

CONSTRUCTION OF FUSION FRAME IN CARTESIAN PRODUCT OF TWO HILBERT SPACES

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ABSTRACT. We study the concept of fusion frame in Cartesian product of two Hilbert spaces as Cartesian product of two Hilbert spaces is again a Hilbert space and see that the Cartesian product of two fusion frames is also a fusion frame. The concept of fusion frame operator on Cartesian product of two Hilbert spaces is being given and results of it are being presented. A perturbation result on fusion frame in Cartesian product of two Hilbert spaces is being discussed.

1. INTRODUCTION AND PRELIMINARIES

The notion of frame in Hilbert space was born in 1952 in the work of Duffin and Schaeffer [3] and their idea did not appear to make much general interest outside of non-harmonic Fourier series. Later on, after some innovative work of Daubechies, Grossman, Meyer [4], the theory of frames began to be studied more widely.

The theory of frames has been generalized rapidly and various generalizations of frames in Hilbert spaces namely, K -frames, G -frames, fusion frames etc. have been introduced in recent times. General frame theory of subspaces were introduced by P. Casazza and G. Kutyniok [2]. In recent times, the frames of subspaces have been renamed as fusion frame. P. Casazza, G. Kutyniok and S. Li [1] studied fusion frame and disturbing processing and they also discussed some perturbation results on fusion frame. Fusion frame is a natural generalization of the frame theory in Hilbert space and it has so many application in data processing, coding theory, signal processing and so on. Fusion frame and its alternative dual in tensor product of Hilbert spaces was studied by P. Ghosh and T. K. Samanta [7].

In this paper, fusion frame in Cartesian product of two Hilbert spaces is introduced and some of its properties are going to be established. It is verified that the Cartesian product of two fusion frames is a fusion frame. Perturbation of a fusion frame is a interesting topic in frame theory. In this perspective, we discuss a perturbation result on fusion frame in Cartesian product of two Hilbert spaces.

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Throughout this paper, H and K are considered to be separable Hilbert spaces with associated inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$. $\mathcal{B}(H, K)$ is a collection of all bounded linear operators from H to K . In particular $\mathcal{B}(H)$ denote the space of all bounded linear operators on H . P_V denote the orthogonal projection onto the closed subspace $V \subset H$. $\{V_i\}_{i \in I}$ and $\{W_j\}_{j \in J}$ are sequence of closed subspaces of Hilbert spaces H and K , where I, J are subsets of integers $\mathbb{Z}$. Define the space

$$l^2(\{V_i\}_{i \in I}) = \left\{ \{f_i\}_{i \in I} : f_i \in V_i, \sum_{i \in I} \|f_i\|_1^2 < \infty \right\}$$

with inner product is given by $\langle \{f_i\}_{i \in I}, \{g_i\}_{i \in I} \rangle = \sum_{i \in I} \langle f_i, g_i \rangle_1$.

Clearly $l^2(\{V_i\}_{i \in I})$ is a Hilbert space with the above inner product [2].

Similarly, we can define the space $l^2(\{W_j\}_{j \in J})$.

Now, we recall some basic definitions and results needed in this paper.

Theorem 1.1. [6] *Let $V \subset H$ be a closed subspace and $T \in \mathcal{B}(H)$. Then $P_V T^* = P_V T^* P_{\overline{TV}}$. If T is an unitary operator (i.e $T^* T = I_H$), then $P_{\overline{TV}} T = T P_V$.*

Theorem 1.2. [8] *The set $\mathcal{S}(H)$ of all self-adjoint operators on H is a partially ordered set with respect to the partial order $\leq$ which is defined as for $T, S \in \mathcal{S}(H)$*

$$T \leq S \Leftrightarrow \langle T f, f \rangle_1 \leq \langle S f, f \rangle_1 \quad \forall f \in H.$$

Definition 1.3. [2] Let H be a Hilbert space. Consider a collection of closed subspaces $\{V_i\}_{i \in I}$ of H and a collection of positive weights $\{v_i\}_{i \in I}$. A family of weighted closed subspaces $\{(V_i, v_i) : i \in I\}$ is called a fusion frame for H if there exist constants $0 < A \leq B < \infty$ such that

$$A \|f\|_1^2 \leq \sum_{i \in I} v_i^2 \|P_{V_i}(f)\|_1^2 \leq B \|f\|_1^2 \quad \forall f \in H. \quad (1.1)$$

The constants A, B are called fusion frame bounds. If $A = B$ then the fusion frame is called a tight fusion frame, if $A = B = 1$ then it is called a Parseval fusion frame. If the family $\{(V_i, v_i)\}_{i \in I}$ satisfies the right inequality of (1.1), it is called a fusion Bessel sequence in H with bound B .

