Gulf Journal of Mathematics
Vol 24, Issue 2 (2026) 1-19

ISSN: 2309-4966
https://doi.org/10.56947/3qqc6y80

LANGEVIN-TYPE q-DIFFERENCE FRACTIONAL BOUNDARY VALUE PROBLEM

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Abstract. In this paper, we investigate a Langevin-type generalized q- difference fractional boundary value problem with infinite-point boundary conditions. Using fixed-point theorems, we establish sufficient conditions for the existence and uniqueness of solutions. Furthermore, we analyze the Ulam-Hyers stability of the problem considered, ensuring the robustness of solutions under small perturbations. The results obtained extend and complement recent works on fractional and q-fractional boundary value problems, particularly in the context of Langevin-type equations.

Keywords. Langevin equation, q-difference equations, Fractional order, Boundary value problem, Existence, Uniqueness, Stability.
2020 Mathematics Subject Classification. Primary 34A08; Secondary 39A13, 34B10, 34B15

1. Introduction

Fractional calculus has emerged as a powerful generalization of classical calculus, providing an effective mathematical framework for modeling processes that exhibit memory, hereditary properties, and nonlocal behaviors. Unlike traditional integer-order derivatives, fractional operators are capable of capturing the intrinsic complexity of dynamic systems characterized by long-range temporal or spatial dependencies. This feature makes fractional calculus particularly suitable for describing anomalous diffusion, viscoelastic materials, biological systems with memory, and control processes with feedback mechanisms [[7], [18], [20], [25]].

Over the past two decades, the study of fractional boundary value problems has attracted growing interest among mathematicians and applied scientists. Numerous works have been devoted to establishing the existence, uniqueness, and stability of solutions for nonlinear fractional differential equations under various types of fractional operators such as Caputo, Riemann–Liouville, Hadamard, and generalized fractional derivatives, and a wide range of boundary conditions [[1],[6], [15], [29]]. These developments not only deepen the theoretical understanding of

Date: Received: Aug 10, 2026; Revised: Sep 7, 2026; Accepted: Sep 15, 2026.
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fractional models, but also enhance their applicability in real-world phenomena where classical integer-order models fail to provide sufficient accuracy.

In this context, q-calculus and its fractional extensions have played a fundamental role in bridging discrete and continuous approaches to analysis. The development of generalized q-fractional operators has enriched the study of nonlinear difference equations, particularly in problems involving infinite-point boundary conditions, which naturally arise in models where the current state depends on an infinite sequence of historical states [[4], [5], [12]-[14], [16], [19]]. Such problems are not only theoretically interesting but also relevant in discrete models of population dynamics, viscoelastic materials, and signal processing, where memory plays a crucial role.

The Langevin equation serves as a fundamental framework for analyzing the temporal evolution of stochastic or fluctuating physical systems. It has proven particularly useful in modeling anomalous diffusion and transport phenomena observed in complex environments. Nevertheless, the classical form of the Langevin equation often fails to adequately capture the intricate dynamics that arise in heterogeneous or memory-dependent media. To address these shortcomings, various extensions and modified versions of the Langevin equation have been developed. Among them, the generalized Langevin equation stands out, as it incorporates both fractal characteristics and memory effects through a dissipative memory kernel, thus providing a more realistic description of the underlying processes. There are some papers that deal with the existence of solutions for the nonlinear Langevin equation for a variety of boundary conditions using various methods (contraction mapping and Krasnoselskii's fixed point, standard fixed point theorems, variational methods, etc.) [[2], [10], [21], [22], [33]]. In recent years, increasing attention has been devoted to fractional-order versions of the Langevin equation, where derivatives of non-integer order are employed to better represent fractal and anomalous behaviors. For detailed discussions and recent advances on this topic, the reader is referred to [[8], [9]] and the references therein. However, the Langevin equation involving q-fractional derivatives of different orders has not been extensively studied so far.

Ulam-Hyers stability [[27], [28], [31]] is an important qualitative property of differential equations, describing the robustness of solutions under small perturbations. This concept has been extensively studied for fractional differential equations and related boundary value problems; see, for example, [[17], [23], [32]]. Recent studies have addressed the existence and Ulam-Hyers stability of nonlinear Langevin equations involving different fractional operators and boundary conditions [24]. More recently, Slimane et al. [30] investigated the existence theory for nonlinear Langevin-type fractional differential equations involving a generalized Caputo proportional fractional derivative. However, the combined effect of Caputo fractional q-derivatives of different orders, infinite-point fractional integral boundary conditions, and the dependence of the nonlinear term on the unknown function, a fractional q-derivative, and a fractional q-integral remains insufficiently explored. These results further highlight the importance of investigating stability properties for fractional Langevin-type problems.

In [11], the authors studied the existence and uniqueness of solutions for the following nonlinear Langevin fractional differential equation.

$$\begin{aligned} {}^{c}D^{\beta}\left({}^{c}D^{\alpha}+\lambda\right)\ x(t) &= f\left(t,x(t),\ I^{\varphi}x(t)\right), \qquad t\in[0,1]\\ x\left(0\right) &= 0, \qquad {}^{c}D^{\alpha}x\left(0\right)=0\\ {}^{c}D^{\alpha}x\left(\eta\right)+{}^{c}D^{\alpha}x\left(\xi\right) &= \ k\ I^{\varphi}x\left(\mu\right), \end{aligned}$$

that involve fractional derivatives of different orders, subject to nonlocal integral and anti-periodic boundary conditions. In addition, they examined the Hyers-Ulam stability of the obtained solutions. The existence results are obtained using the Krasnoselskii fixed point theorem, while the uniqueness is established through the Banach contraction principle.

