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ON GENERALIZED REDUCED AND AUGMENTED SOMBOR INDICES

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Abstract. This paper investigates two relatively less studied Sombor-type topological indices: the generalized reduced Sombor index and the augmented Sombor index. We establish several upper and lower bounds for these indices in terms of graph parameters such as order, size, maximum and minimum degrees. Furthermore, we characterize the extremal graphs that attain these bounds. Several new inequalities relating these indices to classical Zagreb indices and the general Sombor index are also derived.

Keywords. Chemical graph theory, Topological indices, Generalized reduced Sombor index, Augmented Sombor index
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1. Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$, where $|V(G)| = n$ and $|E(G)| = m$. The degree of a vertex $u \in V(G)$ is denoted by $d(u)$. The maximum and minimum vertex degrees are denoted by Δ and δ, respectively.

The Sombor index, introduced by Gutman in 2021 [9], is defined as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d(u)^2 + d(v)^2}.$$

Since its inception, numerous variants have been proposed and studied [7, 10, 11]. Among these variants, the generalized reduced Sombor index and the augmented Sombor index have received comparatively less attention. This paper aims to fill this gap by establishing new theoretical results for these indices.

The reduced Sombor index, also introduced by Gutman [9], is obtained by subtracting 1 from each vertex degree. It is defined as

$$SO_{red}(G) = \sum_{uv \in E(G)} \sqrt{(d(u) - 1)^2 + (d(v) - 1)^2}.$$

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The generalized reduced Sombor index was introduced by Deng et al. [8] and is defined as

$$SO_{red}^{(p)}(G) = \sum_{uv \in E(G)} \sqrt{[d(u)-p]^2 + [d(v)-p]^2},$$

where p is a real number.

The augmented Sombor index was introduced by Das et al. [6] and is defined as

$$SO_{aug}(G) = \sum_{uv \in E(G)} \sqrt{\frac{d(u)^2 + d(v)^2}{d(u)+d(v)-2}}.$$

Throughout this paper, we assume $p > 0$ for the generalized reduced Sombor index, and for the augmented Sombor index, we consider graphs with $\delta \geq 2$ to avoid singularities.

2. Main results

In this section, we present our main findings. We first establish several upper and lower bounds for the generalized reduced Sombor index $SO_{red}^{(p)}(G)$ for both the special case $p = 1$ and for general $p > 0$. We then derive analogous bounds for the augmented Sombor index $SO_{aug}(G)$. Finally, we explore the relationship between these indices.

2.1. Bounds for the Generalized Reduced Sombor Index.

Theorem 2.1. *Let G be a connected graph with n vertices, m edges, maximum degree Δ, minimum degree δ, and average degree $\bar{d} = \frac{2m}{n}$. Then*

$$\sqrt{2}m(\bar{d}-1) \leq SO_{red}^{(1)}(G) \leq \sqrt{2}m \max\{\Delta - 1, \delta - 1\}.$$

Moreover, equality holds in both bounds if and only if G is regular.

Proof. **Lower Bound:**

For each edge $uv \in E(G)$, since $d(u), d(v) \geq 1$, we have $d(u)-1, d(v)-1 \geq 0$. It is well known that for any two non-negative numbers a and b, $\sqrt{a^2+b^2} \geq \frac{a+b}{\sqrt{2}}$. Using this inequality, we obtain

$$\sqrt{(d(u)-1)^2 + (d(v)-1)^2} \geq \frac{(d(u)-1)+(d(v)-1)}{\sqrt{2}}.$$

Summing over all edges $uv \in E(G)$,

$$SO_{red}^{(1)}(G) \geq \frac{1}{\sqrt{2}} \sum_{uv \in E(G)} (d(u)+d(v)-2). \tag{2.1}$$

Now, since each vertex u appears in exactly $d(u)$ edges, we have

$$\sum_{uv \in E(G)} (d(u)+d(v)) = \sum_{u \in V(G)} d(u)^2 = M_1(G),$$

where $M_1(G)$ denotes the first Zagreb index.

Hence, from (2.1), we have

$$SO_{red}^{(1)}(G) \geq \frac{1}{\sqrt{2}} \left(M_1(G) - 2m \right).$$

By the Cauchy-Schwarz inequality, we have

$$\left(\sum_{i=1}^{n} d(v_i) \right)^2 \leq \left(\sum_{i=1}^{n} d(v_i)^2 \right) \left(\sum_{i=1}^{n} 1^2 \right).$$

Since $\sum_{i=1}^{n} d(v_i) = 2m$, it follows that

$$4m^2 \leq M_1(G) \cdot n.$$

Hence,

$$M_1(G) = \sum_{u \in V(G)} d(u)^2 \geq \frac{4m^2}{n}.$$

Therefore,

$$SO_{red}^{(1)}(G) \geq \frac{1}{\sqrt{2}} \left(\frac{4m^2}{n} - 2m \right) = \sqrt{2}m \left(\frac{2m}{n} - 1 \right) = \sqrt{2}m(\bar{d} - 1).$$

This proves the lower bound.