Definition 1.4. [2] Let $\{(V_i, v_i)\}_{i \in I}$ be a fusion Bessel sequence in H with a bound B . The synthesis operator $T_\Lambda : l^2(\{V_i\}_{i \in I}) \rightarrow H$, is defined as

$$T_V(\{f_i\}_{i \in I}) = \sum_{i \in I} v_i f_i, \quad \forall \{f_i\}_{i \in I} \in l^2(\{V_i\}_{i \in I})$$

and the analysis operator is given by

$$T_V^* : H \rightarrow l^2(\{V_i\}_{i \in I}), \quad T_V^*(f) = \{v_i P_{V_i}(f)\}_{i \in I} \quad \forall f \in H.$$

The fusion frame operator $S_V : H \rightarrow H$ is defined as follows:

$$S_V f = T_V T_V^* f = \sum_{i \in I} v_i^2 P_{V_i}(f) \quad \forall f \in H.$$

Remark 1.5. [2] Let $\{(V_i, v_i)\}_{i \in I}$ be a fusion frame for H with a bounds A, B . Then it is easy to verify that

$$\langle S_V f, f \rangle_1 = \sum_{i \in I} v_i^2 \|P_{V_i}(f)\|_1^2, \text{ and from (1.1),}$$

$$\langle Af, f \rangle_1 \leq \langle S_V f, f \rangle_1 \leq \langle Bf, f \rangle_1 \quad \forall f \in H.$$

The operator S_V is bounded, self-adjoint, positive and invertible. Now, according to the Theorem (1.2), we can write, $AI_H \leq S_V \leq BI_H$ and this gives $B^{-1}I_H \leq S_V^{-1} \leq A^{-1}I_H$.

2. FUSION FRAME IN CARTESIAN PRODUCT OF TWO HILBERT SPACES

In this section, fusion frame in Cartesian product of two Hilbert spaces is described and some results are going to be established.

Let H and K be two Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$. Then the space define by $H \oplus K = \{(f, g) : f \in H, g \in X\}$ is a linear space with respect to the addition and scalar multiplication defined by

$$(f_1, g_1) + (f_2, g_2) = (f_1 + f_2, g_1 + g_2), \text{ and}$$

$$\lambda(f, g) = (\lambda f, \lambda g) \quad \forall f, f_1, f_2 \in H, g, g_1, g_2 \in K \text{ and } \lambda \in \mathbb{K}.$$

Now, $H \oplus K$ is an inner product space with respect to the inner product given by

$$\langle (f, g), (f', g') \rangle = \langle f, f' \rangle_1 + \langle g, g' \rangle_2 \quad \forall f, f' \in H \text{ and } \forall g, g' \in X.$$

The norm on $H \oplus K$ is defined by

$$\|(f, g)\| = \|f\|_1 + \|g\|_2 \quad \forall f \in H, g \in K,$$

where $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms generated by $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, respectively. The space $H \oplus K$ is complete with respect to the above inner product. Therefore the space $H \oplus K$ is a Hilbert space.

Remark 2.1. Let $U \in \mathcal{B}(H)$, $V \in \mathcal{B}(K)$. Then for all $f \in H$, $g \in X$, define

$$U \oplus V \in \mathcal{B}(H \oplus K) \text{ by } (U \oplus V)(f, g) = (Uf, Vg), \text{ and}$$

$$(U \oplus V)^*(f, g) = (U^*f, V^*g).$$

Furthermore, if U, V and $(U \oplus V)$ are invertible operators, then we define

$$(U \oplus V)^{-1}(f, g) = (U^{-1}f, V^{-1}g)$$

and for the orthogonal projection, we define $P_{M \oplus N}(f, g) = (P_M f, P_N g)$, where P_M, P_N and $P_{M \oplus N}$ are the orthogonal projection onto the closed subspaces $M \subset H, N \subset X$ and $M \oplus N \subset H \oplus X$, respectively.

Theorem 2.2. Suppose $U, U' \in \mathcal{B}(H)$ and $V, V' \in \mathcal{B}(K)$. Then

- (1) $(U + U') \oplus V = U \oplus V + U' \oplus V$, $\lambda U \oplus \lambda V = \lambda(U \oplus V)$ and $U \oplus (V + V') = U \oplus V + U \oplus V'$.
- (2) $I_H \oplus I_K = I_{H \oplus K}$, where I_H, I_K and $I_{H \oplus K}$ are identity operators on H, K and $H \oplus K$, respectively.

- (3) $(U \oplus V)(U' \oplus V') = (UU' \oplus VV')$.
- (4) $(U \oplus V)^* = U^* \oplus V^*$.
- (5) If U and V are invertible, then $(U \oplus V)$ is invertible and moreover $(U \oplus V)^{-1} = U^{-1} \oplus V^{-1}$.
- (6) If U and V are unitary then so is $U \oplus V$.
- (7) $P_{M \oplus N} = P_M \oplus P_N$, where P_M, P_N and $P_{M \oplus N}$ are the orthogonal projection onto the closed subspaces $M \subset H, N \subset X$ and $M \oplus N \subset H \oplus X$, respectively.

Proof. (1) For each $(f, g) \in H \oplus K$, we have

$$\begin{aligned}
 [(U + U') \oplus V](f, g) &= (Uf + U'f, Vg) \\
 &= (Uf, Vg) + (U'f, Vg) \\
 &= (U \oplus V)(f, g) + (U' \oplus V)(f, g) \\
 &= [U \oplus V + U' \oplus V](f, g).
 \end{aligned}$$

Similarly, other results can be easily verified.