In [3], Bashir Ahmad established the existence of solutions to a nonlinear fractional q-difference integral equation, considered as a q-variant of the Langevin equation, with two different fractional orders and nonlocal four-point boundary conditions

$$\begin{aligned} {}^{c}D_{q}^{\beta}\left({}^{c}D_{q}^{\gamma}+\lambda\right)\ x(t) &= pf(t,x(t))+\ I_{q}^{\xi}g(t,x(t)), \qquad t\in[0,1]\\ \alpha_{1}x\left(0\right)+\beta_{1}(t^{(1-\gamma)}D_{q}x\left(0\right))|_{t=0} &= \sigma_{1}\ x\left(\eta_{1}\right)\\ \alpha_{2}x\left(1\right)+\beta_{2}D_{q}x\left(1\right) &= \sigma_{2}\ x\left(\eta_{2}\right) \end{aligned}$$

by applying a fixed-point theorem.

The above studies indicate that fractional Langevin boundary value problems have been investigated under various local and nonlocal boundary conditions. However, the combination of a Langevin-type operator involving Caputo fractional q-derivatives of different orders with infinite-point fractional integral boundary conditions appears to be considerably less explored. In particular, the simultaneous dependence of the nonlinear term on the unknown function, a Caputo fractional q-derivative of an intermediate order, and a fractional q-integral introduces an additional analytical structure that is not present in the related problems considered in [[3], [11]].

Motivated by these observations, we consider a generalized fractional q-Langevin boundary value problem in which the nonlocality enters both through the nonlinear operator and through infinite-point integral boundary conditions of the form

$$\begin{aligned} {}^{c}D_{q}^{\beta}\left({}^{c}D_{q}^{\alpha}+\lambda\right)x(t) &= f\left(t,x(t),{}^{c}D_{q}^{\theta}x(t),I_{q}^{\varphi}x(t)\right), \qquad t\in[0,1],\\ \delta_{1}x(0)+\delta_{2}{}^{c}D_{q}^{\alpha}x(0) &= \sum_{i=1}^{\infty}\sigma_{i}I_{q}^{\gamma}x(\eta_{i}),\\ \delta_{3}x(1)-\delta_{4}{}^{c}D_{q}^{\alpha}x(1) &= \sum_{i=1}^{\infty}\mu_{i}I_{q}^{\gamma}x(\eta_{i}). \end{aligned} \tag{1.1}$$

where ${}^{c}D_{q}^{\beta}$, ${}^{c}D_{q}^{\alpha}$, ${}^{c}D_{q}^{\theta}$ are Caputo fractional q-derivatives of order $\alpha,\ \beta,\ \theta\in(0,1]$ such that $\alpha-\theta>0$, and I_{q}^{φ}, I_{q}^{γ} are q-fractional integrals of order $\varphi,\ \gamma>0$, $x\in\mathcal{C}\left([0,1],\mathbb{R}\right)$ and $f:[0,1]\times\mathbb{R}^{3}\to\mathbb{R}$ is a given function.

The main contributions of this work are as follows. We derive an explicit integral representation for the linear counterpart of problem (1.1), incorporating

the infinite-point fractional integral boundary conditions, and formulate the nonlinear problem as a fixed-point problem. We then establish sufficient conditions for the existence and uniqueness of solutions using the Krasnosel'skii and Banach fixed-point theorems, respectively, and investigate their Ulam–Hyers stability. An example is also provided to illustrate the applicability of the obtained results. The novelty lies in the specific analytical framework developed for problem (1.1), while the fixed-point theorems are used as standard analytical tools.

2. Preliminaries

Throughout this paper, we use the standard notation and basic properties of q-calculus and fractional q-calculus. The definitions of the q-number, q-analogue of the power function, q-gamma function, q-derivative, and q-integral, together with their standard properties, can be found in [26]. We shall use the Riemann–Liouville fractional q-integral and the Caputo fractional q-derivative in their standard forms; see [26] for their definitions and fundamental properties.

In particular, for $\alpha > 0$ and $p = \lceil \alpha \rceil$, we use the standard identity

$$I_q^{\alpha}, {}^cD_q^{\alpha} f(t) = f(t) - \sum_{k=0}^{p-1} \frac{t^k}{\Gamma_q(k+1)} (D_q^k f)(0),$$

whenever the involved quantities are well defined.

We also use the standard properties of fractional q-integrals and q-derivatives given in [26] without further mention.

In order to define the solution for problem (1), we need the following lemma.

Lemma 2.1. *For any $\alpha,\ \beta,\ \theta \in (0,1]$ such that $\alpha - \theta > 0$, $\varphi,\ \gamma > 0$ and $h \in \mathcal{C}([0,1],\ \mathbb{R})$, the function $x \in \mathcal{C}([0,1],\ \mathbb{R})$ is a solution to the linear problem*

$$\begin{array}{l} {}^cD_q^{\beta}\left({}^cD_q^{\alpha} + \lambda\right)\ x(t) = h(t), \qquad t \in [0,1] \\ \delta_1 x(0) + \delta_2 {}^cD_q^{\alpha} x(0) = \sum_{i=1}^{\infty} \sigma_i\ I_q^{\gamma} x(\eta_i) \\ \delta_3 x(1) - \delta_4 {}^cD_q^{\alpha} x(1) = \sum_{i=1}^{\infty} \mu_i\ I_q^{\gamma} x(\eta_i), \end{array}$$

if and only if

$$\begin{aligned} x(t) =& I_q^{\alpha+\beta} h(t) - \lambda I_q^{\alpha} x(t) \\ & - \frac{t^{\alpha}}{\Delta \Gamma_q(\alpha+1)} \Bigg\{ \sum_{i=1}^{\infty} (A_3\mu_i - A_1\sigma_i) \left(I_q^{\alpha+\beta+\gamma} h(\eta_i) - \lambda I_q^{\alpha+\gamma} x(\eta_i) \right) \\ & - A_3\delta_3 \left(I_q^{\alpha+\beta} h(1) - \lambda I_q^{\alpha} x(1) \right) + A_3\delta_4 \left(I_q^{\beta} h(1) - \lambda x(1) \right) \Bigg\} \\ & - \frac{1}{\Delta} \Bigg\{ \sum_{i=1}^{\infty} (A_4\sigma_i - A_2\mu_i) \left(I_q^{\alpha+\beta+\gamma} h(\eta_i) - \lambda I_q^{\alpha+\gamma} x(\eta_i) \right) \\ & \quad + A_2\delta_3 \left(I_q^{\alpha+\beta} h(1) - \lambda I_q^{\alpha} x(1) \right) - A_2\delta_4 \left(I_q^{\beta} h(1) - \lambda x(1) \right) \Bigg\}, \end{aligned}$$

