Gulf Journal of Mathematics
Vol 24, Issue 2 (2026) 1-23

ISSN: 2309-4966
https://doi.org/10.56947/a1830862

ANTIADJACENCY AND HERMITIAN ANTIADJACENCY MATRICES OF MIXED GRAPHS

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Abstract. The complement of a graph records the adjacencies absent from the original graph, and its adjacency matrix coincides with the antiadjacency matrix of the original graph. In this paper, this relation is extended to mixed graphs by introducing the Hermitian antiadjacency matrix and proving that it coincides with the Hermitian adjacency matrix of the complement mixed graph. Formulas for its powers, determinant, and characteristic polynomial are obtained, together with spectral relations between a mixed graph and its complement. The antiadjacency and Hermitian antiadjacency matrices are also compared, clarifying their corresponding algebraic and spectral properties.

Keywords. Antiadjacency matrix; characteristic polynomial; complement of a mixed graph; determinant; Hermitian antiadjacency matrix; mixed graph.
2020 Mathematics Subject Classification. Primary 05C30; Secondary 05C20

1. Introduction and preliminaries

1.1. **Motivation.** Spectral graph theory provides a useful framework for studying the structural properties of graphs through matrices associated with them. In chemical graph theory, for example, spectral quantities have been used to study molecular structures, and complement graphs have also been considered through indices such as Zagreb indices; see [2]. This indicates that graph complements are not merely auxiliary objects but may also carry structural information relevant to spectral investigations.

The antiadjacency matrix was introduced by Bapat [3] as a matrix representation associated with graph complements. For a digraph, this matrix can be viewed as the adjacency matrix of its complement. Several studies have developed this idea for particular classes of digraphs, including directed cyclic sun graphs, line digraphs, digraphs with directed digons, and complement digraphs; see, for example, [14, 15, 18, 19].

Further developments on antiadjacency matrices have considered their powers and spectral properties. Prayitno and Sugeng [16] studied the second and third

Date: Received: Aug 3, 2026; Revised: Sep 1, 2026; Accepted: Sep 7, 2026.
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powers of adjacency and antiadjacency matrices of digraphs, while Sugeng et al. [20] investigated determinants, characteristic polynomial coefficients, eigenvalues, and energy of antiadjacency matrices, particularly in relation to directed paths in acyclic digraphs. These works show that antiadjacency matrices provide a useful way to study missing adjacencies in directed graphs.

On the other hand, the Hermitian adjacency matrix of a mixed graph was introduced by Liu and Li [12], and several of its spectral properties were later studied in [10]. Mohar [13] proposed a broader formulation allowing directed loops, parallel arcs, and parallel edges. In this paper, we follow the formulation of Abudayah et al. [1], which allows directed loops but excludes parallel arcs and parallel edges, and which reduces to the Liu–Li matrix when $\alpha = i$.

A recent study on the Hermitian adjacency matrix of mixed graphs developed a generalised walk method for describing powers, determinants, characteristic polynomials, eigenvalues, and Hermitian energy; see [17]. In that framework, arcs may be traversed in either direction, and the orientation of each arc is represented through complex Hermitian weights. This method provides a useful basis for extending spectral results from adjacency-type matrices to their complement-based counterparts.

Motivated by these developments, the present paper studies antiadjacency and Hermitian antiadjacency matrices of mixed graphs. First, we extend earlier results on antiadjacency matrices from digraphs to the broader setting of mixed graphs and undirected graphs. We then introduce the Hermitian antiadjacency matrix of a mixed graph by defining it as the Hermitian adjacency matrix of its complement. In this way, the Hermitian antiadjacency matrix becomes a natural Hermitian analogue of the ordinary antiadjacency matrix.

This paper is organised into two main parts. The first part studies the antiadjacency matrix of a mixed graph, including its powers, determinant, characteristic polynomial, eigenvalues, and complement energy. The second part develops the corresponding theory for the Hermitian antiadjacency matrix. The methods used in both parts are adapted from the approach developed by Prayitno et al. [18]. In particular, the determinant formulae are obtained by identifying the spanning sub-mixed graphs corresponding to the terms in the Leibniz formula. The characteristic polynomial coefficients are then derived from the principal minors, whose values follow from the determinant formulae. For the powering results, new types of walks, namely mixed disconnected walks and undirected disconnected walks, are introduced, and the corresponding formulae are proved by mathematical induction. The Hermitian antiadjacency case is treated using an analogous strategy, with the additional Hermitian weights determined by the orientations of arcs.

1.2. **Preliminaries.** Throughout this paper, all digraphs are non-multidigraphs. Directed loops are permitted, whereas parallel arcs and directed digons are not. Such a digraph is denoted by $\overrightarrow{D}$, with vertex set $V_{\overrightarrow{D}}$ and arc set $A_{\overrightarrow{D}}$. A mixed graph is obtained by replacing each directed digon with a single undirected edge. Thus, a mixed graph may contain directed loops, but has neither parallel arcs nor parallel edges. We write $M = (V_M, A_M)$, where V_M is the vertex set and A_M

contains all arcs and undirected edges of M. An undirected graph is regarded as a mixed graph without directed arcs and is denoted by $G = (V_G, E_G)$. For vertices u and v, (u, v) denotes an arc directed from u to v, whereas $\{u, v\}$ denotes an undirected edge joining u and v. Throughout the paper, M and $\overrightarrow{D}$ are assumed not to be isomorphic to the null digraph $\overrightarrow{N_n}$, and G is assumed not to be isomorphic to the null graph N_n.

For $u \in V_M$, the out-neighbourhood and in-neighbourhood of u are defined by $N_M^+(u) = \{v : (u, v) \in A_M \text{ or } \{u, v\} \in A_M\}$ and $N_M^-(u) = \{v : (v, u) \in A_M \text{ or } \{u, v\} \in A_M\}$, respectively, with $od_M(u) = |N_M^+(u)|$ and $id_M(u) = |N_M^-(u)|$. For digraphs and undirected graphs, the analogous notation $N_{\overrightarrow{D}}^+(u)$, $N_{\overrightarrow{D}}^-(u)$, $od_{\overrightarrow{D}}(u)$, $id_{\overrightarrow{D}}(u)$, $N_G(u)$, and $deg_G(u)$ is used in the standard sense; see [6, 7].

For a mixed graph M, the associated underlying graph is formed by ignoring the directions of all directed adjacencies. Thus, an arc $(v_i, v_j) \in A_M$ contributes the undirected edge $\{v_i, v_j\}$, while each undirected edge already present in M remains unchanged. This agrees with the usual underlying-graph construction for digraphs; see [6].

Although directed walks on digraphs and undirected walks on graphs are well known, they are not directly suited to the Hermitian antiadjacency matrix $T(M)$, whose entries are determined by missing adjacencies and orientations. We therefore use the following notion. Let $\overrightarrow{D}$ be a digraph with a set of vertices $V_{\overrightarrow{D}} = \{v_1, v_2, \ldots, v_n\}$. A directed disconnected walk from v_i to v_j is a sequence $(v_i = u_{r_0}, u_{r_1}, \ldots, u_{r_k} = v_j)$ such that, for each $0 \le p \le k-1$, one of the following holds:

$$(u_{r_p}, u_{r_{p+1}}) \in A_{\overrightarrow{D}}, \quad (u_{r_{p+1}}, u_{r_p}) \in A_{\overrightarrow{D}}, \quad \text{or} \quad (u_{r_p}, u_{r_{p+1}}), (u_{r_{p+1}}, u_{r_p}) \notin A_{\overrightarrow{D}}.$$

Its length is the number of consecutive pairs of vertices in the sequence. We denote by $w_{ij}^{\text{dir dis-}k,l}$ the number of directed disconnected walks from v_i to v_j of length k that contain exactly $l \ge 0$ disconnected consecutive pairs. When $l = 0$, a disconnected walk reduces to a generalised directed walk.

Let M be a mixed graph with a set of vertices $V_M = \{v_1, v_2, \ldots, v_n\}$. A mixed disconnected walk from v_i to v_j is a sequence $(v_i = u_{r_0}, u_{r_1}, \ldots, u_{r_k} = v_j)$ such that for each $0 \le p \le k-1$, one of the following holds:

$$\begin{gathered}(u_{r_p}, u_{r_{p+1}}) \in A_M, \quad (u_{r_p+1}, u_{r_p}) \in A_M, \\ \{u_{r_p}, u_{r_{p+1}}\} \in A_M, \quad \text{or} \quad (u_{r_p}, u_{r_{p+1}}), (u_{r_{p+1}}, u_{r_p}), \{u_{r_p}, u_{r_{p+1}}\} \notin A_M.\end{gathered}$$

The length of a mixed disconnected walk is the number of successive vertex pairs appearing in the sequence. Let $w_{ij}^{\text{mix dis-}k,l}$ denote the number of mixed disconnected walks from v_i to v_j of length k with exactly $l \ge 0$ disconnected successive pairs. In the special case $l = 0$, this notion coincides with a generalised walk; see [17]. The terms generalised path and generalised cycle are used in the sense of [17].

A mixed disconnected walk from v_i to v_j of length k with exactly $l \ge 0$ disconnected consecutive pairs is denoted by $D_{ij}^{(k,l)}$, and the set of all such walks is denoted by $\mathcal{D}_{ij}^{(k,l)}$.

Let G be an undirected graph with a set of vertices $V_G = \{v_1, v_2, \ldots, v_n\}$. For vertices $v_i, v_j \in V_G$, an undirected disconnected walk from v_i to v_j is a sequence $(v_i = u_{r_0}, u_{r_1}, \ldots, u_{r_k} = v_j)$ such that, for each $0 \leq p \leq k-1$, either $\{u_{r_p}, u_{r_{p+1}}\} \in E_{\overrightarrow{D}}$ or $\{u_{r_p}, u_{r_{p+1}}\} \notin A_{\overrightarrow{D}}$. The length of an undirected disconnected walk is the number of consecutive pairs of vertices in the sequence. We denote by $w_{ij}^{\text{und dis-}k,l}$ the number of undirected disconnected walks from v_i to v_j of length k containing exactly $l \geq 0$ disconnected consecutive pairs. When $l = 0$, an undirected disconnected walk reduces to a generalised walk; see [17].

The terminologies and notion of a cycle digraph $\overrightarrow{C_n}$ are well known; see [9]. According to that definition, it does not allow an arc in the opposite direction. Hence, we modified the definition to define it as the generalised cycle digraph. A generalised cycle digraph $\overrightarrow{C_n^p}$ is defined as a digraph obtained by changing p arcs of the form $(v_{(i \mod n)}, v_{((i+1) \mod n)})$ in a cycle digraph into p arcs of the form $(v_{((i+1) \mod n)}, v_{(i \mod n)})$, where $1 \leq i \leq n$ and $0 \leq p \leq n$. If $p = 0$, then it is a cycle digraph $\overrightarrow{C_n}$; whenever $p = n$, it is isomorphic to the cycle digraph $\overrightarrow{C_n}$.

The terminology for the path digraph $\overrightarrow{P_n}$ is standard; see [9]. Following [17], an isolated path digraph $\overrightarrow{P_n^{\text{iso}}}$ is obtained from $\overrightarrow{P_n}$ by deleting p arcs so that at least one isolated vertex occurs. A non-isolated k-path digraph $\overrightarrow{P_i^{n,k}}$ is an induced subdigraph of $\overrightarrow{P_n}$ of order i, where $4 \leq i \leq n$ consists of $k \geq 2$ path digraphs $\overrightarrow{P_{i_\iota}}$, $1 \leq \iota \leq k$, and $i_\iota \geq 2$. In the usual path digraph, disconnected vertex pairs and reversed arcs of the form $(v_{(l+1) \bmod n}, v_{l \bmod n})$, $1 \leq l \leq n$ are excluded. This motivates a broader notion of a generalised path digraph, which includes $\overrightarrow{P_n}$, $\overrightarrow{P_n^{\text{iso}}}$, and $\overrightarrow{P_i^{n,k}}$ as special cases.