- (2) For each $(f, g) \in H \oplus K$, we have

$$(I_H \oplus I_K)(f, g) = (I_H f, I_K g) = (f, g) = I_{H \oplus K}(f, g).$$

- (3) For each $(f, g) \in H \oplus K$, we have

$$\begin{aligned}
 (U \oplus V)(U' \oplus V')(f, g) &= (U \oplus V)(U'f, V'g) \\
 &= (UU'f, VV'g) = (UU' \oplus VV')(f, g).
 \end{aligned}$$

- (4) For $(f, g), (f', g') \in H \oplus K$, we have

$$\begin{aligned}
 \langle (U \oplus V)^*(f, g), (f', g') \rangle &= \langle (f, g), (U \oplus V)(f', g') \rangle \\
 &= \langle (f, g), (Uf', Vg') \rangle = \langle f, Uf' \rangle_1 + \langle g, Vg' \rangle_2 \\
 &= \langle U^*f, f' \rangle_1 + \langle V^*g, g' \rangle_2 = \langle (U^*f, V^*g), (f', g') \rangle \\
 &= \langle (U^* \oplus V^*)(f, g), (f', g') \rangle.
 \end{aligned}$$

This shows that $(U \oplus V)^* = U^* \oplus V^*$.

- (5) If U and V are invertible, then for every $(f, g) \in H \oplus K$, we have

$$\begin{aligned}
 (U \oplus V)(U^{-1} \oplus V^{-1})(f, g) &= (U \oplus V)(U^{-1}f, V^{-1}g) \\
 &= (UU^{-1}f, VV^{-1}g) = (f, g)
 \end{aligned}$$

and similarly it can be shown that

$$(U^{-1} \oplus V^{-1})(U \oplus V)(f, g) = (f, g).$$

Hence, $U \oplus V$ is invertible and $(U \oplus V)^{-1} = U^{-1} \oplus V^{-1}$.

(6) If U and V are unitary, i.e., if $U^*U = I_H$ and $V^*V = I_K$ then

$$\begin{aligned} (U \oplus V)^* (U \oplus V) (f, g) &= (U^* \oplus V^*) (Uf, Vg) \\ &= (U^*Uf, V^*Vg) = (f, g) \quad \forall (f, g) \in H \oplus K. \end{aligned}$$

(7) For each $(f, g) \in H \oplus K$, we have

$$P_{M \oplus N} (f, g) = (P_M f, P_N g) = (P_M \oplus P_N) (f, g).$$

This completes the proof. $\square$

Definition 2.3. Let $\{v_i\}_{i \in I}$ be a family of positive weights i.e., $v_i > 0 \quad \forall i \in I$. Let $\{V_i\}_{i \in I}$ and $\{W_i\}_{i \in I}$ be the families of closed subspaces of H and K , respectively. Then the family of closed subspaces $\{V_i \oplus W_i\}_{i \in I}$ of the Hilbert space $H \oplus K$ is said to be a fusion frame for $H \oplus K$ with respect to $\{v_i\}_{i \in I}$, if there exist constants $0 < A \leq B < \infty$ such that

$$A \|(f, g)\|^2 \leq \sum_{i \in I} v_i^2 \|P_{V_i \oplus W_i} (f, g)\|^2 \leq B \|(f, g)\|^2,$$

for all $(f, g) \in H \oplus K$, where $P_{V_i \oplus W_i}$ is the orthogonal projection of $H \oplus K$ onto $V_i \oplus W_i$. The constants A and B are called the frame bounds. If $A = B$ then it is called a tight fusion frame and it is called a Parseval fusion frame if $A = B = 1$. If the family $\{(V_i \oplus W_i, v_i)\}_{i \in I}$ satisfies

$$\sum_{i \in I} v_i^2 \|P_{V_i \oplus W_i} (f, g)\|^2 \leq B \|(f, g)\|^2 \quad \forall (f, g) \in H \oplus K,$$

then it is called a fusion Bessel sequences in $H \oplus K$ with bound B .

Fusion frame in Cartesian product of two Hilbert spaces can be studied to rich the existing literature of fusion frames and their applications in coding theory, sensor network, etc.

Remark 2.4. Let $\{(V_i \oplus W_i, v_i)\}_{i \in I}$ be a tight fusion frame for $H \oplus K$ with bound A . Then for all $(f, g) \in H \oplus K$, we have

$$\begin{aligned} \sum_{i \in I} v_i^2 \|P_{V_i \oplus W_i} (f, g)\|^2 &= A \|(f, g)\|^2 \\ \Rightarrow \sum_{i \in I} \frac{v_i^2}{A} \|P_{V_i \oplus W_i} (f, g)\|^2 &= \|(f, g)\|^2. \end{aligned}$$

This verify that $\left\{ \left(V_i \oplus W_i, \frac{v_i^2}{\sqrt{A}} \right) \right\}_{i \in I}$ is a Parseval fusion frame for $H \oplus K$.

