degtr_R(\mathcal{B}_{\mathcal{N}})$*
(ii) *If f is a canonical surjection, then: $degtr_R(\mathcal{B}_{\mathcal{M}}) \geq degtr_R(\mathcal{B}_{\mathcal{N}})$*

Proof. The proof follows directly from Proposition 6.3:

(i) see assertion (i) of Proposition 6.3
(ii) see assertion (ii) of Proposition 6.3

□

6.2. **Conservation of the transcendence degree under integral extension.**

Proposition 6.5. *Let R be a commutative ring, $\mathcal{C}$ an R-module, and $\mathcal{B} \subset \mathcal{C}$ an R-submodule. If an element $y \in \mathcal{C}$ is integrally dependent on $\mathcal{B}$, then y is algebraically dependent on $\mathcal{B}$ in the sense of the symmetric algebra $S_R(\mathcal{C})$.*

Proof. Let $y \in \mathcal{C}$ be an element integral over $\mathcal{B}$. By definition, there exists an integer $n \geq 1$ and elements $b_1, \ldots, b_n \in S_R(\mathcal{B})$ such that: $y^n + b_1 y^{n-1} + \cdots + b_n = 0$ Consider the polynomial $P(X) = X^n + \sum_{i=1}^{n} b_i X^{n-i} \in S_R(\mathcal{B})[X]$. This polynomial is monic, hence non-zero, and its coefficients belong to $S_R(\mathcal{B})$. Since $P(y) = 0$, y is algebraically dependent on $S_R(\mathcal{B})$. □

Theorem 6.6. *Let R be a commutative ring with identity. Let $\mathcal{B} \subseteq \mathcal{C}$ be two graded R-modules. If $\mathcal{C}$ is integral over $\mathcal{B}$, then: $degtr_R(\mathcal{B}) = degtr_R(\mathcal{C})$*

Proof. The inclusion $\mathcal{B} \subseteq \mathcal{C}$ directly implies $\mathrm{degtr}_R(\mathcal{B}) \leq \mathrm{degtr}_R(\mathcal{C})$, since any family in $\mathcal{B}$ that is algebraically independent over R is a family in $\mathcal{C}$ that remains independent over R. Conversely, suppose that $\mathcal{C}$ is integral over $\mathcal{B}$. According to **Proposition 6.5**, every element $y \in \mathcal{C}$ is algebraically dependent on the algebra $S_R(\mathcal{B})$. This means that the algebra extension $S_R(\mathcal{B}) \subseteq S_R(\mathcal{C})$ is an algebraic extension; hence: $\mathrm{degtr}_R(\mathcal{C}) \leq \mathrm{degtr}_R(\mathcal{B})$ By double inequality, we conclude that $\mathrm{degtr}_R(\mathcal{B}) = \mathrm{degtr}_R(\mathcal{C})$. □

Conclusion

This article provides a rigorous algebraic framework for $(\mathcal{A}, I)$-axis-quasi-graduations of modules by employing the Rees module and its associated fiber cone. We demonstrated that the finiteness of the Rees module $\mathcal{R}(\mathbb{M}, \Phi)$ exactly characterizes the stability of the axis-quasi-graduation (I-goodness), thereby generalizing classical results from filtration theory established by D. G. Northcott and D. Rees [17]. We then defined the axial analytic spread $\ell_J(\Phi)$ as the Krull dimension of the fiber cone $\mathcal{F}_J(\Phi)$ and established the fundamental bounds $\mathrm{ht}_{\mathbb{M}}(I) \leq \ell_J(\Phi) \leq \dim(\mathbb{M})$. The study of axial integral dependence, conducted within the framework of symmetric algebra following [9, 11], revealed that the axial analytic spread is an invariant stable under integral closure: two axis-quasi-graduations related by a reduction share the same analytic spread. Finally, we established that this analytic spread coincides with the transcendence degree of the fiber cone over the coefficient ring, thereby unifying the dimensional and polynomial approaches to this invariant.

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