We define a generalised path digraph $\overrightarrow{P_{n,p,q}}$ as a digraph obtained by deleting p arcs in $\overrightarrow{P_n}$ or changing q arcs of the form $(v_{(i \mod n)}, v_{((i+1) \mod n)})$ into q arcs of the form $(v_{((i+1) \mod n)}, v_{(i \mod n)})$, where $1 \leq i \leq n$, $0 \leq p \leq n-1$, and $0 \leq q \leq n-1$. However, the arcs that are removed and changed cannot be the same. Obviously, if $p = n-1$ and $q = 0$ are equal to a null digraph $\overrightarrow{N_n}$, $\overrightarrow{P_n^{\text{iso}}}$ is a generalised path digraph $\overrightarrow{P_{n,p,q}}$ whenever we remove the specified p arcs for $0 < p \leq n-1$ and $0 \leq q \leq n-1$, and $\overrightarrow{P_i^{n,k}}$ is also a generalised path digraph $\overrightarrow{P_{n,p,q}}$ whenever we remove the specified p arcs for $0 < p \leq n-1$ and $0 \leq q \leq n-1$.

A mixed graph Ψ_M is a spanning sub-mixed graph of a mixed graph M if it is obtained by removing some arcs and/or edges from M while keeping all vertices of M. Furthermore, a spanning subgraph of an undirected graph G is denoted by Ψ_G and is defined as a digraph obtained by removing some edges while keeping all vertices of G.

For a mixed graph M, let $\equiv_M$ be the relation on V_M defined by $x \equiv_M y$ if $x = y$, or if x and y are mutually reachable by a generalised path. An equivalence class of a vertex $x \in V_M$ is defined as a set $[x]_{\equiv_M} = \{y \in V_M : x \equiv_M y\}$. A component-class of a spanning sub-mixed graph of M is defined as the sub-mixed graph corresponding to an equivalence class of each vertex equivalence class of a vertex $x \in V_M$. If the equivalence class is not a singleton, then the component-class is

the sub-mixed graph induced by an equivalence class of a vertex $x \in V_M$, with directed loops excluded. If the equivalence class is a singleton x and there is a directed loop at $x \in V_M$, then the component-class is a directed loop. Otherwise, the component-class consists of only a null digraph with a single vertex $x \in V_M$.

A spanning sub-mixed graph does not always have to be an induced sub-mixed graph of the original graph. Furthermore, in this paper, we classify spanning sub-mixed graphs according to their component-classes. A spanning sub-mixed graph with exactly one component class is either a generalised cycle digraph $\overrightarrow{C_n^p}$, where $0 \leq p \leq n$, or a generalised path digraph $\overrightarrow{P_{n,p,q}}$, where $0 \leq p, q \leq n-1$. Spanning sub-mixed graphs with more than one component-classes are classified as spanning non-elementary sub-mixed graphs.

A spanning sub-mixed graph that is a generalised cycle $\overrightarrow{C_n^p}$, where $0 \leq p \leq n$, is denoted by $\Psi_{M_{\overrightarrow{C_n^p}}}$. For $\Psi_{M_{\overrightarrow{C_n^p}}}$, the numbers of arcs of the forms $(\vartheta_{r_p}, \vartheta_{r_{p+1}})$ and $(\vartheta_{r_{p+1}}, \vartheta_{r_p})$ are denoted by $a(\vartheta_{r_p}, \vartheta_{r_{p+1}})_{\Psi_{M_{\overrightarrow{C_n^p}}}}$ and $a(\vartheta_{r_{p+1}}, \vartheta_{r_p})_{\Psi_{M_{\overrightarrow{C_n^p}}}}$, respectively. For any mixed graph M, exactly one of the following two cases occurs: either $\Psi_{M_{\overrightarrow{C_n^p}}}$ uniquely exists, or it does not exist.

A spanning sub-mixed graph that is a generalised path $\overrightarrow{P_{n,p,q}}$, where $0 \leq p \leq n-1$ and $0 \leq q \leq n-1$, is denoted by $\Psi_{M_{\overrightarrow{P_{n,p,q}}}}$. In $\Psi_{M_{\overrightarrow{P_{n,p,q}}}}$, the numbers of arcs of the form $(\vartheta_{r_p}, \vartheta_{r_{p+1}})$ and $(\vartheta_{r_{p+1}}, \vartheta_{r_p})$ are denoted by $a(\vartheta_{r_p}, \vartheta_{r_{p+1}})_{\Psi_{M_{\overrightarrow{P_{n,p,q}}}}}$ and $a(\vartheta_{r_{p+1}}, \vartheta_{r_p})_{\Psi_{M_{\overrightarrow{P_{n,p,q}}}}}$, respectively. For any mixed graph M, exactly one of the following two cases occurs: either $\Psi_{M_{\overrightarrow{P_{n,p,q}}}}$ uniquely exists, or it does not exist.

The spanning sub-mixed graphs $\Psi_{M_{\overrightarrow{C_n^p}}}$ and $\Psi_{M_{\overrightarrow{P_{n,p,q}}}}$ are mutually exclusive. That is, if $\Psi_{M_{\overrightarrow{C_n^p}}}$ exists, then $\Psi_{M_{\overrightarrow{P_{n,p,q}}}}$ does not exist, and vice versa. Nevertheless, it is possible that neither $\Psi_{M_{\overrightarrow{C_n^p}}}$ nor $\Psi_{M_{\overrightarrow{P_{n,p,q}}}}$ exists.

A collection of all spanning sub-mixed graphs of M that are generalised cycle digraphs is denoted by $\mathcal{C}_n(M) = \left\{\Psi_{M_{\overrightarrow{C_n^p}}} : 0 \leq p \leq n\right\}$. On the other hand, a collection of all spanning sub-mixed graphs of M that are generalised path digraphs is denoted by $\mathcal{P}_n(M) = \left\{\Psi_{M_{\overrightarrow{P_{n,p,q}}}} : 0 \leq p, q \leq n-1\right\}$. Furthermore, the collection of all spanning sub-mixed graphs of M that are generalised cycles or generalised paths is denoted by $\mathcal{G}_n(M) = \mathcal{C}_n(M) \cup \mathcal{P}_n(M)$. In $\Psi_M \in \mathcal{G}_n(M)$, the numbers of arcs of the form $(\vartheta_{r_p}, \vartheta_{r_{p+1}})$ and $(\vartheta_{r_{p+1}}, \vartheta_{r_p})$ are denoted by $a(\vartheta_{r_p}, \vartheta_{r_{p+1}})_{\Psi_{\mathcal{G}_n(M)}}$ and $a(\vartheta_{r_{p+1}}, \vartheta_{r_p})_{\Psi_{\mathcal{G}_n(M)}}$, respectively.

A spanning non-elementary sub-mixed graph is denoted by Ψ_M^{non} and is defined as the spanning sub-mixed graph M whose component-classes are generalised path digraphs or generalised cycle digraphs. Now, assume that Ψ_M^{non} has k component-classes. Then, after relabelling, its component-classes can be decomposed as $\mathcal{E}(\Psi_M^{\text{non}}) = \left\{\overrightarrow{P_{j_a, j_{p_a}, j_{q_a}}}\right\}_{a=1}^{r} \bigcup \left\{\overrightarrow{C_{l_b}^{l_{p_b}}}\right\}_{b=1}^{t}$, where $r + t = k$. The respective orders of these component-classes are j_a and l_b. Consequently, $|\mathcal{E}(\Psi_M^{\text{non}})| = r + t = k$. For each component-class of Ψ_M^{non} that is $\overrightarrow{P_{j_a, j_{p_a}, j_{q_a}}}$, define

$\mathcal{A}_{j_a} = \{(j_a - 1, 0), (0, j_a - 1), (0, 0)\}$ and for each component-class that is $\overrightarrow{C_{l_b}^{l_{p_b}}}$, define $\mathcal{B}_{l_b} = \{0, l_b\}$.

Let Ψ_M^{non} be a spanning non-elementary sub-mixed graph of M whose component-classes are $\mathcal{E}_1, \ldots, \mathcal{E}_\kappa$. For each $1 \leq i \leq \kappa$, denote by $a(\vartheta_{r_p}, \vartheta_{r_{p+1}})_{(\mathcal{E}_i)_{\Psi_M^{\text{non}}}}$ and $a(\vartheta_{r_{q+1}}, \vartheta_{r_q})_{(\mathcal{E}_i)_{\Psi_M^{\text{non}}}}$ the numbers of arcs of the forms $(\vartheta_{r_p}, \vartheta_{r_{p+1}})$ and $(\vartheta_{r_{q+1}}, \vartheta_{r_q})$ in the component-class $\mathcal{E}_i$, respectively. Define

$$\delta(\vartheta_{r_p}, \vartheta_{r_{p+1}})_{\Psi_M^{\text{non}}} = \sum_{i=1}^{\kappa} a(\vartheta_{r_p}, \vartheta_{r_{p+1}})_{(\mathcal{E}_i)_{\Psi_M^{\text{non}}}}$$

and

$$\delta(\vartheta_{r_{q+1}}, \vartheta_{r_q})_{\Psi_M^e} = \sum_{i=1}^{\kappa} a(\vartheta_{r_{q+1}}, \vartheta_{r_q})_{(\mathcal{E}_i)_{\Psi_M^{\text{non}}}}.$$

Now, for an undirected graph G, let $\equiv_G$ be the relation on V_G defined by $x \equiv_G y$ if $x = y$, or if x and y are joined by an undirected path. An equivalence class of a vertex $x \in V_G$ is defined as a set $[x]_{\equiv_G} = \{y \in V_G : x \equiv_G y\}$. If the equivalence class is not a singleton, then the component-class is the subgraph induced by an equivalence class of a vertex $x \in V_M$, with undirected loops excluded. If the equivalence class is a singleton x and there is an undirected loop at $x \in V_M$, then the component-class is an undirected loop. Otherwise, the component-class consists of only a null graph with a single vertex $x \in V_M$.

A spanning non-elementary subgraph is denoted by Ψ_G^{non} and is defined as the spanning subgraph of G whose classes are null graphs. Now, assume that Ψ_G^{non} has k component-classes. Then after relabelling, its component-classes can be expressed as $\mathcal{E}(\Psi_G^{\text{non}}) = \{N_{j_a}\}_{a=1}^k$. The orders of these component-classes are j_a. Consequently, $|\mathcal{E}(\Psi_G^{\text{non}})| = k$.

The classical definitions of adjacency and antiadjacency matrices are given by Bapat [3]. In particular, the antiadjacency matrix is obtained by subtracting the adjacency matrix from the all-one matrix J. Accordingly, $B(M) = J - A(M)$, $B(\overrightarrow{D}) = J - A(\overrightarrow{D})$, and $B(G) = J - A(G)$. Since the formulation in [3] is given for loopless digraphs and undirected graphs, we extend the definition here to the classes of mixed graphs, digraphs, and undirected graphs considered in this paper, where loops are allowed.