where $\Delta = A_1A_2 - A_3A_4$ *such that*

$$A_1 = \delta_3 - \frac{\sum_{i=1}^{\infty} \mu_i {\eta_i}^{\gamma}}{\Gamma_q(\gamma+1)}, \qquad A_2 = \delta_2 - \frac{\sum_{i=1}^{\infty} \sigma_i {\eta_i}^{\alpha+\gamma}}{\Gamma_q(\alpha+\gamma+1)},$$

$$A_3 = \delta_1 - \lambda\delta_2 - \frac{\sum_{i=1}^{\infty} \sigma_i {\eta_i}^{\gamma}}{\Gamma_q(\gamma+1)}, \qquad A_4 = \frac{\delta_3}{\Gamma_q(\alpha+1)} - \delta_4 - \frac{\sum_{i=1}^{\infty} \mu_i {\eta_i}^{\alpha+\gamma}}{\Gamma_q(\alpha+\gamma+1)}.$$

Proof. Applying I_q^{β} to the differential equation and using the standard properties of the Caputo fractional q-derivative, we obtain

$$\begin{aligned} x(t) =& \int_0^t \frac{(t-qu)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(\int_0^u \frac{(u-qr)^{(\beta-1)}}{\Gamma_q(\beta)} h(r)\, d_q r - \lambda x(u) \right) d_q u \\ & - \frac{t^{\alpha}}{\Gamma_q(\alpha+1)} c_0 - c_1 \\ =& I_q^{\alpha}\left(I_q^{\beta} h - \lambda x\right)(t) - \frac{t^{\alpha}}{\Gamma_q(\alpha+1)} c_0 - c_1 \\ =& I_q^{\alpha+\beta} h(t) - \lambda I_q^{\alpha} x(t) - \frac{t^{\alpha}}{\Gamma_q(\alpha+1)} c_0 - c_1. \end{aligned} \tag{2.1}$$

Using (2.1), we have the following equalities

$$\begin{aligned} {}^{c}D_q^{\alpha} x(t) =& I_q^{\beta} h(t) - \lambda x(t) - c_0 \\ =& I_q^{\beta} h(t) - \lambda \left[I_q^{\alpha+\beta} h(t) - \lambda I_q^{\alpha} x(t) - \frac{t^{\alpha}}{\Gamma_q(\alpha+1)} c_0 - c_1 \right] - c_0 \end{aligned}$$

and

$$\begin{aligned} I_q^{\gamma} x(t) =& I_q^{\alpha+\beta+\gamma} h(t) - \lambda I_q^{\alpha+\gamma} x(t) - \frac{c_0}{\Gamma_q(\alpha+\gamma+1)} t^{\alpha+\gamma} - \frac{c_1}{\Gamma_q(\gamma+1)} t^{\gamma} \\ =& \int_0^t \frac{(t-qs)^{(\alpha+\beta+\gamma-1)}}{\Gamma_q(\alpha+\beta+\gamma)} h(s)\, d_q s - \lambda \int_0^t \frac{(t-qs)^{(\alpha+\gamma-1)}}{\Gamma_q(\alpha+\gamma)} x(s)\, d_q s \\ & - \frac{c_0}{\Gamma_q(\alpha+\gamma+1)} t^{\alpha+\gamma} - \frac{c_1}{\Gamma_q(\gamma+1)} t^{\gamma}. \end{aligned}$$

From the boundary conditions, we have the system according to c_0 and c_1. If we define

$$\begin{aligned} \Delta :=& \begin{vmatrix} -\delta_2 + \frac{\sum_{i=1}^{\infty} \sigma_i {\eta_i}^{\alpha+\gamma}}{\Gamma_q(\alpha+\gamma+1)} & -\left(\delta_1 - \lambda\delta_2 - \frac{\sum_{i=1}^{\infty} \sigma_i {\eta_i}^{\gamma}}{\Gamma_q(\gamma+1)}\right) \\ -\left(\frac{\delta_3}{\Gamma_q(\alpha+1)} - \delta_4 - \frac{\sum_{i=1}^{\infty} \mu_i {\eta_i}^{\alpha+\gamma}}{\Gamma_q(\alpha+\gamma+1)}\right) & -\left(\delta_3 - \frac{\sum_{i=1}^{\infty} \mu_i {\eta_i}^{\gamma}}{\Gamma_q(\gamma+1)}\right) \end{vmatrix} \\ =& \left(\delta_3 - \frac{\sum_{i=1}^{\infty} \mu_i {\eta_i}^{\gamma}}{\Gamma_q(\gamma+1)}\right)\left(\delta_2 - \frac{\sum_{i=1}^{\infty} \sigma_i {\eta_i}^{\alpha+\gamma}}{\Gamma_q(\alpha+\gamma+1)}\right) \\ & - \left(\delta_1 - \lambda\delta_2 - \frac{\sum_{i=1}^{\infty} \sigma_i {\eta_i}^{\gamma}}{\Gamma_q(\gamma+1)}\right)\left(\frac{\delta_3}{\Gamma_q(\alpha+1)} - \delta_4 - \frac{\sum_{i=1}^{\infty} \mu_i {\eta_i}^{\alpha+\gamma}}{\Gamma_q(\alpha+\gamma+1)}\right) \\ =& A_1.A_2 - A_3.A_4, \end{aligned}$$