Now, define $l^2(\{V_i \oplus W_i\}_{i \in I})$

$$= \left\{ \{(f_i, g_i)\}_{i \in I} : f_i \in V_i, g_i \in W_i, \text{ and } \sum_i \|(f_i, g_i)\|^2 < \infty \right\}$$

with inner product

$$\begin{aligned}
\langle \{ (f_i, g_i) \}_{i \in I}, \{ (f'_i, g'_i) \}_{i \in I} \rangle_{l^2} &= \sum_{i \in I} \langle (f_i, g_i), (f'_i, g'_i) \rangle \\
&= \sum_{i \in I} [\langle f_i, f'_i \rangle_1 + \langle g_i, g'_i \rangle_2] = \sum_{i \in I} \langle f_i, f'_i \rangle_1 + \sum_{i \in I} \langle g_i, g'_i \rangle_2 \\
&= \langle \{ f_i \}_{i \in I}, \{ f'_i \}_{i \in I} \rangle_{l^2(\{ V_i \}_{i \in I})} + \langle \{ g_i \}_{i \in I}, \{ g'_i \}_{i \in I} \rangle_{l^2(\{ W_i \}_{i \in I})}.
\end{aligned}$$

The space $l^2(\{ V_i \oplus W_i \})$ is complete with respect to the above inner product. Therefore the space $l^2(\{ V_i \oplus W_i \})$ is a Hilbert space.

In the following theorem, we show a sufficient condition for a Cartesian product of fusion frames be also a fusion frame.

Theorem 2.5. *If $\{ (V_i, v_i) \}_{i \in I}$ and $\{ (W_i, v_i) \}_{i \in I}$ are fusion frames for H and K , then $\{ (V_i \oplus W_i, v_i) \}_{i \in I}$ is a fusion frame for $H \oplus K$.*

Proof. Since $\{ (V_i, v_i) \}_{i \in I}$ and $\{ (W_i, v_i) \}_{i \in I}$ are fusion frames for H and K , there exist positive constants A, B and C, D such that

$$A \|f\|_1^2 \leq \sum_{i \in I} v_i^2 \|P_{V_i}(f)\|_1^2 \leq B \|f\|_1^2 \quad \forall f \in H \quad (2.1)$$

$$C \|g\|_2^2 \leq \sum_{i \in I} v_i^2 \|P_{W_i}(g)\|_2^2 \leq D \|g\|_2^2 \quad \forall g \in K. \quad (2.2)$$

Now, for each $(f, g) \in H \oplus K$, we have

$$\begin{aligned}
\sum_{i \in I} v_i^2 \|P_{V_i \oplus W_i}(f, g)\|^2 &= \sum_{i \in I} v_i^2 \langle P_{V_i \oplus W_i}(f, g), P_{V_i \oplus W_i}(f, g) \rangle \\
&= \sum_{i \in I} v_i^2 \langle (P_{V_i}(f), P_{W_i}(g)), (P_{V_i}(f), P_{W_i}(g)) \rangle \\
&= \sum_{i \in I} v_i^2 \{ \langle P_{V_i}(f), P_{V_i}(f) \rangle_1 + \langle P_{W_i}(g), P_{W_i}(g) \rangle_2 \} \\
&= \sum_{i \in I} v_i^2 \|P_{V_i}(f)\|_1^2 + \sum_{i \in I} v_i^2 \|P_{W_i}(g)\|_2^2 \\
&\leq B \|f\|_1^2 + D \|g\|_2^2 \quad [\text{by (2.1) and (2.2)}] \\
&\leq \max\{B, D\} \{ \|f\|_1^2 + \|g\|_2^2 \} = \max\{B, D\} \|(f, g)\|^2.
\end{aligned} \quad (2.3)$$

On the other hand, from (2.3)

$$\begin{aligned}
\sum_{i \in I} v_i^2 \|P_{V_i \oplus W_i}(f, g)\|^2 &= \sum_{i \in I} v_i^2 \|P_{V_i}(f)\|_1^2 + \sum_{i \in I} v_i^2 \|P_{W_i}(g)\|_2^2 \\
&\geq A \|f\|_1^2 + C \|g\|_2^2 \quad [\text{by (2.1) and (2.2)}] \\
&\geq \min\{A, C\} \{ \|f\|_1^2 + \|g\|_2^2 \} = \min\{A, C\} \|(f, g)\|^2.
\end{aligned}$$

Therefore, for all $(f, g) \in H \oplus K$, we have

$$\min\{A, C\} \|(f, g)\|^2 \leq \sum_{i \in I} v_i^2 \|P_{V_i \oplus W_i}(f, g)\|^2 \leq \max\{B, D\} \|(f, g)\|^2.$$

Hence, $\{(V_i \oplus W_i, v_i)\}_{i \in I}$ is a fusion frame for $H \oplus K$ with bounds $\min\{A, C\}$ and $\max\{B, D\}$. $\square$