The antiadjacency matrix of M is denoted by $B(M) = (b_{ij}) \in \mathbb{C}^{n \times n}$, where $b_{ij} = 0$ if $\{v_i, v_j\} \in A_M$ or $(v_i, v_j) \in A_M$, and $b_{ij} = 1$ otherwise. For the diagonal entries, $b_{ii} = 0$ whenever M contains a directed loop at v_i, while $b_{ii} = 1$ otherwise. For a digraph $\overrightarrow{D}$, its antiadjacency matrix is denoted by $B(\overrightarrow{D}) = (\beta_{ij}) \in \mathbb{C}^{n \times n}$, where $\beta_{ij} = 0$ if $(v_i, v_j) \in A_{\overrightarrow{D}}$, and $\beta_{ij} = 1$ otherwise. In particular, $\beta_{ii} = 0$ whenever $\overrightarrow{D}$ contains a directed loop at v_i, and $\beta_{ii} = 1$ otherwise. Lastly, for an undirected graph G of order n, its antiadjacency matrix is denoted by $B(G) = (\varrho_{ij}) \in \mathbb{C}^{n \times n}$, where $\varrho_{ij} = 0$ if $\{v_i, v_j\} \in E_G$, and $\varrho_{ij} = 1$ otherwise. Thus, $\varrho_{ii} = 0$ whenever G contains an undirected loop at v_i, while $\varrho_{ii} = 1$ otherwise.

Let M be a mixed graph where $|V_M| = n$, and choose $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$. The Hermitian adjacency matrix associated with M, denoted by $H(M) = (h_{ij}) \in \mathbb{C}^{n \times n}$, is defined by $h_{ij} = 1$ if $\{v_i, v_j\} \in A_M$, $h_{ij} = \alpha$ if $(v_i, v_j) \in A_M$ and

$(v_j, v_i) \notin A_M$, $h_{ij} = \overline{\alpha}$ if $(v_j, v_i) \in A_M$ and $(v_i, v_j) \notin A_M$, and $h_{ij} = 0$ otherwise. For the diagonal entries, $h_{ii} = 1$ whenever M contains a directed loop at v_i, while $h_{ii} = 0$ otherwise.

Let $G = (V_G, E_G)$ be an undirected graph. The complement of G, denoted by $\overline{G}$, is the undirected graph with vertex set $V_{\overline{G}} = V_G$ such that, for every pair of distinct vertices $u, v \in V_G$, we have $\{u, v\} \in E_{\overline{G}}$ if and only if $\{u, v\} \notin E_G$; see [5]. This classical definition is stated for simple undirected graphs and therefore excludes loops from the complement.

Since this paper also considers undirected graphs whose complements may contain loops, we extend the usual notion of graph complement as follows. For an undirected graph G, its complement $\overline{G}$ has vertex set $V_{\overline{G}} = V_G$, and, for any $u, v \in V_G$, not necessarily distinct, $\{u, v\} \in E_{\overline{G}}$ if and only if $\{u, v\} \notin E_G$.

A corresponding convention is adopted for mixed graphs, which may contain directed loops. Let M be a mixed graph. Its complement $\overline{M}$ has vertex set $V_{\overline{M}} = V_M$, with adjacencies determined by those absent from M. More precisely, whenever an ordered or unordered pair of vertices does not represent an adjacency in M, the corresponding adjacency is included in $\overline{M}$. Under this convention, the complement of a digraph need not remain a digraph, and similarly, the complement of a mixed graph may belong to a more restricted graph class.

These definitions agree with the matrix formulation adopted throughout this paper. Let J denote the all-one matrix. Then $A(G)+B(G) = J$, $A(\overrightarrow{D})+B(\overrightarrow{D}) = J$, and $A(M) + B(M) = J$. Therefore, the antiadjacency matrix of each graph coincides with the adjacency matrix of its complement; namely, $B(G) = A(\overline{G})$, $B(\overrightarrow{D}) = A(\overrightarrow{\overline{D}})$, and $B(M) = A(\overline{M})$.

Following the same principle, the Hermitian antiadjacency matrix of a mixed graph is defined as the Hermitian adjacency matrix of its complement. Thus, $T(M) = H(\overline{M})$. For the admissible choices of α considered in this paper, the corresponding matrices satisfy $H(M) + T(M) = J$.

The Hermitian antiadjacency matrix of a mixed graph is defined as the Hermitian adjacency matrix of its complement. A formal definition of a Hermitian antiadjacency matrix of a mixed graph is given in the following definition.

Definition 1.1. For $\alpha \in \mathbb{C}$ with $|\alpha| = 1$, the Hermitian antiadjacency matrix of M is denoted by $T(M) = (t_{ij}) \in \mathbb{C}^{n\times n}$ and satisfies $T(M) = H(\overline{M})$. Equivalently, for $i \neq j$, we have $t_{ij} = 0$ if $\{v_i, v_j\} \in A_M$, $t_{ij} = \overline{\alpha}$ if $(v_i, v_j) \in A_M$ and $(v_j, v_i) \notin A_M$, $t_{ij} = \alpha$ if $(v_j, v_i) \in A_M$ and $(v_i, v_j) \notin A_M$, and $t_{ij} = 1$ otherwise. In addition, $t_{ii} = 0$ whenever M contains a directed loop at v_i, and $t_{ii} = 1$ otherwise.

Recall that the entry b_{ij} of $B(M)$ satisfies $b_{ij} = 0$ whenever $(v_i, v_j) \in A_M$ but $(v_j, v_i) \notin A_M$ or $\{v_i, v_j\} \in A_M$ while $b_{ij} = 1$ elsewhere. Therefore, in order to transform $T(M) = (t_{ij})$ into $B(M) = (b_{ij})$, substitute $\overline{\alpha} = 1$ and $\alpha = 0$ in $T(M)$. Furthermore, substituting $\overline{\alpha} = 0$ and $\alpha = 0$ in $T(M)$ will reduce $T(M)$ to the antiadjacency matrix of the underlying graph of M. Moreover, if we substitute $\alpha = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\alpha = \frac{1}{2} - \frac{\sqrt{3}}{2}i$, $\alpha = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, or $\alpha = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$,

our definition of Hermitian antiadjacency also holds a similar relation to the antiadjacency and adjacency matrices, which is $T(M)+H(M) = J$. In particular, this definition includes digraphs $\overrightarrow{D}$ and undirected graphs G as special cases in which $T\left(\overrightarrow{D}\right) + H\left(\overrightarrow{D}\right) = J$ and $T(G) + H(G) = J$.

With the above conventions, digraphs and undirected graphs are treated as particular cases of mixed graphs. Hence, the results on Hermitian antiadjacency matrices of mixed graphs also cover the corresponding digraph case. In the undirected case, the Hermitian antiadjacency matrix reduces to the ordinary antiadjacency matrix. Therefore, this paper studies the powers, determinants, characteristic polynomials, and eigenvalues of Hermitian antiadjacency matrices for mixed graphs and separately discusses the ordinary antiadjacency matrix for mixed graphs and undirected graphs.

For the antiadjacency matrix $B(M)$ of a mixed graph M, its k-th power, determinant, spectral radius, spectrum, and characteristic polynomial are denoted by $B^k(M) = (b_{ij}^{(k)})$, $\det(B(M))$, $\rho(B(M))$, $\sigma(B(M))$, and $P(\lambda; B(M)) = \lambda^n + \sum_{j=1}^{n}(-1)^j \cdot c_j \cdot \lambda^{n-j}$, respectively. For the antiadjacency matrix $B(G)$ of an undirected graph G, the corresponding notation is $B^k(G) = (\varrho_{ij}^{(k)})$, $\det(B(G))$, $\rho(B(G))$, $\sigma(B(G))$, and $P(\lambda; B(G)) = \lambda^n + \sum_{j=1}^{n}(-1)^j \cdot \chi_j \cdot \lambda^{n-j}$. Similarly, for the Hermitian antiadjacency matrix $T(M)$ of a mixed graph M, we write $T^k(M) = (t_{ij}^{(k)})$, $\det(T(M))$, $\rho(T(M))$, $\sigma(T(M))$, and $P(\lambda; T(M)) = \lambda^n + \sum_{k=1}^{n}(-1)^k \cdot a_k \lambda^{n-k}$.

We use the standard Leibniz expansion of the determinant; see [11]. For later use, let s_0 denote the identity permutation in S_n, and let

$$s_1 = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix}, \qquad s_2 = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & 1 & \cdots & n-2 & n-1 \end{pmatrix}.$$

By the Leibniz formula, $\det(B(M))$ and $\det(T(M))$ are determined by the products $\prod_{i=1}^{n} b_{i\,s(i)}$ and $\prod_{i=1}^{n} t_{i\,s(i)}$, where $s \in S_n$. Hence, it is enough to identify each Ψ_M that corresponds to nonzero such products. Furthermore, these spanning sub-mixed graphs have at least one component-class. This is why we only focus on the spanning sub-mixed graphs that have at least one component-class.

In particular, the terms $b_{11}b_{22}\cdots b_{nn}$ and $t_{11}t_{22}\cdots t_{nn}$ vanish whenever M contains a directed loop. The remaining nonzero terms are obtained by the structure of the corresponding spanning sub-mixed graphs, which will be analysed in the following sections.

2. Antiadjacency matrix of mixed graphs, digraphs, and undirected graphs

2.1. Powers of the antiadjacency matrix.

The following Theorem 2.1 extends the result of Prayitno and Sugeng [16] on $B^k(M)$, for $k = 2$ and $k = 3$. Thus, the earlier result can be considered as a special case of the following Theorem 2.1.

Theorem 2.1. For $1 \le i, j \le n$ and $k \ge 2$, it follows that $b_{ij}^{(k)} = w_{ji}^{\text{mix dis-}k,l}$, where $0 \le l \le k$.

Proof. For $k = 2$, we obtain that $b_{ij}^{(2)} = \sum_{k=1}^{n} b_{ik} b_{kj}$ and each nonzero term corresponds to a mixed disconnected walk of length 2 from v_j to v_i. Hence, $b_{ij}^{(2)} = w_{ji}^{\text{mix dis-}2,l}$, where $0 \leq l \leq 2$.

Assume that for $k = \varphi$, $b_{ij}^{(\varphi)} = w_{ji}^{\text{mix dis-}\varphi,l}$, where $0 \leq l \leq \varphi$. Since $b_{ij}^{(\varphi+1)} = \sum_{k=1}^{n} b_{ik}^{(\varphi)} b_{kj}$, each nonzero summand extends a mixed disconnected walk of length φ by one admissible step. Therefore, $b_{ij}^{(\varphi+1)} = w_{ji}^{\text{mix dis-}(\varphi+1),l}$, where $0 \leq l \leq \varphi+1$. Hence, the theorem follows. □

The undirected case is obtained similarly since it is also an immediate consequence of the mixed graph result.

Corollary 2.2. For $1 \leq i, j \leq n$ and $k \geq 2$, we acquire $\varrho_{ij}^{(k)} = w_{ji}^{\text{und dis-}k,k}$.

Proof. Since G may be viewed as a mixed graph with no arcs, the mixed disconnected walks considered in Theorem 2.1 reduce to undirected disconnected walks. Therefore, the result follows directly from that theorem. □

Now, recall one of the results on the main diagonal of $B^2(M)$ shown in [16] which is given in the following lemma.

Lemma 2.3. [16] For $1 \leq i \leq n$, we obtain $b_{ii}^{(2)} = |\left(N_M^+(v_i)\right)^C \bigcup \left(N_M^-(v_i)\right)^C|$.

Let $l(M)$ denote the number of directed loops in M, and let $d(M)$ denote the number of arcs and edges of M excluding directed loops. The Lemma 2.3 gives an expression for the diagonal entries $b_{ii}^{(2)}$, $1 \leq i \leq n$, of $B^2(M)$. Summing these entries shows that $\sum_{i=1}^{n} b_{ii}^{(2)}$ can be expressed in terms of $l(M)$ and $d(M)$, as stated in the following corollary.