Upper Bound:
For any edge $uv \in E(G)$, since $\delta \leq d(u), d(v) \leq \Delta$, we have

$$d(u) - 1 \leq \max\{\Delta - 1, \delta - 1\}, \quad d(v) - 1 \leq \max\{\Delta - 1, \delta - 1\}.$$

Thus,

$$\sqrt{(d(u) - 1)^2 + (d(v) - 1)^2} \leq \sqrt{2} \cdot \max\{\Delta - 1, \delta - 1\}.$$

Summing over all m edges,

$$SO_{red}^{(1)}(G) \leq \sqrt{2}m \cdot \max\{\Delta - 1, \delta - 1\}.$$

This proves the upper bound.

Equality Conditions:
Equality in the lower bound requires equality in Cauchy-Schwarz, which occurs if and only if all degrees are equal, i.e., G is regular. Equality in the upper bound requires $d(u) = \Delta = \delta$ for all vertices, which again means G is regular. Thus equality holds in both bounds if and only if G is regular. □

Theorem 2.2. *Let G be a connected graph with m edges, maximum degree Δ, and minimum degree δ. For any real number $p > 0$,*

$$SO_{red}^{(p)}(G) \leq \sqrt{2}m \max\{|\Delta - p|, |\delta - p|\}.$$

Equality holds if and only if G is regular.

Proof. For any edge $uv \in E(G)$, since $\delta \leq d(u), d(v) \leq \Delta$, we have

$$\sqrt{(d(u) - p)^2 + (d(v) - p)^2} \leq \sqrt{2} \cdot \max\{|\Delta - p|, |\delta - p|\}.$$

Summing over all m edges,

$$SO_{red}^{(p)}(G) \leq \sqrt{2}m \cdot \max\{|\Delta - p|, |\delta - p|\}.$$

Equality holds if and only if $d(u) = d(v) = \Delta = \delta$ for every edge, i.e., if and only if G is regular. □

Theorem 2.3. *Let G be a connected graph. For any real number $p > 0$,*

$$SO_{red}^{(p)}(G) \geq \frac{1}{\sqrt{2}} \sum_{u \in V(G)} d(u)|d(u) - p|.$$

Equality holds if and only if for every edge $uv \in E(G)$, $|d(u) - p| = |d(v) - p|$.

Proof. For any edge $uv \in E(G)$, by the inequality $\sqrt{a^2 + b^2} \geq \frac{|a|+|b|}{\sqrt{2}}$, we obtain

$$\sqrt{(d(u) - p)^2 + (d(v) - p)^2} \geq \frac{|d(u) - p| + |d(v) - p|}{\sqrt{2}}. \tag{2.2}$$

Summing over all edges,

$$SO_{red}^{(p)}(G) \geq \frac{1}{\sqrt{2}} \sum_{uv \in E(G)} \left(|d(u) - p| + |d(v) - p|\right).$$

Using the following identity from [12], we obtain

$$\sum_{uv \in E(G)} \left(|d(u) - p| + |d(v) - p|\right) = \sum_{u \in V(G)} d(u)|d(u) - p|.$$

Therefore,

$$SO_{red}^{(p)}(G) \geq \frac{1}{\sqrt{2}} \sum_{u \in V(G)} d(u)|d(u) - p|.$$

Equality in (2.2) occurs if and only if $|d(u) - p| = |d(v) - p|$ for every edge $uv \in E(G)$. □

The following result is obvious.

Corollary 2.4. *If G is a k-regular graph, then for any $p > 0$,*

$$SO_{red}^{(p)}(G) = m\sqrt{2}|k - p|.$$

In particular, for $p = 1$,

$$SO_{red}^{(1)}(G) = m\sqrt{2}(k - 1).$$

Theorem 2.5. *Let T be a tree with $n \geq 2$ vertices. Then*

$$SO_{red}^{(1)}(T) \geq \frac{\sqrt{2}(n-1)(n-2)}{n}.$$

Equality holds if and only if $n = 2$. For $n \geq 3$, the inequality is strict.

Proof. For a tree T, we have $m = n-1$ and $\bar{d} = \frac{2(n-1)}{n}$. By Theorem 2.1,

$$SO_{red}^{(1)}(T) \geq \sqrt{2}(n-1)(\bar{d}-1) = \sqrt{2}(n-1)\left(\frac{2(n-1)}{n} - 1\right) = \frac{\sqrt{2}(n-1)(n-2)}{n}.$$

Equality in Theorem 2.1 requires T to be regular. However, no tree with $n \geq 3$ is regular (since trees have at least two leaves of degree 1). Therefore, the inequality is strict for all trees with $n \geq 3$. □

2.2. **Bounds for the Augmented Sombor Index.**

Theorem 2.6. *Let G be a graph with m edges, $\delta \geq 2$, and maximum degree Δ. Then*

$$m \cdot \frac{\delta}{\sqrt{\Delta - 1}} \leq SO_{aug}(G) \leq m \cdot \frac{\Delta}{\sqrt{\delta - 1}}.$$

The bounds are attained if and only if G is regular.