Remark 2.6. Let $\{(V_i \oplus W_i, v_i)\}_{i \in I}$ be fusion frame for $H \oplus K$. According to the definition (1.4), the synthesis operator $T_{V \oplus W} : l^2(\{V_i \oplus W_i\}_{i \in I}) \rightarrow H \oplus K$ is described by

$$T_{V \oplus W}(\{(f_i, g_i)\}_{i \in I}) = \sum_{i \in I} v_i(f_i, g_i)$$

for all $\{(f_i, g_i)\}_{i \in I} \in l^2(\{V_i \oplus W_i\}_{i \in I})$, and the corresponding frame operator $S_{V \oplus W} : H \oplus K \rightarrow H \oplus K$ is given by

$$S_{V \oplus W}(f, g) = \sum_{i \in I} v_i^2 P_{V_i \oplus W_i}(f, g) \quad \forall (f, g) \in H \oplus K.$$

The following theorem demonstrates that the synthesis operator, analysis operator and frame operator associated with the Cartesian product of two fusion frames are exactly the Cartesian product of their respective synthesis operators, analysis operators and frame operators.

Theorem 2.7. *If S_V , S_W and $S_{V \oplus W}$ are the corresponding fusion frame operators and T_V , T_W and $T_{V \oplus W}$ are the synthesis operators of fusion frames $\{(V_i, v_i)\}_{i \in I}$, $\{(W_i, v_i)\}_{i \in I}$ and $\{(V_i \oplus W_i, v_i)\}_{i \in I}$, respectively, then*

$$S_{V \oplus W} = S_V \oplus S_W, \quad S_{V \oplus W}^{-1} = S_V^{-1} \oplus S_W^{-1}, \quad \text{and} \\ T_{V \oplus W} = T_V \oplus T_W, \quad T_{V \oplus W}^* = T_V^* \oplus T_W^*.$$

Proof. For each $(f, g) \in H \oplus K$, we have

$$\begin{aligned} S_{V \oplus W}(f, g) &= \sum_{i \in I} v_i^2 P_{V_i \oplus W_i}(f, g) = \sum_{i \in I} v_i^2 (P_{V_i} f, P_{W_i} g) \\ &= \left(\sum_{i \in I} v_i^2 P_{V_i} f, \sum_{i \in I} v_i^2 P_{W_i} g \right) = (S_V(f), S_W(g)) = (S_V \oplus S_W)(f, g). \end{aligned}$$

This shows that $S_{V \oplus W} = S_V \oplus S_W$. Since S_V and S_W are invertible, by the Theorem (2.2) (5), $S_{V \oplus W}$ is also invertible and $S_{V \oplus W}^{-1} = S_V^{-1} \oplus S_W^{-1}$.

On the other hand, for all $\{(f_i, g_i)\}_{i \in I} \in l^2(\{V_i \oplus W_i\}_{i \in I})$, we have

$$\begin{aligned} T_{V \oplus W}(\{(f_i, g_i)\}_{i \in I}) &= \sum_{i \in I} v_i(f_i, g_i) = \left(\sum_{i \in I} v_i f_i, \sum_{i \in I} v_i g_i \right) \\ &= (T_V(\{f_i\}_{i \in I}), T_W(\{g_i\}_{i \in I})) = (T_V \oplus T_W)(\{(f_i, g_i)\}_{i \in I}). \end{aligned}$$

Hence, $T_{V \oplus W} = T_V \oplus T_W$ and therefore by the Theorem (2.2) (4), $T_{V \oplus W}^* = T_V^* \oplus T_W^*$. $\square$

Theorem 2.8. *Let $V = \{(V_i, v_i)\}_{i \in I}$ and $W = \{(W_i, v_i)\}_{i \in I}$ be fusion frames for H and K with frame bounds A, B and C, D having their associated frame operators S_V and S_W , respectively. Then*

$$\min(A, C) I_{H \oplus K} \leq S_{V \oplus W} \leq \max(B, D) I_{H \oplus K}.$$

Proof. Since S_V and S_W are frame operators for V and W , respectively,

$$A \|f\|_1^2 \leq \langle S_V(f), f \rangle_1 \leq B \|f\|_1^2 \quad \forall f \in H, \text{ and}$$

$$C \|g\|_2^2 \leq \langle S_W(g), g \rangle_2 \leq D \|g\|_2^2 \quad \forall g \in K.$$