Corollary 2.4. For $1 \leq i \leq n$, we have $\sum_{i=1}^{n} b_{ii}^{(2)} = n^2 - (l(M) + 2\,d(M))$.

Proof. Recall that $b_{ii}^{(2)} = |\left(N_M^+(v_i)\right)^C \bigcup \left(N_M^-(v_i)\right)^C| = n - |N_M^+(v_i) \bigcup N_M^-(v_i)|$. Moreover, $|N_M^+(v_i) \bigcup N_M^-(v_i)| = 1 + d(M)$ whenever the vertex $v_i \in V_M$ is loopless, while $|N_M^+(v_i) \bigcup N_M^-(v_i)| = d(M)$ otherwise. Then, by summing the main diagonal entries $b_{ii}^{(2)}$ for $1 \leq i \leq n$, the corollary follows. □

Trivially, the coefficient of the characteristic polynomial of $B(M)$, $c_1 = \text{tr}(B(M)) = \sum_{i=1}^{n} b_{ii} = n - l(M)$, or in other words, c_1, is equal to the number of loopless vertices in M. Now, the coefficient c_2 is given in the following theorem.

Theorem 2.5. If $P(\lambda; B(M)) = \lambda^n + \sum_{k=1}^{n} (-1)^k\, c_k\, \lambda^{n-k}$, then

$$c_2 = \frac{(l(M))^2 - (2n-1)\, l(M) + 2d(M)}{2}.$$

Proof. Recall that $\left(\sum_{i=1}^{n} \lambda_i\right)^2 = \left(\sum_{i=1}^{n} \lambda_i^2\right) + 2\left(\sum_{1 \leq i,j \leq |V_M|} \lambda_i \lambda_j\right)$, which implies $c_2 = \frac{1}{2}\left((\text{tr}(B(M)))^2 - \text{tr}\left(B(M)^2\right)\right)$. Since $\text{tr}(B(M)) = n - l(M)$ and by Corollary 2.4, we obtain $tr\left(B(M)^2\right) = n^2 - (l(M) + 2\,d(M))$; the required result follows. □

After finding the first and second coefficients of $P(\lambda; B(M))$, c_1, and c_2, we now want to generalise the coefficients of $P(\lambda; B(M))$ and c_i for $1 \leq i \leq n$. We observe that to find c_i, it would be more convenient for us to use the principal minor method instead of the trace method. Therefore, we need to find $\det(B(M))$, which we will discuss in the next subsection.

2.2. **Determinant of the antiadjacency matrix.** Recall from the Leibniz formula that we need to find the other terms of $\prod_{i=1}^{n} b_{i\,s(i)}$ other than $b_{11}b_{22}\cdots b_{nn}$. Furthermore, we need to find every cases of spanning sub-mixed graphs that correspond to the term $\prod_{i=1}^{n} b_{i\,s(i)}$. Some of the cases in which the terms do not vanish are when there exists a spanning sub-mixed graph that is a generalised cycle $\Psi_{M_{\overrightarrow{C_n^p}}}$, where $p = n$ or $p = 0$. This is shown in the following lemma.

Lemma 2.6. If $\Psi_{M_{\overrightarrow{C_n^p}}}$ exists, then

$$b_{12}b_{23}\cdots b_{(n-1)n}b_{n1} = \begin{cases} 1, & \text{if } p = n; \\ 0, & \text{if } 0 < p < n; \\ 0, & \text{if } p = 0; \end{cases}$$

and

$$b_{1n}b_{21}b_{32}\cdots b_{n(n-1)} = \begin{cases} 1, & \text{if } p = 0; \\ 0, & \text{if } 0 < p < n; \\ 0, & \text{if } p = n. \end{cases}$$

Proof. Suppose that there exists $\Psi_{M_{\overrightarrow{C_n^p}}}$. For $0 < p < n$, there exist both arcs $(v_{(l \mod n)}, v_{((l+1) \mod n)}) \in V_{\Psi_{M_{\overrightarrow{C_n^p}}}}$ and $(v_{((j+1) \mod n)}, v_{(j \mod n)}) \in V_{\Psi_{M_{\overrightarrow{C_n^p}}}}$ such that $b_{((l \mod n))((l+1) \mod n)} = 0$ and $b_{((j+1) \mod n)(j \mod n)} = 0$ (where $1 \leq l, j \leq n$ and $l \neq j$), which implies $b_{12}b_{23}\cdots b_{(n-1)n}b_{n1} = 0$ and $b_{1n}b_{21}b_{32}\cdots b_{n(n-1)} = 0$. For $p = 0$, we obtain $\overrightarrow{C_n^p} = \overrightarrow{C_n}$. This means that all the arcs are in the form of $(v_{(l \mod n)}, v_{((l+1) \mod n)}) \in V_{\Psi_{M_{\overrightarrow{C_n^p}}}}$, implying $b_{12}b_{23}\cdots b_{(n-1)n}b_{n1} = 0$ and $b_{1n}b_{21}b_{32}\cdots b_{n(n-1)} = 1$. Lastly, for $p = n$, all the arcs are in the form of $(v_{((j+1) \mod n)}, v_{(j \mod n)}) \in V_{\Psi_{M_{\overrightarrow{C_n^p}}}}$, implying $b_{12}b_{23}\cdots b_{(n-1)n}b_{n1} = 1$ and $b_{1n}b_{21}b_{32}\cdots b_{n(n-1)} = 0$. □

The next types of spanning sub-mixed graphs are generalised path digraphs. The value of the term that corresponds to this spanning sub-mixed graph is given in the following lemma.

Lemma 2.7. If $\Psi_{M_{\overrightarrow{P_{n,p,q}}}}$ exists, then

$$b_{12}b_{23}\cdots b_{(n-1)n}b_{n1} = \begin{cases} 1, & \text{if } p = n-1, q = 0; \\ 1, & \text{if } p = 0, q = n-1; \\ 0, & \text{elsewhere;} \end{cases}$$

and

$$b_{1n}b_{21}b_{32}\cdots b_{n(n-1)} = \begin{cases} 1, & \text{if } 0 \leq p \leq n-1, q = 0; \\ 0, & \text{elsewhere.} \end{cases}$$

Proof. Suppose that there exists $\Psi_{M_{\overrightarrow{P_{n,p,q}}}}$. For $p = n-1$ and $q = 0$, $\Psi_{M_{\overrightarrow{P_{n,p,q}}}}$ is a null digraph $\overrightarrow{N_n}$ which implies that for $1 \leq i \leq n$, $b_{(i \mod n)((i+1) \mod n)} = 1$ and $b_{((i+1) \mod n)(i \mod n)} = 1$. Therefore, $b_{12}b_{23}\cdots b_{(n-1)n}b_{n1} = 1$ and $b_{1n}b_{21}b_{32}\cdots b_{n(n-1)} = 1$. Now, for $0 \leq p < n-1$ and $q = 0$, each undeleted arc of $\Psi_{M_{\overrightarrow{P_{n,p,q}}}}$ is of the form (v_j, v_{j+1}) where $1 \leq j \leq n-1$. Therefore, for $1 \leq i \leq n$, $b_{(i \mod n)((i+1) \mod n)} = 0$, and $b_{((i+1) \mod n)(i \mod n)} = 1$. This implies $b_{12}b_{23}\cdots b_{(n-1)n}b_{n1} = 1$ and $b_{1n}b_{21}b_{32}\cdots b_{n(n-1)} = 1$. On the other hand, for $p = 0$ and $q = n-1$, each undeleted arc of $\Psi_{M_{\overrightarrow{P_{n,p,q}}}}$ is of the form (v_{j+1}, v_j) where $1 \leq j \leq n-1$. Therefore, for $1 \leq i \leq n$, $b_{(i \mod n)((i+1) \mod n)} = 1$, and $b_{((i+1) \mod n)(i \mod n)} = 1$. This implies $b_{12}b_{23}\cdots b_{(n-1)n}b_{n1} = 1$ and $b_{1n}b_{21}b_{32}\cdots b_{n(n-1)} = 1$. Lastly, for the remaining cases, there are arcs of the forms (v_j, v_{j+1}) and (v_{l+1}, v_l) where $1 \leq j, l \leq n-1$ and $j \neq l$. Hence, $b_{12}b_{23}\cdots b_{(n-1)n}b_{n1} = 0$ and $b_{1n}b_{21}b_{32}\cdots b_{n(n-1)} = 0$. □

Next, we find that if the component-classes of the spanning sub-mixed graphs are undirected cycles, undirected paths, arcs, or directed loops, for all $s \in S_n$, the term $\prod_{i=1}^{n} b_{i\,s(i)} = 0$. This result is shown in the following lemma.

Lemma 2.8. Suppose that Ψ_M corresponds to the term $\prod_{i=1}^{n} b_{is(i)}$, where undirected cycles, undirected paths, arcs, or directed loops are its component-classes. Then for every $s \in S_n$, $\prod_{i=1}^{n} b_{i\,s(i)} = 0$.

Proof. Each component-class listed in the statement forces at least one zero factor in the corresponding permutation term. Indeed, if a component-class contains an edge $\{v_{r_i}, v_{r_{i+1}}\}$, then $b_{r_i r_{i+1}} = b_{r_{i+1} r_i} = 0$. If it contains an arc $(v_{r_i}, v_{r_{i+1}})$, then the factor corresponding to this arc is $b_{r_i r_{i+1}} = 0$. If it contains a directed loop (v_{r_i}, v_{r_i}), then $b_{r_i r_i} = 0$. Hence, in each case, the product $\prod_{i=1}^{n} b_{i,s(i)}$ contains a zero factor. Therefore, for every $s \in S_n$, we have $\prod_{i=1}^{n} b_{i,s(i)} = 0$. □

We now consider the remaining case, that is Ψ_M^{non}. The value of the terms associated with Ψ_M^{non} is shown by the following lemma.

Lemma 2.9. For any term of the form $\prod_{i=1}^{n} b_{is(i)}$, where $s \in S_n$, of Ψ_M^{non}, the following holds.

(a) If for all $a \in \{1, \ldots, r\}$ and $b \in \{1, \ldots, t\}$, $(j_{p_a}, j_{q_a}) \in \mathcal{A}_{j_a}$ and $l_{p_b} \in \mathcal{B}_{l_b}$, then there exists $s \in S_n \setminus \{s_0, s_1, s_2\}$ such that $\prod_{i=1}^{n} b_{is(i)} = 1$.
(b) If for all $a \in \{1, \ldots, r\}$, $(j_{p_a}, j_{q_a}) \in \mathcal{A}_{j_a}$, there exists $b \in \{1, \ldots, s\}$ such that $0 < l_{p_b} < l_b$, then for every $s \in S_n \setminus \{s_0, s_1, s_2\}$, $\prod_{i=1}^{n} b_{is(i)} = 0$.
(c) If there exists $a \in \{1, \ldots, r\}$ such that $(j_{p_a}, j_{q_a}) \notin \mathcal{A}_{j_a}$, then for every $s \in S_n \setminus \{s_0, s_1, s_2\}$, $\prod_{i=1}^{n} b_{is(i)} = 0$.

Proof. For every $a \in \{1, \ldots, r\}$ and $b \in \{1, \ldots, t\}$, suppose that $(j_{p_a}, j_{q_a}) \in \mathcal{A}_{j_a}$ and $l_{p_b} \in \mathcal{B}_{l_b}$. Then each $\overrightarrow{P_{j_a, j_{p_a}, j_{q_a}}}$ satisfies one of the non-vanishing cases in

Lemma 2.7. Similarly, each $E_{l_b}^{\text{cycle}} \cong \overrightarrow{C_{l_b}^{l_{p_b}}}$ satisfies one of the non-vanishing cases in Lemma 2.6. Since the component-classes are pairwise disjoint, the corresponding local permutations may be combined into a permutation $s \in S_n \setminus \{s_0, s_1, s_2\}$ such that no factor in the product vanishes. Hence $\prod_{i=1}^{n} b_{is(i)} = 1$. This proves (a).