and suppose $\Delta \neq 0$, we have

$$c_0 = \frac{1}{\Delta}\Bigg\{ \sum_{i=1}^{\infty} \left(A_3\mu_i - A_1\sigma_i\right)\Big(I_q^{\alpha+\beta+\gamma}h(\eta_i) - \lambda I_q^{\alpha+\gamma}x(\eta_i)\Big) \\ - A_3\delta_3\Big(I_q^{\alpha+\beta}h(1) - \lambda I_q^{\alpha}x(1)\Big) + A_3\delta_4\Big(I_q^{\beta}h(1) - \lambda x(1)\Big)\Bigg\}$$

and

$$c_1 = \frac{1}{\Delta}\Bigg\{ \sum_{i=1}^{\infty} \left(A_4\sigma_i - A_2\mu_i\right)\left(I_q^{\alpha+\beta+\gamma}h\left(\eta_i\right) - \lambda I_q^{\alpha+\gamma}x\left(\eta_i\right)\right) \\ + A_2\delta_3\left(I_q^{\alpha+\beta}h\left(1\right) - \lambda I_q^{\alpha}x\left(1\right)\right) - A_2\delta_4\left(I_q^{\beta}h\left(1\right) - \lambda x\left(1\right)\right)\Bigg\}.$$

Substituting these expressions into (2.1) yields the desired integral representation. □

Let us define the operator $T : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ by

$$\begin{aligned} Tx(t) := {} & I_q^{\alpha+\beta}f\big(t, x(t), {}^cD_q^{\theta}x(t), I_q^{\varphi}x(t)\big) - \lambda I_q^{\alpha}x(t) \\ & - \frac{t^{\alpha}}{\Delta\Gamma_q(\alpha+1)}\Bigg\{ \sum_{i=1}^{\infty}(A_3\mu_i - A_1\sigma_i)\Big(I_q^{\alpha+\beta+\gamma}f(\eta_i, x(\eta_i), {}^cD_q^{\theta}x(\eta_i), I_q^{\varphi}x(\eta_i)) \\ & \qquad - \lambda I_q^{\alpha+\gamma}x(\eta_i)\Big) \\ & - A_3\delta_3\Big(I_q^{\alpha+\beta}f(1, x(1), {}^cD_q^{\theta}x(1), I_q^{\varphi}x(1)) - \lambda I_q^{\alpha}x(1)\Big) \\ & + A_3\delta_4\Big(I_q^{\beta}f(1, x(1), {}^cD_q^{\theta}x(1), I_q^{\varphi}x(1)) - \lambda x(1)\Big)\Bigg\} \\ & - \frac{1}{\Delta}\Bigg\{ \sum_{i=1}^{\infty}(A_4\sigma_i - A_2\mu_i)\Big(I_q^{\alpha+\beta+\gamma}f(\eta_i, x(\eta_i), {}^cD_q^{\theta}x(\eta_i), I_q^{\varphi}x(\eta_i)) \\ & \qquad - \lambda I_q^{\alpha+\gamma}x(\eta_i)\Big) \\ & + A_2\delta_3\Big(I_q^{\alpha+\beta}f(1, x(1), {}^cD_q^{\theta}x(1), I_q^{\varphi}x(1)) - \lambda I_q^{\alpha}x(1)\Big) \\ & - A_2\delta_4\Big(I_q^{\beta}f(1, x(1), {}^cD_q^{\theta}x(1), I_q^{\varphi}x(1)) - \lambda x(1)\Big)\Bigg\}, \end{aligned} \tag{2.2}$$

where Δ and A_i $(i = 1, 2, 3, 4)$ are given in Lemma 2.1.

The integral representation obtained in Lemma 2.1 is not merely a reformulation of the differential problem. It explicitly incorporates the infinite-point fractional integral boundary conditions through the coefficients A_i and the determinant Δ. Consequently, the structure of the resulting integral operator depends essentially on the particular boundary conditions of problem (1.1). This representation constitutes the basis for the subsequent existence, uniqueness, and stability analysis.

Corollary 2.2. *Let $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function. Then the solutions of problem (1.1) are the fixed points of the operator T.*

The following fixed point theorems are fundamental and important to the proof of our main results.

Theorem 2.3. [18] *Let X be a Banach space, and K a closed, convex, nonempty subset of X. Let T_1, T_2 be operators such that $T_1(x) + T_2(x) \in K$, T_1 is compact and continuous, and T_2 is a contraction. Then there exists $x \in K$ such that $T_1(x) + T_2(x) = x$.*

Theorem 2.4. [1] *Let T be a contraction mapping from a closed subset K of a Banach space X into K. Then there exists a unique fixed point x in K such that $T(x) = x$.*

3. Main result

In this section, we establish the uniqueness and existence of solutions for problem (1.1) through the utilization of some fixed-point theorems. To accomplish this, we ensure that all the required conditions are prepared.

(H_1) $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous and there exists a constant $C > 0$ such that

$$|f(t_1,u_1,v_1,w_1) - f(t_2,u_2,v_2,w_2)| \leq C\{|u_1-u_2| + |v_1-v_2| + |w_1-w_2|\},$$

$$(t_i,u_i,v_i,w_i) \in [0,1] \times \mathbb{R}^3, \quad (i=1,2)$$

(H_2) $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous and there exists $y(t) \in \mathcal{C}[0,1]$ with $\sup_{t\in[0,1]} |y(t)| = \tilde{y}$ such that

$$|f(t,u(t),v(t),w(t))| \leq y(t), \qquad (t,u,v,w) \in [0,1] \times \mathbb{R}^3.$$

Let $\mathcal{C} = \mathcal{C}([0,1],\mathbb{R})$ be a Banach space of all continuous functions of $[0,1]$ endowed with the usual supremum norm. Define the space $E = \{x : x \in \mathcal{C},\ {}^cD_q^\theta x \in \mathcal{C}\}$ equipped with the norm $\|x\|_E = \|x\|_\infty + \|{}^cD_q^\theta x\|_\infty$ such that $\|x\|_\infty = \sup_{t\in[0,1]} |x(t)|$.