Proof. For each edge $uv \in E(G)$, we have $\delta \leq d(u), d(v) \leq \Delta$. Therefore,

$$\frac{d(u)^2 + d(v)^2}{d(u) + d(v) - 2} \geq \frac{2\delta^2}{\Delta + \Delta - 2} = \frac{\delta^2}{\Delta - 1}, \tag{2.3}$$

and

$$\frac{d(u)^2 + d(v)^2}{d(u) + d(v) - 2} \leq \frac{2\Delta^2}{\delta + \delta - 2} = \frac{\Delta^2}{\delta - 1}. \tag{2.4}$$

Taking square roots and summing over all m edges we have,

$$m \cdot \frac{\delta}{\sqrt{\Delta - 1}} \leq SO_{aug}(G) \leq m \cdot \frac{\Delta}{\sqrt{\delta - 1}}.$$

For sharpness, equality must hold in (2.3) and (2.4) for every edge $uv \in E(G)$. This requires $d(u) = d(v) = \Delta$ and $d(u) = d(v) = \delta$, which implies $\Delta = \delta$. Hence G is regular.

Conversely, if G is r-regular ($r \geq 2$), then $\Delta = \delta = r$, and

$$SO_{aug}(G) = m \cdot \frac{r}{\sqrt{r - 1}}.$$

Substituting $\Delta = \delta = r$ into the bounds gives

$$m \cdot \frac{r}{\sqrt{r - 1}} \leq SO_{aug}(G) \leq m \cdot \frac{r}{\sqrt{r - 1}},$$

so equality holds in both.

This completes the proof. □

Theorem 2.7. *Let G be a graph with m edges and $\delta \geq 2$. Then*

$$SO_{aug}(G) \leq \sqrt{m \cdot \sum_{uv \in E(G)} \frac{d(u)^2 + d(v)^2}{d(u) + d(v) - 2}}.$$

Proof. For each edge $uv \in E(G)$, define

$$x_{uv} = \sqrt{\frac{d(u)^2 + d(v)^2}{d(u) + d(v) - 2}}.$$

Then

$$SO_{aug}(G) = \sum_{uv \in E(G)} x_{uv}.$$

By the Cauchy-Schwarz inequality,

$$\left(\sum_{uv \in E(G)} x_{uv} \right)^2 \leq \left(\sum_{uv \in E(G)} 1^2 \right) \left(\sum_{uv \in E(G)} x_{uv}^2 \right).$$

Taking square roots and substituting $x_{uv}^2 = \dfrac{d(u)^2 + d(v)^2}{d(u) + d(v) - 2}$, we have

$$SO_{aug}(G) \leq \sqrt{m \cdot \sum_{uv \in E(G)} \frac{d(u)^2 + d(v)^2}{d(u) + d(v) - 2}}.$$

This proves the desired inequality. □

As a direct consequence of Theorem 2.7, we have the following result.

Corollary 2.8. *For any graph G with $\delta \geq 2$,*

$$SO_{aug}(G) \leq \sqrt{\frac{m}{2(\delta - 1)} \cdot \sum_{uv \in E(G)} (d(u)^2 + d(v)^2)}.$$

2.3. **Relation Between the Two Indices.**

Theorem 2.9. *Let G be a graph with $\delta \geq 2$ and let p be a real number such that $1 \leq p \leq \delta$. Then*

$$SO_{red}^{(p)}(G) \leq \sqrt{2(\Delta - 1)} \cdot SO_{aug}(G).$$

Proof. For each edge $uv \in E(G)$, since $p \leq \delta \leq d(u), d(v)$, we have $d(u) - p \leq d(u)$ and $d(v) - p \leq d(v)$. Therefore,

$$\sqrt{[d(u) - p]^2 + [d(v) - p]^2} \leq \sqrt{d(u)^2 + d(v)^2}.$$

Also, since $d(u) + d(v) - 2 \leq 2(\Delta - 1)$,

$$\frac{d(u)^2 + d(v)^2}{d(u) + d(v) - 2} \geq \frac{d(u)^2 + d(v)^2}{2(\Delta - 1)}.$$

Hence,

$$\sqrt{d(u)^2 + d(v)^2} \leq \sqrt{2(\Delta - 1)} \cdot \sqrt{\frac{d(u)^2 + d(v)^2}{d(u) + d(v) - 2}}.$$

Combining these inequalities,

$$\sqrt{[d(u)-p]^2+[d(v)-p]^2} \leq \sqrt{2(\Delta-1)} \cdot \sqrt{\frac{d(u)^2+d(v)^2}{d(u)+d(v)-2}}.$$

Summing over all edges $uv \in E(G)$ gives

$$SO_{red}^{(p)}(G) \leq \sqrt{2(\Delta-1)} \cdot SO_{aug}(G).$$

This completes the proof. □

3. Conclusion

In this paper, we have established several new bounds for the generalized reduced Sombor index and the augmented Sombor index. These results contribute to the growing body of literature on Sombor-type topological indices [1, 2, 3, 4, 5]. Future work could explore analogous results for other Sombor variants listed in the survey [10], such as the elliptic Sombor index, hyperbolic Sombor index, or the various multiplicative versions. Additionally, investigating the chemical applicability of these new bounds through QSPR studies would be of significant interest.

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