Next, suppose that for all $a \in \{1, \ldots, r\}$ and $(j_{p_a}, j_{q_a}) \in \mathcal{A}_{j_a}$, there exists $b \in \{1, \ldots, s\}$ such that $0 < l_{p_b} < l_b$. Then $E_{l_{b_0}}^{\text{cycle}} \cong \overrightarrow{C_{l_{b_0}}^{l_{p_{b_0}}}}$ falls into a vanishing case of Lemma 2.6. Therefore, every Leibniz term associated with such a component contains a zero factor. Consequently, for every $s \in S_n \setminus \{s_0, s_1, s_2\}$, $\prod_{i=1}^{n} b_{is(i)} = 0$. This proves (b).

Finally, there exists $a \in \{1, \ldots, r\}$ such that $(j_{p_a}, j_{q_a}) \notin \mathcal{A}_{j_a}$. Then $E_{j_{a_0}}^{\text{path}} \cong \overrightarrow{P_{j_{a_0}, j_{p_{a_0}}, j_{q_{a_0}}}}$ falls into a vanishing case of Lemma 2.7. Hence, every Leibniz term associated with such a component contains a zero factor. Thus, for every $s \in S_n \setminus \{s_0, s_1, s_2\}$, $\prod_{i=1}^{n} b_{is(i)} = 0$. This proves (c) and completes the proof. □

Having determined the contribution of spanning non-elementary sub-mixed graphs, we now combine it with the preceding cases to obtain a formula for $\det(B(M))$.

Theorem 2.10. Given a mixed graph M of order n. Let $\mathcal{M}^{\text{non}}$ be the collection of all spanning non-elementary sub-mixed graphs Ψ_M^{non} of M. For any $\Psi_M^{\text{non}} \in \mathcal{M}^{\text{non}}$, let $\mathcal{E}(\Psi_M^{\text{non}}) = \{E_{j_a}^{\text{path}}\}_{a=1}^{r} \bigcup \{E_{l_b}^{\text{cycle}}\}_{b=1}^{t}$, where $(j_{p_a}, j_{q_a}) \in \mathcal{A}_{j_a}$ and $l_{p_b} \in \mathcal{B}_{l_b}$. Let $c(\Psi_M^{\text{non}})$ be the number of component-classes of order ≥ 2 in $\Psi_M^{\text{non}} \in \mathcal{M}^{\text{non}}$, $N_{\Psi_M^{\text{non}}}^{l}$ be the number of isolated vertices in $\Psi_M^{\text{non}} \in \mathcal{M}^{\text{non}}$, and let $\varpi(\Psi_M^{\text{non}})$ be the number of generalised cycle digraphs of order ≥ 3 in $\Psi_M^{\text{non}} \in \mathcal{M}^{\text{non}}$. Let $\varepsilon(M) = 1$ if M is loopless, and let $\varepsilon(M) = 0$ otherwise. Then

$$\det(B(M)) = \varepsilon(M) + (-1)^{n-1} + \sum_{\Psi_M^{\text{non}} \in \mathcal{M}^{\text{non}}} (-1)^{\left(c(\Psi_M^{\text{non}}) - N_{\Psi_M^{\text{non}}}^{l}\right)} \cdot 2^{\varpi(\Psi_M^{\text{non}})}.$$

If $\Psi_{M'\overrightarrow{C_n^0}}$, $\Psi_{M\overrightarrow{C_n^n}}$, $\Psi_{M\overrightarrow{P_{n,0,n-1}}}$, and $\Psi_{M\overrightarrow{P_{n,p,q}}}$, where $0 \leq p < n-1$ do not exist, then the term $(-1)^{n-1}$ is taken to be zero.

Proof. Consider first the term corresponding to the identity permutation. If M is loopless, then $b_{11} = b_{22} = \cdots = b_{nn} = 1$, and hence $\prod_{i=1}^{n} b_{ii} = 1$. If M contains at least one directed loop, then at least one diagonal entry of $B(M)$ is zero, so this product vanishes. Next, by Lemmas 2.6 and 2.7, the contribution associated with $\Psi_{M\overrightarrow{C_n}}$, $\Psi_{M\overrightarrow{C_n^n}}$, $\Psi_{M\overrightarrow{P_{n,0,n-1}}}$, or $\Psi_{M\overrightarrow{P_{n,p,q}}}$ (where $0 \leq p < n-1$) is $(-1)^{n-1}$. If none of these spanning sub-mixed graphs exist, this contribution is zero. By Lemma 2.8, every spanning sub-mixed graph containing an undirected path, an undirected cycle, an edge, or a directed loop as a component-class contributes zero. Thus, only spanning non-elementary sub-mixed graphs remain. For each such graph Ψ_M^{non}, Lemma 2.9 gives $\prod_{i=1}^{n} b_{is(i)} = 1$. The sign of the associated permutation is $(-1)^{c(\Psi_M^{\text{non}})}$, while each generalised cycle component of order at least 3 contributes

two possible cyclic orientations. Hence, the total contribution of all spanning non-elementary sub-mixed graphs is $\sum_{\Psi_M^{\rm non}\in\mathcal{M}^{\rm non}}(-1)^{\left(c(\Psi_M^{\rm non})-N^l_{\Psi_M^{\rm non}}\right)}\cdot 2^{\varpi(\Psi_M^{\rm non})}$. If no such $\Psi_M^{\rm non}$ exists, this sum is zero. Combining these contributions yields the stated determinant formula. □

The preceding theorem already contains the formula for $\det(B(\overrightarrow{D}))$ as a special case, since every digraph may be regarded as a mixed graph with no edges. In the undirected case, the structure of the corresponding spanning non-elementary subgraphs is slightly different. The following corollary is therefore obtained as an immediate consequence of Theorem 2.10.

Corollary 2.11. Given an undirected graph G of order n. Let $\mathcal{G}^{\rm non}$ be the collection of all spanning non-elementary subgraphs $\Psi_G^{\rm non}$ of G. For any $\Psi_G^{\rm non}\in\mathcal{G}^{\rm non}$, let $\mathcal{E}(\Psi_G^{\rm non})=\{N_{j_a}\}_{a=1}^{k}$. Let $c(\Psi_G^{\rm non})$ be the number of component-classes of order ≥ 2 in $\Psi_G^{\rm non}\in\mathcal{G}^{\rm non}$, $N^l_{\Psi_G^{\rm non}}$ be the number of isolated vertices in $\Psi_G^{\rm non}\in\mathcal{G}^{\rm non}$, and let $\varpi(\Psi_G^{\rm non})$ be the number of cycle components of order ≥ 3 in $\Psi_G^{\rm non}\in\mathcal{G}^{\rm non}$. Let $\varepsilon(G)=1$ whenever G is loopless, and let $\varepsilon(G)=0$ otherwise. Then

$$\det\left(B\left(G\right)\right)=\varepsilon(G)+(-1)^{n-1}+\sum_{\Psi_G^{\rm non}\in\mathcal{G}^{\rm non}}(-1)^{\left(c(\Psi_G^{\rm non})-N^l_{\Psi_G^{\rm non}}\right)}\cdot 2^{\varpi(\Psi_G^{\rm non})}.$$

Furthermore, if $\Psi_{G_{P_{n,n-1,0}}}$ does not exist, then the term $(-1)^{n-1}$ vanishes.

Proof. The result follows from Theorem 2.10 by taking the mixed graph to have no directed arcs. Then the spanning non-elementary sub-mixed graphs are precisely the spanning non-elementary subgraphs of G, and the corresponding component contributions remain unchanged. Hence, the stated result follows. □

2.3. **Characteristic polynomial of antiadjacency matrix.** We now consider $P(\lambda;B(M))$. Since its coefficients are determined by the principal minors of $B(M)$, the determinant formula obtained above can be applied to the corresponding induced sub-mixed graphs.

Theorem 2.12. Given a mixed graph M of order n. For $1\leq j\leq n$, let $\mathcal{M}_j=\{M' : M' \text{is an induced sub-mixed graph of } M \text{ of order } j\}$. Let $(\mathcal{M}')^{\rm non}$ be the collection of all spanning non-elementary sub-mixed graphs $\Psi_{M'}^{\rm non}$ of $M'\in\mathcal{M}$. For any $\Psi_{M'}^{\rm non}\in(\mathcal{M}')^{\rm non}$, let $\mathcal{E}(\Psi_{M'}^{\rm non})=\{E_{j_a}^{\rm path}\}_{a=1}^{r}\bigcup\{E_{l_b}^{\rm cycle}\}_{b=1}^{t}$, where $(j_{p_a},j_{q_a})\in\mathcal{A}_{j_a}$ and $l_{p_b}\in\mathcal{B}_{l_b}$. Let $c(\Psi_{M'}^{\rm non})$ denote the number of component-classes of order ≥ 2 in $\Psi_{M'}^{\rm non}\in(\mathcal{M}')^{\rm non}$, $N^l_{\Psi_{M'}^{\rm non}}$ be the number of isolated vertices in $\Psi_{M'}^{\rm non}\in\mathcal{M}^{\rm non}$, and let $\varpi(\Psi_{M'}^{\rm non})$ denote the number of its generalised cycle digraph components of order ≥ 3. Let $\varepsilon(M')=1$ if $M'\in\mathcal{M}$ is loopless, and let $\varepsilon(M')=0$ otherwise. Then, for every $1\leq j\leq n$,

$$c_j=\sum_{M'\in\mathcal{M}_j}\left(\varepsilon(M')+(-1)^{j-1}+\sum_{\Psi_{M'}^{\rm non}\in(\mathcal{M}')^{\rm non}}(-1)^{\left(c(\Psi_{M'}^{\rm non})-N^l_{\Psi_{M'}^{\rm non}}\right)}2^{\varpi(\Psi_{M'}^{\rm non})}\right).$$

Furthermore, if $\Psi_{M'\overrightarrow{C_j^0}}$, $\Psi_{M'\overrightarrow{C_j^j}}$, $\Psi_{M'\overrightarrow{P_{j,0,j-1}}}$, and $\Psi_{M'\overrightarrow{P_{j,p,q}}}$, where $0 \leq p < j-1$, do not exist, then the term $(-1)^{j-1}$ is taken to be zero.

Proof. By Theorem 2.10, each coefficient c_j is obtained by summing the determinants $\det(B(M'))$ over the corresponding induced sub-mixed graphs M'. Hence, the required expression follows. □

As in the mixed graph case, the coefficients of $P(\lambda; B(G))$ are determined by the principal minors of $B(G)$. Hence, the determinant formula for induced subgraphs of G yields the following corollary.