Theorem 3.1. *Assume that (H_1) and (H_2) hold. Then the problem (1.1) has at least one solution on E, provided that $1 - |\lambda| H(0) > 0$ and $M < 1$, where*

$$M := \frac{1}{|\Delta|}\Bigg\{\left[\sum_{i=1}^{\infty}\left(\frac{1}{\Gamma_q(\alpha+1)} + \frac{1}{\Gamma_q(\alpha-\theta+1)}\right)\left(|A_1\sigma_i - A_3\mu_i| + |A_2\mu_i - A_4\sigma_i|\right)\right] G_{\alpha,\gamma}$$
$$+ \left[\left(\frac{1}{\Gamma_q(\alpha+1)} + 1\right)|A_3| + |A_2|\right]\left[|\delta_3| G_{\alpha,0} + |\delta_4| G_{0,0}\right]\Bigg\},$$

such that

$$G_{a,b} := \frac{C}{\Gamma_q(a+\beta+b+1)}\left(1 + \frac{1}{\Gamma_q(\varphi+1)}\right) + \frac{|\lambda|}{\Gamma_q(a+b+1)}$$

and

$$H(c) := \frac{1}{|\Delta|}\Bigg\{\frac{1}{\Gamma_q(\alpha+c+\gamma+1)}\sum_{i=1}^{\infty}\Bigg[\left(\frac{1}{\Gamma_q(\alpha+1)}+\frac{1}{\Gamma_q(\alpha-\theta+1)}\right)\left|A_1\sigma_i - A_3\mu_i\right| + \left|A_2\mu_i - A_4\sigma_i\right|\Bigg] + \frac{|\delta_3|}{\Gamma_q(\alpha+c+1)}\left(\frac{|A_3|}{\Gamma_q(\alpha+1)}+\frac{|A_3|}{\Gamma_q(\alpha-\theta+1)}+|A_2|\right) + \frac{|\delta_4|}{\Gamma_q(c+1)}\left(1+\frac{|A_3|}{\Gamma_q(\alpha-\theta+1)}\right)\Bigg\} + \frac{1}{\Gamma_q(\alpha+c+1)}+\frac{1}{\Gamma_q(\alpha+c-\theta+1)}, \tag{3.1}$$

for all $a, b, c \geq 0$.

Proof. Consider the continuous operator $T : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$, which is defined by (2.2). Let B_ℓ be a closed ball defined by $B_\ell = \{x \in E : \|x\|_E \leq \ell\}$ with $\ell \geq \dfrac{\widetilde{y}\, H(\beta)}{1-|\lambda|\, H(0)}$. For all $x \in B_\ell$, we provide $Tx \in B_\ell$.
Let $T_1,\ T_2$ be two operators such that $T_1 + T_2 = T$, where

$$T_1x(t) := I_q^{\alpha+\beta} f\left(t, x(t), {}^cD_q^{\theta}x(t),\ I_q^{\varphi}x(t)\right) - \lambda I_q^{\alpha}x(t),$$

and

$$T_2x(t) := \frac{1}{\Delta}\Bigg\{\sum_{i=1}^{\infty}\left(\frac{(A_1\sigma_i - A_3\mu_i)\, t^{\alpha}}{\Gamma_q(\alpha+1)} + (A_2\mu_i - A_4\sigma_i)\right) \times \left(I_q^{\alpha+\beta+\gamma} f(\eta_i,\ x(\eta_i), {}^cD_q^{\theta}x(\eta_i),\ I_q^{\varphi}x(\eta_i)) - \lambda I_q^{\alpha+\gamma}x(\eta_i)\right) + \delta_3\left(\frac{A_3t^{\alpha}}{\Gamma_q(\alpha+1)} - A_2\right)\left(I_q^{\alpha+\beta} f(1, x(1), {}^cD_q^{\theta}x(1),\ I_q^{\varphi}x(1)) - \lambda I_q^{\alpha}x(1)\right) + \delta_4\left(-\frac{A_3t^{\alpha}}{\Gamma_q(\alpha+1)} + A_2\right)\left(I_q^{\beta} f(1, x(1), {}^cD_q^{\theta}x(1),\ I_q^{\varphi}x(1)) - \lambda x(1)\right)\Bigg\}.$$

By (H_2), we have

$$|T_1x(t)| \leq \frac{\tilde{y}}{\Gamma_q(\alpha+\beta+1)} + \frac{|\lambda|\,\|x\|_{\infty}}{\Gamma_q(\alpha+1)}$$

and

$$\left|{}^cD_q^{\theta}T_1x(t)\right| \leq \frac{\tilde{y}}{\Gamma_q(\alpha+\beta-\theta+1)} + \frac{|\lambda|\,\|x\|_{\infty}}{\Gamma_q(\alpha-\theta+1)},$$

for each $t \in [0,1]$.

So, we get

$$\begin{aligned}\|T_1x\|_E &= \|T_1x\|_\infty + \left\|{}^cD_q^\theta T_1x\right\|_\infty \\ &\leq \frac{\tilde{y}}{\Gamma_q(\alpha+\beta+1)} + \frac{|\lambda|\,\|x\|_\infty}{\Gamma_q(\alpha+1)} + \frac{\tilde{y}}{\Gamma_q(\alpha+\beta-\theta+1)} + \frac{|\lambda|\,\|x\|_\infty}{\Gamma_q(\alpha-\theta+1)} \\ &= \tilde{y}\left(\frac{1}{\Gamma_q(\alpha+\beta+1)} + \frac{1}{\Gamma_q(\alpha+\beta-\theta+1)}\right) \\ &\quad + |\lambda|\,\|x\|_\infty\left(\frac{1}{\Gamma_q(\alpha+1)} + \frac{1}{\Gamma_q(\alpha-\theta+1)}\right).\end{aligned}$$