Adding above two inequalities, we get

$$\begin{aligned} A \|f\|_1^2 + C \|g\|_2^2 &\leq \langle S_V(f), f \rangle_1 + \langle S_W(g), g \rangle_2 \leq B \|f\|_1^2 + D \|g\|_2^2 \\ \Rightarrow E \{ \|f\|_1^2 + \|g\|_2^2 \} &\leq \langle S_V(f), f \rangle_1 + \langle S_W(g), g \rangle_2 \leq F \{ \|f\|_1^2 + \|g\|_2^2 \}, \end{aligned}$$

where $E = \min(A, C)$ and $F = \max(B, D)$. Thus, for $(f, g) \in H \oplus K$,

$$\begin{aligned} E \langle (f, g), (f, g) \rangle &\leq \langle (S_V f, S_W g), (f, g) \rangle \leq F \langle (f, g), (f, g) \rangle \\ \Rightarrow E \langle (I_H f, I_K g), (f, g) \rangle &\leq \langle (S_V f, S_W g), (f, g) \rangle \leq F \langle (I_H f, I_K g), (f, g) \rangle \\ \Rightarrow E \langle (I_H \oplus I_K)(f, g), (f, g) \rangle &\leq \langle (S_V \oplus S_W)(f, g), (f, g) \rangle \\ &\leq F \langle (I_H \oplus I_K)(f, g), (f, g) \rangle \\ \Rightarrow E \langle I_{H \oplus K}(f, g), (f, g) \rangle &\leq \langle S_{V \oplus W}(f, g), (f, g) \rangle \leq F \langle I_{H \oplus K}(f, g), (f, g) \rangle. \end{aligned}$$

Now, according to the Theorem (1.2), we can write

$$\min(A, C) I_{H \oplus K} \leq S_{V \oplus W} \leq \max(B, D) I_{H \oplus K}.$$

□

In the next theorem, we establish an image of the Cartesian product of two fusion frames under a bounded linear operator will be a fusion frame if the operator is invertible and unitary.

Theorem 2.9. *Let $V = \{(V_i, v_i)\}_{i \in I}$ and $W = \{(W_i, v_i)\}_{i \in I}$ be fusion frames for H and K with frame bounds A_1, B_1 and A_2, B_2 having their associated frame operators S_V and S_W , respectively. If T_1 and T_2 are invertible and unitary operators on H and K , then $\{(T_1 \oplus T_2)(V_i \oplus W_i), v_i\}_{i \in I}$ is a fusion frame for $H \oplus K$.*

Proof. Since T_1 and T_2 are invertible, by Theorem (2.2) (5), $T_1 \oplus T_2$ is invertible and $(T_1 \oplus T_2)^{-1} = (T_1^{-1} \oplus T_2^{-1})$. By Theorem (1.1),

$$\begin{aligned} \|P_{V_i} T_1^*(f)\|_1 &\leq \|T_1^*\| \|P_{T_1 V_i}(f)\|_1, \text{ and} \\ \|P_{W_i} T_2^*(g)\|_2 &\leq \|T_2^*\| \|P_{T_2 W_i}(g)\|_2, \text{ for any } i \in I. \end{aligned} \quad (2.4)$$

Also, for each $f \in H$ and $g \in K$, we have

$$\|f\|_1 \leq \|(T_1^{-1})^*\| \|T_1^*(f)\|_1, \quad \|g\|_2 \leq \|(T_2^{-1})^*\| \|T_2^*(g)\|_2. \quad (2.5)$$

Now, for each $(f, g) \in H \oplus K$,

$$\begin{aligned}
 \sum_{i \in I} v_i^2 \|P_{(T_1 \oplus T_2)(V_i \oplus W_i)}(f, g)\|^2 &= \sum_{i \in I} v_i^2 \|P_{(T_1 V_i \oplus T_2 W_i)}(f, g)\|^2 \\
 &= \sum_{i \in I} v_i^2 \|(P_{T_1 V_i} \oplus P_{T_2 W_i})(f, g)\|^2 = \sum_{i \in I} v_i^2 \|(P_{T_1 V_i} f, P_{T_2 W_i} g)\|^2 \\
 &= \sum_{i \in I} v_i^2 \|P_{T_1 V_i}(f)\|_1^2 + \sum_{i \in I} v_i^2 \|P_{T_2 W_i}(g)\|_2^2 \tag{2.6} \\
 &\geq \frac{1}{\|T_1\|^2} \sum_{i \in I} v_i^2 \|P_{V_i}(T_1^* f)\|_1^2 + \frac{1}{\|T_2\|^2} \sum_{i \in I} v_i^2 \|P_{W_i}(T_2^* g)\|_2^2 \quad [\text{by (2.4)}] \\
 &\geq \frac{A_1}{\|T_1\|^2} \|T_1^*(f)\|_1^2 + \frac{A_2}{\|T_2\|^2} \|T_2^*(g)\|_2^2 \quad [\text{since } V, W \text{ are fusion frames}] \\
 &\geq \frac{A_1}{\|T_1\|^2 \|T_1^{-1}\|^2} \|f\|_1^2 + \frac{A_2}{\|T_2\|^2 \|T_2^{-1}\|^2} \|g\|_2^2 \quad [\text{by (2.5)}] \\
 &\geq C \{\|f\|_1^2 + \|g\|_2^2\} = C \|(f, g)\|^2,
 \end{aligned}$$

where $C = \min \left\{ \frac{A_1}{\|T_1\|^2 \|T_1^{-1}\|^2}, \frac{A_2}{\|T_2\|^2 \|T_2^{-1}\|^2} \right\}$.