Corollary 2.13. Given an undirected graph G of order n. For $1 \leq j \leq n$, let $\mathcal{G}_j = \{G' : G' \text{ is an induced subgraph of } G \text{ of order } j\}$. Let $(\mathcal{G}')^{\text{non}}$ be the collection of all spanning non-elementary subgraphs $\Psi_{G'}^{\text{non}}$ of $G' \in \mathcal{G}$. For any $\Psi_{G'}^{\text{non}} \in (\mathcal{G}')^{\text{non}}$, let $\mathcal{E}\left(\Psi_{G'}^{\text{non}}\right) = \{N_{j_a}\}_{a=1}^{k}$. Let $c(\Psi_{G'}^{\text{non}})$ denote the number of component-classes of order ≥ 2 in $\Psi_{G'}^{\text{non}} \in (\mathcal{G}')^{\text{non}}$, let $N^l_{\Psi_{G'}^{\text{non}}}$ be the number of isolated vertices in $\Psi_{G'}^{\text{non}} \in (\mathcal{G}')^{\text{non}} \in (\mathcal{G}')^{\text{non}}$, and let $\varpi(\Psi_{G'}^{\text{non}})$ denote the number of cycle components of order ≥ 3 in $\Psi_{G'}^{\text{non}} \in (\mathcal{G}')^{\text{non}}$. Let $\varepsilon(G') = 1$ if G' is loopless, and let $\varepsilon(G') = 0$ otherwise. Then, for every $1 \leq j \leq n$,

$$c_j = \sum_{G' \in \mathcal{G}_j} \left(\varepsilon(G') + (-1)^{j-1} + \sum_{\Psi_{G'}^{\text{non}} \in (\mathcal{G}')^{\text{non}}} (-1)^{\left(c(\Psi_{G'}^{\text{non}}) - N^l_{\Psi_{G'}^{\text{non}}}\right)} \cdot 2^{\varpi(\Psi_{G'}^{\text{non}})} \right).$$

Moreover, if $\Psi_{G'P_{j,j-1,0}}$ does not exist, then the term $(-1)^{j-1}$ is taken to be zero.

2.4. **Eigenvalues of antiadjacency matrix.** We now discuss the eigenvalue properties of $B(M)$. We begin by recalling known spectral-radius results for the antiadjacency matrix in the acyclic and cyclic cases.

Theorem 2.14. [18] Let M_a be an acyclic mixed graph of order n. Then

$$\min\left\{\left|\left(N^+(\nu_i)\right)^C\right| : \nu_i \in V_{M_a}\right\} \leq \rho\left(B(M_a)\right) \leq \max\left\{\left|\left(N^+(\nu_i)\right)^C\right| : \nu_i \in V_{M_a}\right\}.$$

Theorem 2.15. [18] Let M_c be a cyclic mixed graph containing $\overrightarrow{C_n}$ as an induced sub-mixed graph. Then $\eta - 1 \leq \rho\left(B(M_c)\right) \leq \max\left\{\left|\left(N^+(\nu_i)\right)^C\right| : \nu_i \in V_{M_c}\right\}$, where η denotes the order of the largest cycle in M_c.

We define the complement energy of a mixed graph M by $\varepsilon_{B(M)} = \sum_{i=1}^{n} |\lambda_i|$, where $\lambda_i \in \sigma(B(M))$ for $1 \leq i \leq n$. An upper bound for $\varepsilon_{B(M)}$ follows directly from Theorem 2.14 and Theorem 2.15, and is stated in Corollary 2.16.

Corollary 2.16. If $\lambda_i \in \sigma(B(M))$ for $1 \leq i \leq n$, then $\varepsilon_{B(M)} \leq n \max\left\{|(N^+(\nu_i))^C| : \nu_i \in V_M\right\}$.

Proof. By Theorems 2.14 and 2.15, for every $\lambda_i \in \sigma(B(M))$, $|\lambda_i| \leq \max\left\{\left|\left(N^+(\nu_i)\right)^C\right| : \nu_i \in V_M\right\}$. Summing over $1 \leq i \leq n$ yields the required bound. □

Having established bounds for $\sigma(B(M))$ in Theorem 2.14 and Theorem 2.15, we now derive a structural sufficient condition for $0 \in \sigma(B(M))$, as shown in Theorem 2.17 below.

Theorem 2.17. $0 \in \sigma(B(M))$, where M is a mixed graph of order n, whenever:

(a) M has a directed loop;
(b) $\Psi_{M_{\overrightarrow{C_n}}}$; $\Psi_{M_{\overrightarrow{C_n^n}}}$, $\Psi_{M_{\overrightarrow{P_{n,0,n-1}}}}$ and $\Psi_{M_{\overrightarrow{P_{n,p,q}}}}$ (where $0 \leq p < n-1$) do not exist; and
(c) a spanning non-elementary sub-mixed graph does not exist.

Proof. By Theorem 2.10, the stated conditions imply that $\det(B(M)) = 0$. Since a matrix is singular if and only if 0 is one of its eigenvalues, it follows that $0 \in \sigma(B(M))$ whenever these conditions are satisfied. □

3. Hermitian antiadjacency matrix of mixed graphs, digraphs, and undirected graphs

We now turn to the matrix $T(M)$. Having studied $B^k(M) = (b_{ij}^{(k)})$, $\det(B(M))$, $P(\lambda; B(M))$, and eigenvalues of $B(M)$, we develop the corresponding results for $T(M)$. This allows us not only to extend the preceding analysis to the Hermitian setting, but also to compare the results on $T(M)$ and $B(M)$ in the combinatorial and spectral properties. In particular, we examine $T^k(M) = (t_{ij}^{(k)})$, $\det(T(M))$, $P(\lambda; T(M))$, and eigenvalues of $T(M)$, and identify the extent to which these results parallel or differ from those established in the previous sections.

3.1. Powers of Hermitian antiadjacency matrix.

Motivated by the connection between disconnected walks in M and the powers of $B(M)$, we establish analogous results for $T(M)$.

A mixed disconnected walk from v_i to v_j of length k, containing $l \geq 0$ disconnected consecutive pairs of vertices, is denoted by $D_{ij}^{(k,l)}$. Moreover, for $1 \leq i, j \leq n$, the collection of all mixed disconnected walks from v_i to v_j of length k, containing $l \geq 0$ disconnected consecutive pairs of vertices, is denoted by $\mathcal{D}_{ij}^{(k,l)}$.

For a mixed disconnected walk $D_{ij}^{(k,l)} = (v_i = u_{r_0}, u_{r_1}, \ldots, u_{r_k} = v_j) \in \mathcal{D}_{ij}^{(k,l)}$, its weight is defined by $wt\left(D_{ij}^{(k,l)}\right) = t_{u_{r_0},u_{r_1}} t_{u_{r_1},u_{r_2}} \cdots t_{u_{r_{k-1}},u_{r_k}}$. Each consecutive pair $u_{r_s}, u_{r_{s+1}} \in A_M$ (where $0 \leq s \leq k-1$), contributes to the above product according to its adjacency relation in M. More precisely, a step following an arc $(u_{r_s}, u_{r_{s+1}}) \in A_M$ contributes $\overline{\alpha}$, a step taken opposite to an arc $(u_{r_{s+1}}, u_{r_s}) \in A_M$ contributes α, an edge $\{u_{r_s}, u_{r_{s+1}}\} \in A_M$ contributes 0, and a disconnected consecutive pair contributes 1.

Let $f(D_{ij}^{(k,l)})$ and $b(D_{ij}^{(k,l)})$ denote, respectively, the numbers of forward and backward arc steps occurring in $D_{ij}^{(k,l)}$. Then

$$\mathrm{wt}\left(D_{ij}^{(k,l)}\right) = \begin{cases} 0, & \text{if } D_{ij}^{(k,l)} \text{ contains at least one edge,} \\ \alpha^{b(D_{ij}^{(k,l)})} \overline{\alpha}^{f(D_{ij}^{(k,l)})}, & \text{otherwise.} \end{cases}$$

In particular, if $D_{ij}^{(k,l)}$ is a directed path from v_i to v_j whose k steps all follow the orientations of the corresponding arcs, then $\mathrm{wt}\left(D_{ij}^{(k,l)}\right) = \overline{\alpha}^k$. If exactly x of its arc steps are traversed opposite to their orientations and no disconnected pairs or edges occur, then $\mathrm{wt}\left(D_{ij}^{(k,l)}\right) = \alpha^x \overline{\alpha}^{k-x}$. If the walk contains exactly l disconnected consecutive pairs and all remaining steps follow the corresponding arc orientations, then $\mathrm{wt}\left(D_{ij}^{(k,l)}\right) = \overline{\alpha}^{k-l}$. More generally, if l disconnected consecutive pairs and x backward arc steps occur, with no edge incident, then $\mathrm{wt}\left(D_{ij}^{(k,l)}\right) = \alpha^x \overline{\alpha}^{k-l-x}$.

The preceding weight interpretation yields the following formula for the entries of arbitrary powers of $T(M)$.

Theorem 3.1. For $k \geq 2$ and $1 \leq i, j \leq n$, we have

$$t_{ij}^{(k)} = \sum_{D_{ij}^{(k,l)} \in \mathcal{D}_{ij}^{(k,l)}} \mathrm{wt}\left(D_{ij}^{(k,l)}\right).$$

Proof. For $k = 2$, the result follows directly from matrix multiplication and the definition of $\mathrm{wt}\left(D_{ij}^{(2,l)}\right)$. The proof is analogous to that of Theorem 2.1, with each mixed disconnected walk contributing its corresponding weight determined by the entries of $T(M)$. Hence, the result follows by mathematical induction. □

Recall that $B(M) = (b_{ij})$ is obtained by substituting $\overline{\alpha} = 1$ and $\alpha = 0$ into $T(M) = (t_{ij})$. Applying the same specialisation to Equation (3.1) in Theorem 3.1 shows that the weighted sum of mixed disconnected walks $D_{ij}^{(k,l)}$ reduces to the number of mixed disconnected walks from v_j to v_i. Hence, Equation (3.1) reduces to Equation (2.1) in Theorem 2.1. This establishes the corresponding relation between the powers of $T(M)$ and $B(M)$.

Now, we are interested in discussing the representation of the main diagonals $t_{ii}^{(2)}$ of $T^2(M)$. Theorem 3.2 shows that $t_{ii}^{(2)}$ corresponds to the neighbourhood structures.Furthermore, Theorem 3.3 shows the sum of $t_{ii}^{(2)}$.

Theorem 3.2. For every $1 \leq i \leq n$, we obtain $t_{ii}^{(2)} = \left|\left(N_M^+(v_i)\right)^C \cup \left(N_M^-(v_i)\right)^C\right|$.

Proof. Since $T(M)$ is Hermitian, $t_{ii}^{(2)} = \sum_{q=1}^n t_{iq} t_{qi} = \sum_{q=1}^n |t_{iq}|^2$. By the definition of $T(M)$, $|t_{iq}|^2 = 1$ precisely when $v_q \in \left(N_M^+(v_i)\right)^C \cup \left(N_M^-(v_i)\right)^C$, and $|t_{iq}|^2 = 0$ otherwise. Therefore, $t_{ii}^{(2)}$ is equal to the cardinality of this union, which proves the result. □

Corollary 3.3. For $1 \leq i \leq n$, we have $\sum_{i=1}^n t_{ii}^{(2)} = n^2 - \left(l(M) + 2d(M)\right)$.

Proof. Since $t_{ii}^{(2)} = \left|\left(N_M^+(v_i)\right)^C \bigcup \left(N_M^-(v_i)\right)^C\right| = n - \left|N_M^+(v_i) \bigcup N_M^-(v_i)\right|$. Moreover, $\left|N_M^+(v_i) \bigcup N_M^-(v_i)\right| = 1 + d(M)$ whenever the vertex $v_i \in V_M$ is loopless, while $\left|N_M^+(v_i) \bigcup N_M^-(v_i)\right| = d(M)$. Then, summing the main diagonal entries $t_{ii}^{(2)}$ over $1 \leq i \leq n$ yields the required result. □

Recall that $P(\lambda;T(M)) = \lambda^n + \sum_{k=1}^{n}(-1)^k\, a_k\, \lambda^{n-k}$. Since $t_{ii} = 1$ precisely when v_i has no directed loop, we have $a_1 = \mathrm{tr}(T(M)) = n - l(M)$. For the second coefficient, the identity $a_2 = \frac{1}{2}\left((\mathrm{tr}\, T(M))^2 - \mathrm{tr}(T^2(M))\right)$ is used. By Theorem 3.2, each diagonal entry $t_{ii}^{(2)}$ is determined by the number of vertices that are neither adjacent to nor adjacent from v_i. Consequently, $\mathrm{tr}(T^2(M))$ depends on the non-neighbourhood structure of M, leading to the following result.