Also, for each $t \in [0,1]$, we have

$$\begin{aligned}&|T_2x(t)| \\ &= \frac{1}{|\Delta|}\Bigg\{\sum_{i=1}^{\infty}\left(\frac{|A_1\sigma_i - A_3\mu_i|}{\Gamma_q(\alpha+1)} + |A_2\mu_i - A_4\sigma_i|\right)\left(\frac{\tilde{y}}{\Gamma_q(\alpha+\beta+\gamma+1)} + \frac{|\lambda|\,\|x\|_\infty}{\Gamma_q(\alpha+\gamma+1)}\right) \\ &\qquad + \left(\frac{|A_3|}{\Gamma_q(\alpha+1)} + |A_2|\right)\Bigg[|\delta_3|\left(\frac{\tilde{y}}{\Gamma_q(\alpha+\beta+1)} + \frac{|\lambda|\,\|x\|_\infty}{\Gamma_q(\alpha+1)}\right) \\ &\qquad\qquad + |\delta_4|\left(\frac{\tilde{y}}{\Gamma_q(\beta+1)} + |\lambda|\,\|x\|_\infty\right)\Bigg]\Bigg\}\end{aligned}$$

and

$$\begin{aligned}&\left|{}^cD_q^\theta T_2x(t)\right| \\ &\leq \frac{1}{|\Delta|\,\Gamma_q(\alpha-\theta+1)}\Bigg\{\sum_{i=1}^{\infty}|A_1\sigma_i - A_3\mu_i|\left(\frac{\tilde{y}}{\Gamma_q(\alpha+\beta+\gamma+1)} + \frac{|\lambda|\,\|x\|_\infty}{\Gamma_q(\alpha+\gamma+1)}\right) \\ &\quad + |A_3||\delta_3|\left(\frac{\tilde{y}}{\Gamma_q(\alpha+\beta+1)} + \frac{|\lambda|\,\|x\|_\infty}{\Gamma_q(\alpha+1)}\right) + |A_3||\delta_4|\left(\frac{\tilde{y}}{\Gamma_q(\beta+1)} + |\lambda|\,\|x\|_\infty\right)\Bigg\}.\end{aligned}$$

Thus, we get

$$\begin{aligned}\|T_2x\|_E \leq \frac{\tilde{y}}{|\Delta|}\Bigg\{&\sum_{i=1}^{\infty}\left(\frac{|A_1\sigma_i - A_3\mu_i|}{\Gamma_q(\alpha+1)} + |A_2\mu_i - A_4\sigma_i|\right)\frac{1}{\Gamma_q(\alpha+\beta+\gamma+1)} \\ &+ \frac{|\delta_3|}{\Gamma_q(\alpha+\beta+1)}\left(\frac{|A_3|}{\Gamma_q(\alpha+1)} + |A_2|\right) + \frac{|\delta_4|}{\Gamma_q(\beta+1)} \\ &+ \sum_{i=1}^{\infty}\frac{|A_1\sigma_i - A_3\mu_i|}{\Gamma_q(\alpha-\theta+1)}\frac{1}{\Gamma_q(\alpha+\beta+\gamma+1)} \\ &+ \frac{|A_3||\delta_3|}{\Gamma_q(\alpha+\beta+1)\Gamma_q(\alpha-\theta+1)} + \frac{|A_3||\delta_4|}{\Gamma_q(\beta+1)\Gamma_q(\alpha-\theta+1)}\Bigg\}\end{aligned}$$

$$+\frac{|\lambda|\|x\|_\infty}{|\Delta|}\left\{\sum_{i=1}^{\infty}\left(\frac{|A_1\sigma_i-A_3\mu_i|}{\Gamma_q(\alpha+1)}+|A_2\mu_i-A_4\sigma_i|\right)\frac{1}{\Gamma_q(\alpha+\gamma+1)}\right.$$
$$+\frac{|\delta_3|}{\Gamma_q(\alpha+1)}\left(\frac{|A_3|}{\Gamma_q(\alpha+1)}+|A_2|\right)+|\delta_4|+\sum_{i=1}^{\infty}\frac{|A_1\sigma_i-A_3\mu_i|}{\Gamma_q(\alpha-\theta+1)}\frac{1}{\Gamma_q(\alpha+\gamma+1)}$$
$$\left.+\frac{|A_3||\delta_3|}{\Gamma_q(\alpha+1)\Gamma_q(\alpha-\theta+1)}+\frac{|A_3||\delta_4|}{\Gamma_q(\alpha-\theta+1)}\right\}.$$

Using these inequalities for $\|T_1x\|_E$ and $\|T_2x\|_E$, we have

$$\|Tx\|_E\leq\tilde{y}H(\beta)+|\lambda|.\|x\|_\infty H(0)\leq\tilde{y}H(\beta)+|\lambda|\ \ell\ H(0)\leq\ell,$$

since $\ell\geq\dfrac{\widetilde{y}\,H(\beta)}{1-|\lambda|\,H(0)}$, where $H(\cdot)$ defined by (3.1).

Thus, we have $Tx\in B_\ell$.

We next show that T_1 is compact and continuous. Let (x_n) be a sequence such that $x_n\to x$ in B_ℓ.
Using (H_1), we have

$$\begin{aligned}
&\|T_1x_n-T_1x\|_\infty\\
&=\sup_{t\in[0,1]}\left\{I_q^{\alpha+\beta}\Big[f(t,x_n(t),{}^cD_q^\theta x_n(t),I_q^\varphi x_n(t))-f(t,x(t),{}^cD_q^\theta x(t),I_q^\varphi x(t))\Big]\right.\\
&\qquad\left.-\lambda I_q^\alpha(x_n(t)-x(t))\right\}\\
&\leq CI_q^{\alpha+\beta}\Big[\|x_n-x\|_\infty+\|{}^cD_q^\theta x_n-{}^cD_q^\theta x\|_\infty+I_q^\varphi\|x_n-x\|_\infty\Big]+|\lambda|\frac{\|x_n-x\|_\infty}{\Gamma_q(\alpha+1)}\\
&\leq CI_q^{\alpha+\beta}\Big[\|x_n-x\|_E+\frac{1}{\Gamma_q(\varphi+1)}\|x_n-x\|_E\Big]+|\lambda|\frac{\|x_n-x\|_E}{\Gamma_q(\alpha+1)}\\
&=\left\{\Big(1+\frac{1}{\Gamma_q(\varphi+1)}\Big)\frac{C}{\Gamma_q(\alpha+\beta+1)}+\frac{|\lambda|}{\Gamma_q(\alpha+1)}\right\}\|x_n-x\|_E
\end{aligned}$$