On the other hand, since T_1 and T_2 are unitary operators, by Theorem (1.1), $P_{T_1 V_i} T_1 = T_1 P_{V_i}$ and $P_{T_2 W_j} T_2 = T_2 P_{W_j}$. Then for all $(f, g) \in H \oplus K$,

$$\begin{aligned}
 \sum_{i \in I} v_i^2 \|P_{(T_1 \oplus T_2)(V_i \oplus W_i)}(f, g)\|^2 &= \sum_{i \in I} v_i^2 \|P_{T_1 V_i}(f)\|_1^2 + \sum_{i \in I} v_i^2 \|P_{T_2 W_i}(g)\|_2^2 \quad [\text{by (2.6)}] \\
 &= \sum_{i \in I} v_i^2 \|P_{T_1 V_i} T_1 (T_1^{-1} f)\|_1^2 + \sum_{i \in I} v_i^2 \|P_{T_2 W_i} T_2 (T_2^{-1} g)\|_2^2 \\
 &= \sum_{i \in I} v_i^2 \|T_1 P_{V_i} (T_1^{-1} f)\|_1^2 + \sum_{i \in I} v_i^2 \|T_2 P_{W_i} (T_2^{-1} g)\|_2^2 \\
 &\leq \|T_1\|^2 \sum_{i \in I} v_i^2 \|P_{V_i} (T_1^{-1} f)\|_1^2 + \|T_2\|^2 \sum_{i \in I} v_i^2 \|P_{W_i} (T_2^{-1} g)\|_2^2 \\
 &\leq B_1 \|T_1\|^2 \|T_1^{-1} f\|_1^2 + B_2 \|T_2\|^2 \|T_2^{-1} g\|_2^2 \quad [\text{since } V, W \text{ are fusion frames}] \\
 &\leq B_1 \|T_1\|^2 \|T_1^{-1}\|^2 \|f\|_1^2 + B_2 \|T_2\|^2 \|T_2^{-1}\|^2 \|g\|_2^2 \\
 &\leq D \{\|f\|_1^2 + \|g\|_2^2\} = D \|(f, g)\|^2,
 \end{aligned}$$

where $D = \max \left\{ B_1 \|T_1\|^2 \|T_1^{-1}\|^2, B_2 \|T_2\|^2 \|T_2^{-1}\|^2 \right\}$.

Hence, $\{(T_1 \oplus T_2)(V_i \oplus W_i), v_i\}_{i \in I}$ is a fusion frames for $H \oplus K$. $\square$

Remark 2.10. By Theorem (1.1), for each $(f, g) \in H \oplus K$, we have

$$\begin{aligned}
\sum_{i \in I} v_i^2 P_{(T_1 \oplus T_2)(V_i \oplus W_i)}(f, g) &= \sum_{i \in I} v_i^2 P_{(T_1 V_i \oplus T_2 W_i)}(f, g) \\
&= \sum_{i \in I} v_i^2 (P_{T_1 V_i} \oplus P_{T_2 W_i})(f, g) = \sum_{i \in I} v_i^2 (P_{T_1 V_i} f, P_{T_2 W_i} g) \\
&= \sum_{i \in I} v_i^2 (T_1 P_{V_i}(T_1^{-1} f), T_2 P_{W_i}(T_2^{-1} g)) \\
&= \left(T_1 \sum_{i \in I} v_i^2 P_{V_i}(T_1^{-1} f), T_2 \sum_{i \in I} v_i^2 P_{W_i}(T_2^{-1} g) \right) \\
&= (T_1 S_V(T_1^{-1} f), T_2 S_W(T_2^{-1} g)) \\
&= (T_1 \oplus T_2)(S_V \oplus S_W)(T_1^{-1} \oplus T_2^{-1})(f, g) \\
&= (T_1 \oplus T_2) S_{V \oplus W} (T_1 \oplus T_2)^{-1} (f, g).
\end{aligned}$$

This shows that $(T_1 \oplus T_2) S_{V \oplus W} (T_1 \oplus T_2)^{-1}$ is the corresponding fusion frame operator for $\{(T_1 \oplus T_2)(V_i \oplus W_i), v_i\}_{i \in I}$.

3. PERTURBATION OF FUSION FRAME IN CARTESIAN PRODUCT OF TWO HILBERT SPACES

In frame theory, stability of a frame is an important concept. In this section, we analyze the stability of fusion frame system in Cartesian product of two Hilbert spaces under some perturbations.

Theorem 3.1. *Let $V = \{(V_i, v_i)\}_{i \in I}$ and $W = \{(W_i, v_i)\}_{i \in I}$ be fusion frames for H and K with frame bounds A, B and C, D , respectively. Let $\{X_i \oplus Y_i\}_{i \in I}$ be a family of closed subspaces in $H \oplus K$. If there exists $0 < R < \min\{A, C\}$ such that*

$$\sum_{i \in I} v_i^2 \|P_{V_i \oplus W_i}(f, g) - P_{X_i \oplus Y_i}(f, g)\|^2 \leq R \|(f, g)\|^2$$

for all $(f, g) \in H \oplus K$. Then the family $\{(X_i \oplus Y_i, v_i)\}_{i \in I}$ is a fusion frame for $H \oplus K$.