Theorem 3.4. If $P(\lambda;T(M)) = \lambda^n + \sum_{k=1}^{n}(-1)^k\, a_k\, \lambda^{n-k}$, then

$$a_2 = \frac{(l(M))^2 - (2n-1)l(M) + 2d(M)}{2}.$$

Proof. Using $a_2 = \frac{1}{2}\left((\mathrm{tr}(T(M)))^2 - \mathrm{tr}(T^2(M))\right)$, together with $\mathrm{tr}(T(M)) = n - l(M)$ and Corollary 3.3, $\mathrm{tr}(T^2(M)) = n^2 - \big(l(M) + 2d(M)\big)$, we obtain $a_2 = \frac{1}{2}\left((n - l(M))^2 - n^2 + l(M) + 2d(M)\right)$. Therefore, $a_2 = \frac{(l(M))^2-(2n-1)l(M)+2d(M)}{2}$, and the result follows. □

The preceding theorem determines the second coefficient of $P(\lambda;T(M))$ by using a trace identity. However, as in the derivation of $P(\lambda;B(M))$, the full characteristic polynomial of $T(M)$ is more conveniently obtained through principal minors rather than by continuing with trace identities. In particular, for each $1 \le i \le n$, the coefficient a_i is given by the sum of all $i \times i$ principal minors of $T(M)$. Therefore, to determine $P(\lambda;T(M))$, we apply the determinant formula for $T(M)$ to the corresponding induced sub-mixed graphs.

3.2. **Determinant of the Hermitian antiadjacency matrix.** Next, we examine $\det(T(M))$. In contrast to $B(M)$, the nonzero entries of $T(M)$ may carry Hermitian weights; hence, the contribution of each permutation in the Leibniz expansion depends not only on the associated spanning sub-mixed graph but also on the orientation of all arcs of its component-classes. Thus, to evaluate $\det(T(M))$, it is necessary to identify the spanning sub-mixed graphs whose corresponding permutation terms are nonzero and to determine their Hermitian weights. The first case arises from spanning generalised cycles of the form $\Psi_{M\overrightarrow{C_n^p}}$, where $p = 0$ or $p = n$, as stated in the following lemma.

Lemma 3.5. If $\Psi_{M\overrightarrow{C_n^p}}$ exists, then

$$\prod_{i=1}^{n} t_{(i \mod n)((i+1) \mod n)} = \overline{\alpha}^{\,a(\vartheta_{r_p},\vartheta_{r_{p+1}})\Psi_{M\overrightarrow{C_n^p}}} \cdot \alpha^{\,a(\vartheta_{r_{p+1}},\vartheta_{r_p})\Psi_{M\overrightarrow{C_n^p}}}$$

and

$$\prod_{i=1}^{n} t_{((i+1) \mod n)(i \mod n)} = \overline{\alpha}^{\,a(\vartheta_{r_p},\vartheta_{r_{p+1}})\Psi_{M\overrightarrow{C_n^p}}} \cdot \alpha^{\,a(\vartheta_{r_{p+1}},\vartheta_{r_p})\Psi_{M\overrightarrow{C_n^p}}}$$

Proof. Suppose that there exists $\Psi_{M\overrightarrow{C_n^p}}$. For $0 < p < n$, there exist both arcs $(v_{(l \mod n)}, v_{((l+1) \mod n)}) \in V_{\Psi_{M\overrightarrow{C_n^p}}}$ and $(v_{((j+1) \mod n)}, v_{(j \mod n)}) \in V_{\Psi_{M\overrightarrow{C_n^p}}}$ such that $t_{((l \mod n))((l+1) \mod n)} = \overline{\alpha}$ and $t_{((j+1) \mod n)(j \mod n)} = \alpha$ (where $1 \le l, j \le n$ and $l \neq j$), which implies

$\prod_{i=1}^{n} t_{(i \mod n)((i+1) \mod n)} = \overline{\alpha}^{a(\vartheta_{r_p},\vartheta_{r_{p+1}})_{\Psi_{M_{\overrightarrow{C_n^p}}}}} \cdot \alpha^{a(\vartheta_{r_{p+1}},\vartheta_{r_p})_{\Psi_{M_{\overrightarrow{C_n^p}}}}}$ and $\prod_{i=1}^{n} t_{((i+1) \mod n)(i \mod n)} = \overline{\alpha}^{a(\vartheta_{r_p},\vartheta_{r_{p+1}})_{\Psi_{M_{\overrightarrow{C_n^p}}}}} \cdot \alpha^{a(\vartheta_{r_{p+1}},\vartheta_{r_p})_{\Psi_{M_{\overrightarrow{C_n^p}}}}}$. For $p = 0$, we obtain $\overrightarrow{C_n^p} = \overrightarrow{C_n}$. This means that all the arcs are in the form of $(v_{(l \mod n)}, v_{((l+1) \mod n)}) \in V_{\Psi_{M_{\overrightarrow{C_n^p}}}}$, implying $\prod_{i=1}^{n} t_{(i \mod n)((i+1) \mod n)} = \overline{\alpha}^n$ and $\prod_{i=1}^{n} t_{((i+1) \mod n)(i \mod n)} = \alpha^n$. Lastly, for $p = n$, all the arcs are in the form of $(v_{((j+1) \mod n)}, v_{(j \mod n)}) \in V_{\Psi_{M_{\overrightarrow{C_n^p}}}}$, indicating $\prod_{i=1}^{n} t_{(i \mod n)((i+1) \mod n)} = \alpha^n$ and $\prod_{i=1}^{n} t_{((i+1) \mod n)(i \mod n)} = \overline{\alpha}^n$. □

The next types of spanning sub-mixed graphs are generalised path digraphs. The value of the term that corresponds to this spanning sub-mixed graph is given in the following lemma.

Lemma 3.6. If $\Psi_{M_{\overrightarrow{P_{n,p,q}}}}$ exists, then

$$\prod_{i=1}^{n} t_{(i \mod n)((i+1) \mod n)} = \overline{\alpha}^{a(\vartheta_{r_p},\vartheta_{r_{p+1}})_{\Psi_{M_{\overrightarrow{P_{n,p,q}}}}}} \cdot \alpha^{a(\vartheta_{r_{p+1}},\vartheta_{r_p})_{\Psi_{M_{\overrightarrow{P_{n,p,q}}}}}}$$

and

$$\prod_{i=1}^{n} t_{((i+1) \mod n)(i \mod n)} = \overline{\alpha}^{a(\vartheta_{r_p},\vartheta_{r_{p+1}})_{\Psi_{M_{\overrightarrow{P_{n,p,q}}}}}} \cdot \alpha^{a(\vartheta_{r_{p+1}},\vartheta_{r_p})_{\Psi_{M_{\overrightarrow{P_{n,p,q}}}}}}.$$

Proof. Suppose that there exists $\Psi_{M_{\overrightarrow{P_{n,p,q}}}}$. For $p = n-1$ and $q = 0$, $\Psi_{M_{\overrightarrow{P_{n,p,q}}}}$ is a null digraph $\overrightarrow{N_n}$ which implies $\prod_{i=1}^{n} t_{(i \mod n)((i+1) \mod n)} = 1 = \overline{\alpha}^0 \cdot \alpha^0$ and $\prod_{i=1}^{n} t_{((i+1) \mod n)(i \mod n)} = 1 = \overline{\alpha}^0 \cdot \alpha^0$. Now, for $p \neq n-1$ and $q \neq 0$, each arc of the form (v_j, v_{j+1}), where $1 \leq j \leq n-1$ is associated with the entries $t_{j(j+1)} = \overline{\alpha}$ and $t_{(j+1)j} = \alpha$. If we change an arc of the form (v_j, v_{j+1}) into (v_{j+1}, v_j), then the corresponding entries are $t_{j(j+1)} = \alpha$ and $t_{(j+1)j} = \overline{\alpha}$. Furthermore, if an arc of the form (v_j, v_{j+1}) is reversed to (v_{j+1}, v_j), then the corresponding entries satisfy $t_{j(j+1)} = t_{(j+1)j} = 1$. Hence, the required results follow. □

The following lemma identifies the spanning sub-mixed graphs whose corresponding terms in the determinant expansion of $T(M)$ vanish.

Lemma 3.7. Let Ψ_M be a spanning sub-mixed graph of M associated with the term $\prod_{i=1}^{n} t_{is(i)}$, where $s \in S_n$. If Ψ_M contains a component-class that is an edge, a directed loop, an undirected cycle, or an undirected path, then $\prod_{i=1}^{n} t_{is(i)} = 0$.

We now consider the remaining case, namely spanning non-elementary sub-mixed graphs.

Lemma 3.8. For any term of the form $\prod_{i=1}^{n} t_{is(i)}$, where $s \in S_n$, of Ψ_M^{non}, if $s \neq s_0$, $s \neq s_1$, and $s \neq s_2$, then

$$\prod_{i=1}^{n} t_{is(i)} = \alpha^{\delta\left(\vartheta_{r_p},\vartheta_{r_{p+1}}\right)_{\Psi_M^{\text{non}}}} \overline{\alpha}^{\delta\left(\vartheta_{r_{q+1}},\vartheta_{r_q}\right)_{\Psi_M^{\text{non}}}}.$$

Proof. The component-classes of Ψ_M^{non} can be expressed as

$$\mathcal{E}\left(\Psi_M^{\mathrm{non}}\right) = \left\{\overrightarrow{P_{j_a,j_{p_a},j_{q_a}}}\right\}_{a=1}^{r} \cup \left\{\overrightarrow{C_{l_b}^{l_{p_b}}}\right\}_{b=1}^{t}.$$

Consider first a component-class $\overrightarrow{P_{j_a,j_{p_a},j_{q_a}}}$. Its consecutive pairs consist of forward arcs, backward arcs, and disconnected pairs. In $T(M)$, these contribute $\overline{\alpha}$, α, and 1, respectively. Hence, the contribution of this component-class is determined only by the numbers of forward and backward arcs.

Similarly, for a component-class $\overrightarrow{C_{l_b}^{l_{p_b}}}$, each forward arc contributes $\overline{\alpha}$ and each backward arc contributes α. Therefore, its contribution is again determined by the corresponding numbers of arcs in the two orientations. Multiplying the contributions of all component-classes yields the stated result. □

Since spanning non-elementary sub-mixed graphs constitute the remaining class that may contribute nonzero terms to $\det(T(M))$, the determinant formula can now be established.

For $\Psi \in \mathcal{G}_n(M)$, let $a(\vartheta_{r_p}, \vartheta_{r_{p+1}})_{\Psi_{\mathcal{G}_n(M)}}$ and $a(\vartheta_{r_{p+1}}, \vartheta_{r_p})_{\Psi_{\mathcal{G}_n(M)}}$ denote the numbers of arcs of the forms $(\vartheta_{r_p}, \vartheta_{r_{p+1}})$ and $(\vartheta_{r_{p+1}}, \vartheta_{r_p})$, respectively.