and

$$\begin{aligned}
&\left\|{}^cD_q^\theta T_1x_n-{}^cD_q^\theta T_1x\right\|_\infty\\
&\quad\leq\sup_{t\in[0,1]}\left\{I_q^{\alpha+\beta-\theta}\Big|f(t,x_n(t),{}^cD_q^\theta x_n(t),I_q^\varphi x_n(t))\right.\\
&\qquad\left.-f(t,x(t),{}^cD_q^\theta x(t),I_q^\varphi x(t))\Big|+|\lambda|I_q^\alpha|x_n(t)-x(t)|\right\}\\
&\quad\leq CI_q^{\alpha+\beta-\theta}\Big[\|x_n-x\|_E+\frac{1}{\Gamma_q(\varphi+1)}\|x_n-x\|_E\Big]+|\lambda|\frac{\|x_n-x\|_E}{\Gamma_q(\alpha-\theta+1)}\\
&\quad=\left\{\Big(1+\frac{1}{\Gamma_q(\varphi+1)}\Big)\frac{C}{\Gamma_q(\alpha+\beta-\theta+1)}+\frac{|\lambda|}{\Gamma_q(\alpha-\theta+1)}\right\}\|x_n-x\|_E.
\end{aligned}$$

Thus, we get

$$\begin{aligned}\|T_1x_n - T_1x\|_E =& \|T_1x_n - T_1x\|_\infty + \left\|{}^cD_q^\theta T_1x_n - {}^cD_q^\theta T_1x\right\|_\infty \\ \leq& \left\{C\left(1+\frac{1}{\Gamma_q(\varphi+1)}\right)\left(\frac{1}{\Gamma_q(\alpha+\beta+1)}+\frac{1}{\Gamma_q(\alpha+\beta-\theta+1)}\right)\right. \\ &\left. + |\lambda|\left(\frac{1}{\Gamma_q(\alpha+1)}+\frac{1}{\Gamma_q(\alpha-\theta+1)}\right)\right\}\|x_n - x\|_E \\ =& N\|x_n - x\|_E\end{aligned}$$

and so $\|T_1x_n - T_1x\|_E \to 0$ as $x_n \to x$.
That means T_1 is continuous. Moreover, the operator T_1 is bounded on B_ℓ due to Step 1. Thus, T_1 is uniformly bounded on B_ℓ.

Next, we will prove that T_1 is equicontinuous. Let $t_1, t_2 \in [0,1]$ be such that $t_1 < t_2$. Then

$$\begin{aligned}&\left|T_1x(t_2) - T_1x(t_1)\right| \\ \leq& \frac{\tilde{y}}{\Gamma_q(\alpha+\beta)}\int_0^{t_1}\left|(t_2-qs)^{\alpha+\beta-1} - (t_1-qs)^{\alpha+\beta-1}\right|d_qs + \frac{\tilde{y}}{\Gamma_q(\alpha+\beta)}\int_{t_1}^{t_2}(t_2-qs)^{\alpha+\beta-1}d_qs \\ &+ \frac{|\lambda|}{\Gamma_q(\alpha)}\int_0^{t_1}\left|(t_2-qs)^{\alpha-1} - (t_1-qs)^{\alpha-1}\right||x(s)|d_qs + \frac{|\lambda|}{\Gamma_q(\alpha)}\int_{t_1}^{t_2}(t_2-qs)^{\alpha-1}|x(s)|d_qs.\end{aligned}$$

For $t_2 \to t_1$ and the continuity of the functions $(t-qs)^{\alpha-1}$ and $(t-qs)^{\alpha+\beta-1}$, we obtain $|T_1x(t_2) - T_1x(t_1)| \to 0$ as $t_2 \to t_1$. Hence, T_1 is equicontinuous. As a result of the Arzela-Ascoli theorem, we deduce that the operator T_1 is compact in B_ℓ. Thus, T_1 is completely continuous. Finally, we show that T_2 is the contraction mapping. For $x_1,\ x_2 \in B_\ell,\quad t \in [0,1]$, we obtain

$$\begin{aligned}&|T_2x_1(t) - T_2x_2(t)| \\ &\leq \frac{1}{|\Delta|}\left\{\sum_{i=1}^{\infty}\left(\frac{|A_1\sigma_i - A_3\mu_i|}{\Gamma_q(\alpha+1)} + |A_2\mu_i - A_4\sigma_i|\right)\right. \\ &\qquad\times\left[\frac{C}{\Gamma_q(\alpha+\beta+\gamma+1)}\Big(1+\frac{1}{\Gamma_q(\varphi+1)}\Big) + \frac{|\lambda|}{\Gamma_q(\alpha+\gamma+1)}\right]\|x_1 - x_2\|_E \\ &+\ |\delta_3|\Big(\frac{|A_3|}{\Gamma_q(\alpha+1)} + |A_2|\Big)\left[\frac{C}{\Gamma_q(\alpha+\beta+1)}\Big(1+\frac{1}{\Gamma_q(\varphi+1)}\Big) + \frac{|\lambda|}{\Gamma_q(\alpha+1)}\right]\|x_1 - x_2\|_E \\ &+\ |\delta_4|\Big(\frac{|A_3|}{\Gamma_q(\alpha+1)} + |A_2|\Big)\left[\frac{C}{\Gamma_q(\beta+1)}\Big(1+\frac{1}{\Gamma_q(\varphi+1)}\Big) + |\lambda|\right]\|x_1 - x_2\|_E\Big\}.\end{aligned}$$