Proof. For each $(f, g) \in H \oplus K$, we have

$$\begin{aligned}
\sum_{i \in I} v_i^2 \|P_{V_i \oplus W_i}(f, g) - P_{X_i \oplus Y_i}(f, g)\|^2 &\leq R \|(f, g)\|^2 \\
\Rightarrow \sum_{i \in I} v_i^2 \|(P_{V_i} f, P_{W_i} g) - (P_{X_i} f, P_{Y_i} g)\|^2 &\leq R \|(f, g)\|^2 \\
\Rightarrow \sum_{i \in I} v_i^2 \|(P_{V_i} f - P_{X_i} f, P_{W_i} g - P_{Y_i} g)\|^2 &\leq R \|(f, g)\|^2.
\end{aligned}$$

This implies that for each $f \in H$ and $g \in K$, we have

$$\begin{aligned} & \sum_{i \in I} v_i^2 \|P_{V_i} f - P_{X_i} f\|_1^2 + \sum_{i \in I} v_i^2 \|P_{W_i} g - P_{Y_i} g\|_2^2 \\ & \leq R (\|f\|_1^2 + \|g\|_2^2). \end{aligned} \quad (3.1)$$

Now, by the triangle inequality, we have

$$\begin{aligned} & \sum_{i \in I} v_i^2 \|P_{X_i}(f)\|_1^2 \geq \sum_{i \in I} v_i^2 \|P_{V_i}(f)\|_1^2 - \sum_{i \in I} v_i^2 \|P_{V_i}(f) - P_{X_i}(f)\|_1^2 \\ & \geq A \|f\|_1^2 - \sum_{i \in I} v_i^2 \|P_{V_i}(f) - P_{X_i}(f)\|_1^2, \text{ and} \\ & \sum_{i \in I} v_i^2 \|P_{Y_i}(g)\|_2^2 \geq C \|g\|_2^2 - \sum_{i \in I} v_i^2 \|P_{W_i}(g) - P_{Y_i}(g)\|_2^2. \end{aligned}$$

Adding these above two inequalities, we get

$$\begin{aligned} & \sum_{i \in I} v_i^2 \{ \|P_{X_i}(f)\|_1^2 + \|P_{Y_i}(g)\|_2^2 \} \\ & \geq A \|f\|_1^2 + C \|g\|_2^2 - \left\{ \sum_{i \in I} v_i^2 \|P_{V_i}(f) - P_{X_i}(f)\|_1^2 + \sum_{i \in I} v_i^2 \|P_{W_i}(g) - P_{Y_i}(g)\|_2^2 \right\} \\ & \geq A \|f\|_1^2 + C \|g\|_2^2 - R (\|f\|_1^2 + \|g\|_2^2) \quad [\text{by (3.1)}] \\ & \geq \min(A, C) (\|f\|_1^2 + \|g\|_2^2) - R (\|f\|_1^2 + \|g\|_2^2) \\ & = \{ \min(A, C) - R \} (\|f\|_1^2 + \|g\|_2^2) = \{ \min(A, C) - R \} \|(f, g)\|^2. \\ & \Rightarrow \{ \min(A, C) - R \} \|(f, g)\|^2 \leq \sum_{i \in I} v_i^2 \|P_{X_i \oplus Y_i}(f, g)\|^2 \quad \forall (f, g) \in H \oplus K. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \sum_{i \in I} v_i^2 \|P_{X_i}(f)\|_1^2 \leq \sum_{i \in I} v_i^2 \|P_{V_i}(f)\|_1^2 + \sum_{i \in I} v_i^2 \|P_{V_i}(f) - P_{X_i}(f)\|_1^2 \\ & \leq B \|f\|_1^2 + \sum_{i \in I} v_i^2 \|P_{V_i}(f) - P_{X_i}(f)\|_1^2, \text{ and} \\ & \sum_{i \in I} v_i^2 \|P_{Y_i}(g)\|_2^2 \leq D \|g\|_2^2 + \sum_{i \in I} v_i^2 \|P_{W_i}(g) - P_{Y_i}(g)\|_2^2. \end{aligned}$$

Again adding the above two inequalities, and using (3.1), we get

$$\begin{aligned} & \sum_{i \in I} v_i^2 \|P_{X_i \oplus Y_i}(f, g)\|^2 \leq B \|f\|_1^2 + D \|g\|_2^2 + R (\|f\|_1^2 + \|g\|_2^2) \\ & \leq \{ \max(B, D) + R \} (\|f\|_1^2 + \|g\|_2^2) \\ & = \{ \max(B, D) + R \} \|(f, g)\|^2 \quad \forall (f, g) \in H \oplus K. \end{aligned}$$

Hence, $\{(X_i \oplus Y_i, v_i)\}_{i \in I}$ is a fusion frame for $H \oplus K$ with bounds $\{\min(A, C) - R\}$ and $\{\max(B, D) + R\}$. This completes the proof. $\square$

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