Theorem 3.9. Given a mixed graph M of order n. Let $\mathcal{M}^{\mathrm{non}}$ be the collection of all spanning non-elementary sub-mixed graphs Ψ_M^{non} of M. Let $c(\Psi_M^{\mathrm{non}})$ be the number of component-classes of order ≥ 2 in $\Psi_M^{\mathrm{non}} \in \mathcal{M}^{\mathrm{non}}$, $N^l_{\Psi_M^{\mathrm{non}}}$ be the number of isolated vertices in $\Psi_M^{\mathrm{non}} \in \mathcal{M}^{\mathrm{non}}$, and let $\varpi(\Psi_M^{\mathrm{non}})$ denote the number of its generalised cycle digraphs of order ≥ 3 in $\Psi_M^{\mathrm{non}} \in \mathcal{M}^{\mathrm{non}}$. Let $\varepsilon(M) = 1$ if M is loopless; otherwise, let $\varepsilon(M) = 0$. Then the following holds:

$$\det(T(M)) = \varepsilon(M) + (-1)^{n-1}\, 2\, \mathrm{Re}\left(\alpha^{\left|a(\vartheta_{r_p},\vartheta_{r_{p+1}})_{\Psi_{\mathcal{G}_n(M)}} - a(\vartheta_{r_{p+1}},\vartheta_{r_p})_{\Psi_{\mathcal{G}_n(M)}}\right|}\right) +$$

$$\sum_{\Psi_M^{\mathrm{non}} \in \mathcal{M}^{\mathrm{non}}} (-1)^{\left(c(\Psi_M^{\mathrm{non}}) - N^l_{\Psi_M^{\mathrm{non}}}\right)} \cdot 2^{\varpi(\Psi_M^{\mathrm{non}})} \cdot \mathrm{Re}\left(\alpha^{\left|\delta(\vartheta_{r_p},\vartheta_{r_{p+1}})_{\Psi_M^{\mathrm{non}}} - \delta(\vartheta_{r_{q+1}},\vartheta_{r_q})_{\Psi_M^{\mathrm{non}}}\right|}\right).$$

If no $\Psi_M \in \mathcal{G}_n(M)$ exists, then the term

$$(-1)^{n-1}\, 2\, \mathrm{Re}\left(\alpha^{\left|a(\vartheta_{r_p},\vartheta_{r_{p+1}})_{\Psi_{\mathcal{G}_n(M)}} - a(\vartheta_{r_{p+1}},\vartheta_{r_p})_{\Psi_{\mathcal{G}_n(M)}}\right|}\right)$$

vanishes.

Proof. The combinatorial argument is analogous to that used in the proof of Theorem 2.10. The contribution of the identity permutation is $\varepsilon(M)$. For the two cyclic permutations, Lemmas 3.5 and 3.6 show that whenever $\Psi_M \in \mathcal{G}_n(M)$ exists, their contributions are complex conjugates. Hence, their total contribution is

$$(-1)^{n-1}\, 2\, \mathrm{Re}\left(\alpha^{\left|a(\vartheta_{r_p},\vartheta_{r_{p+1}})_{\Psi_{\mathcal{G}_n(M)}} - a(\vartheta_{r_{p+1}},\vartheta_{r_p})_{\Psi_{\mathcal{G}_n(M)}}\right|}\right).$$

If no $\Psi_M \in \mathcal{G}_n(M)$ exists, this contribution is zero. By Lemma 3.7, the remaining nonzero contributions arise from spanning non-elementary sub-mixed

graphs. Using Lemma 3.8, their total contribution is

$$\sum_{\Psi_M^{\mathrm{non}} \in \mathcal{M}^{\mathrm{non}}} (-1)^{c(\Psi_M^{\mathrm{non}}) - N^l_{\Psi_M^{\mathrm{non}}}} \cdot 2^{\varpi(\Psi_M^{\mathrm{non}})} \cdot \mathrm{Re}\left(\alpha^{\left| \delta(\vartheta_{r_p}, \vartheta_{r_{p+1}})_{\Psi_M^{\mathrm{non}}} - \delta(\vartheta_{r_{q+1}}, \vartheta_{r_q})_{\Psi_M^{\mathrm{non}}} \right|} \right).$$

Combining these contributions yields the stated determinant formula. □

Similar to the relation between the powers of the Hermitian antiadjacency and antiadjacency matrices, substituting $\overline{\alpha} = 1$ and $\alpha = 0$ into the determinant formula of Theorem 3.9 reduces it to the determinant formula of Theorem 2.10. Thus, the determinant of $B(M)$ is obtained as a special case of the determinant of $T(M)$.

3.3. **Characteristic polynomial of Hermitian antiadjacency matrix.** Since each coefficient of $P(\lambda; T(M))$ is determined by the principal minors of $T(M)$, the determinant formula obtained above can be applied to the corresponding induced sub-mixed graphs. This yields the following expression for the coefficients of the characteristic polynomial.

Theorem 3.10. Given a mixed graph M of order n. For $1 \leq j \leq n$, let $\mathcal{M}_j = \{M' : M' \text{ is an induced sub-mixed graph of } M \text{ of order } j\}$. Let $(\mathcal{M}')^{\mathrm{non}}$ be the collection of all spanning non-elementary sub-mixed graphs $\Psi^{\mathrm{non}}_{M'}$ of $M' \in \mathcal{M}$. Let $c(\Psi^{\mathrm{non}}_{M'})$ denote the number of component-classes of order ≥ 2 in $\Psi^{\mathrm{non}}_{M'} \in (\mathcal{M}')^{\mathrm{non}}$, $N^l_{\Psi^{\mathrm{non}}_{M'}}$ be the number of isolated vertices in $\Psi^{\mathrm{non}}_{M'} \in \mathcal{M}^{\mathrm{non}}$, and let $\varpi(\Psi^{\mathrm{non}}_{M'})$ denote the number of its generalised cycle digraph components of order ≥ 3. Let $\varepsilon(M') = 1$ if $M' \in \mathcal{M}$ is loopless, and let $\varepsilon(M') = 0$ otherwise. Then, for every $1 \leq j \leq n$,

$$a_j = \sum_{M' \in \mathcal{M}_j} \left(\varepsilon(M') + (-1)^{j-1}\, 2\ \mathrm{Re}\left(\alpha^{\left| a(\vartheta_{r_p}, \vartheta_{r_{p+1}})_{\Psi_{\mathcal{G}_j(M')}} - a(\vartheta_{r_{p+1}}, \vartheta_{r_p})_{\Psi_{\mathcal{G}_j(M')}} \right|} \right) \right.$$

$$\left. + \sum_{\Psi^{\mathrm{non}}_{M'} \in (\mathcal{M}')^{\mathrm{non}}} (-1)^{\left(c(\Psi^{\mathrm{non}}_{M'}) - N^l_{\Psi^{\mathrm{non}}_{M'}} \right)} 2^{\varpi(\Psi^{\mathrm{non}}_{M'})}\, \mathrm{Re}\left(\alpha^{\left| \delta(\vartheta_{r_p}, \vartheta_{r_{p+1}})_{\Psi^{\mathrm{non}}_{M'}} - \delta(\vartheta_{r_{q+1}}, \vartheta_{r_q})_{\Psi^{\mathrm{non}}_{M'}} \right|} \right) \right),$$

Moreover, if no $\Psi_{M'} \in \mathcal{G}_j(M')$ exists, then the corresponding term

$$(-1)^{j-1}\, 2\ \mathrm{Re}\left(\alpha^{\left| a(\vartheta_{r_p}, \vartheta_{r_{p+1}})_{\Psi_{\mathcal{G}_j(M')}} - a(\vartheta_{r_{p+1}}, \vartheta_{r_p})_{\Psi_{\mathcal{G}_j(M')}} \right|} \right)$$

vanishes.

Proof. For each $1 \leq j \leq n$, the coefficient a_j is the sum of all $j \times j$ principal minors of $T(M)$. Every such principal submatrix is the Hermitian antiadjacency matrix $T(M')$ of an induced sub-mixed graph M' of order j. Hence, $a_j = \sum_{M'}(\det(T(M')))$. Applying Theorem 3.9 to each M' and summing the resulting contributions gives the stated formula. □

Under the specialisation $\alpha = 0$ and $\overline{\alpha} = 1$, the matrix $T(M)$ becomes $B(M)$. Hence, the coefficients of $P(\lambda; T(M))$ in Theorem 3.10 reduce to the corresponding coefficients of $P(\lambda; B(M))$ in Theorem 2.12.

3.4. **Eigenvalues of Hermitian antiadjacency matrix.** Having determined the characteristic polynomial of $T(M)$, we now consider its eigenvalues. Since $T(M)$ is Hermitian, all its eigenvalues are real and may be arranged as

$$\rho(T(M)) = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n;$$

see [11].

For any matrix $A \in \mathbb{C}^{n\times n}$, the Frobenius norm bounds give

$$\sqrt{\frac{1}{n}\sum_{i,j}|a_{ij}|^2} \leq \rho(A) \leq \sqrt{\sum_{i,j}|a_{ij}|^2};$$

see [21, 8]. Applying these inequalities to $T(M)$ yields the following result.

Theorem 3.11. Let M be a mixed graph of order n and size m. Then

$$\sqrt{\frac{n^2-m}{n}} \leq \rho(T(M)) \leq \sqrt{n^2-m}.$$

Proof. From the definition of $T(M)$, each nonzero entry has a modulus of 1, whereas an entry corresponding to an adjacency present in M is zero. Hence, $\sum_{1\leq i,j\leq n}|t_{ij}|^2 = n^2 - m$. Substituting this expression into the Frobenius norm bounds yields the stated result. □

Theorem 3.11 provides bounds for the spectral radius of $T(M)$. We next give a sufficient condition, expressed in terms of the structure of M, for zero to occur as an eigenvalue of $T(M)$.

Theorem 3.12. Suppose that the following conditions hold:

(a) M contains at least one directed loop;
(b) no $\Psi_M \in \mathcal{G}_n(M)$ exists; and
(c) no spanning non-elementary sub-mixed graph Ψ_M^{non} exists.

Then $0 \in \sigma(T(M))$.

Proof. By Theorem 3.9, conditions (a)–(c) eliminate all contributions to the determinant of $T(M)$. Hence, $\det(T(M)) = 0$. Therefore, $T(M)$ is singular, and consequently, $0 \in \sigma(T(M))$. □

Extending the skew-energy concept introduced by Cavers et al. [4], the Hermitian-complement energy of a mixed graph M is defined by $\varepsilon_{T(M)} = \sum_{i=1}^{n}|\lambda_i|$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $T(M)$. The spectral radius bound established in Theorem 3.11 immediately yields the following estimate.

Corollary 3.13. Let M be a mixed graph of order n and size m. Then, we have:

$$\varepsilon_{T(M)} \leq n\sqrt{n^2-m}.$$

Proof. For every $\lambda_i \in \sigma(T(M))$, by Theorem 3.11, we obtain that $|\lambda_i| \leq \rho(T(M)) \leq \sqrt{n^2-m}$. Therefore, $\varepsilon_{T(M)} = \sum_{i=1}^{n}|\lambda_i| \leq n\sqrt{n^2-m}$, which proves the result. □

The bounds for $\varepsilon_{T(M)}$ and $\varepsilon_{B(M)}$ are obtained from different structural properties of M. By Corollary 3.13, the Hermitian-complement energy satisfies $\varepsilon_{T(M)} \leq n\sqrt{n^2 - m}$, so its upper bound is expressed in terms of the order and size of M. In contrast, Corollary 2.16, which follows from Theorem 2.14 and Theorem 2.15, shows that the upper bound for $\varepsilon_{B(M)}$ is determined by the maximum cardinality of the complement of an out-neighbourhood. Thus, the former depends on global parameters of M, whereas the latter depends on a local non-neighbourhood condition.

Acknowledgement. This research was supported by Universitas Gadjah Mada through the RTA Grant 2024 under Grant No. 5286/UN1.P1/PT.01.03/2024. The authors are grateful to the anonymous referees for their valuable comments and suggestions, which helped improve the manuscript.

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