So, we get

$$\|T_2x_1 - T_2x_2\|_\infty \leq \frac{1}{|\Delta|}\Bigg[\Bigg(\sum_{i=1}^{\infty}\Big(\frac{|A_1\sigma_i - A_3\mu_i|}{\Gamma_q(\alpha+1)} + |A_2\mu_i - A_4\sigma_i|\Big)\Bigg)G_{\alpha,\gamma} + \Big(\frac{|A_3|}{\Gamma_q(\alpha+1)} + |A_2|\Big)\big(|\delta_3|\,G_{\alpha,0} + |\delta_4|\,G_{0,0}\big)\Bigg]\|x_1 - x_2\|_E.$$

In addition to this inequality, since

$$\begin{aligned}
&\big|{}^cD_q^\theta T_2x_1(t) - {}^cD_q^\theta T_2x_2(t)\big| \\
&\leq \frac{1}{|\Delta|\,\Gamma_q(\alpha-\theta+1)}\Bigg\{\sum_{i=1}^{\infty}|A_1\sigma_i - A_3\mu_i|\Big[I_q^{\alpha+\beta+\gamma}\big|f(\eta_i, x_1(\eta_i), {}^cD_q^\theta x_1(\eta_i), I_q^\varphi x_1(\eta_i)) \\
&\quad - f(\eta_i, x_2(\eta_i), {}^cD_q^\theta x_2(\eta_i), I_q^\varphi x_2(\eta_i))\big| + |\lambda| I_q^{\alpha+\gamma}|x_1(\eta_i) - x_2(\eta_i)|\Big] \\
&\quad + |A_3||\delta_3|\Big[I_q^{\alpha+\beta}\big|f(1, x_1(1), {}^cD_q^\theta x_1(1), I_q^\varphi x_1(1)) - f(1, x_2(1), {}^cD_q^\theta x_2(1), I_q^\varphi x_2(1))\big| \\
&\qquad + |\lambda| I_q^{\alpha}|x_1(1) - x_2(1)|\Big] \\
&\quad + |A_3||\delta_4|\Big[I_q^{\beta}\big|f(1, x_1(1), {}^cD_q^\theta x_1(1), I_q^\varphi x_1(1)) - f(1, x_2(1), {}^cD_q^\theta x_2(1), I_q^\varphi x_2(1))\big| \\
&\qquad + |\lambda||x_1(1) - x_2(1)|\Big]\Bigg\} \\
&\leq \frac{1}{|\Delta|\,\Gamma_q(\alpha-\theta+1)}\Bigg\{\sum_{i=1}^{\infty}|A_1\sigma_i - A_3\mu_i|\,G_{\alpha,\gamma} + |A_3||\delta_3|\,G_{\alpha,0} + |A_3||\delta_4|\,G_{0,0}\Bigg\}\|x_1 - x_2\|_E
\end{aligned}$$

we have

$$\|{}^cD_q^\theta T_2x_1 - {}^cD_q^\theta T_2x_2\|_\infty \leq \frac{1}{|\Delta|\,\Gamma_q(\alpha-\theta+1)}\Bigg\{\sum_{i=1}^{\infty}|A_1\sigma_i - A_3\mu_i|\,G_{\alpha,\gamma} + |A_3|\Big(|\delta_3|G_{\alpha,0} + |\delta_4|G_{0,0}\Big)\Bigg\}\|x_1 - x_2\|_E.$$

As a result, we get

$$\begin{aligned}
\|T_2x_1 - T_2x_2\|_E &\leq \frac{1}{|\Delta|}\Bigg\{\Bigg[\sum_{i=1}^{\infty}\Big(\frac{1}{\Gamma_q(\alpha+1)} + \frac{1}{\Gamma_q(\alpha-\theta+1)}\Big)|A_1\sigma_i - A_3\mu_i| + |A_2\mu_i - A_4\sigma_i|\Bigg]G_{\alpha,\gamma} \\
&\quad + \Big[\big(\frac{1}{\Gamma_q(\alpha+1)} + 1\big)|A_3| + |A_2|\Big]|\delta_3|G_{\alpha,0} \\
&\quad + \Big[\big(\frac{1}{\Gamma_q(\alpha+1)} + 1\big)|A_3| + |A_2|\Big]|\delta_4|G_{0,0}\Bigg\}\|x_1 - x_2\|_E \\
&= M\,\|x_1 - x_2\|_E.
\end{aligned}$$

Thus, we conclude that T_2 is a contraction, since $M < 1$. Therefore, by Theorem 2.3, we infer that problem (1.1) has at least one solution on $[0, 1]$ □

Theorem 3.2. *Assume that* $(H_1) - (H_2)$ *hold. If*

$$N + M < 1 \tag{3.2}$$

then the problem (1.1) has a unique solution on E, where M is given in Theorem 3.1 and

$$\begin{aligned} N :=& C\left(1 + \frac{1}{\Gamma_q(\varphi+1)}\right)\left(\frac{1}{\Gamma_q(\alpha+\beta+1)} + \frac{1}{\Gamma_q(\alpha+\beta-\theta+1)}\right) \\ &+ |\lambda|\left(\frac{1}{\Gamma_q(\alpha+1)} + \frac{1}{\Gamma_q(\alpha-\theta+1)}\right). \end{aligned}$$

Proof. We note that the fixed points of the operator T defined in (2.2) are the solutions to problem (1.1). Define a closed ball set B_ℓ as in the proof of Theorem 3.1. In this proof, we saw that $T(B_\ell) \subset B_\ell$, for $x \in B_\ell$.
Now, we see that T is a contraction map. Indeed, for $x_1,\ x_2 \in B_\ell, \quad t \in [0, 1]$, we obtain

$$\begin{aligned} \|Tx_1 - Tx_2\|_E \leq& \left\{ C\Big(1 + \frac{1}{\Gamma_q(\varphi+1)}\Big)\Big(\frac{1}{\Gamma_q(\alpha+\beta+1)} + \frac{1}{\Gamma_q(\alpha+\beta-\theta+1)}\Big) \right. \\ &\left. + |\lambda|\Big(\frac{1}{\Gamma_q(\alpha+1)} + \frac{1}{\Gamma_q(\alpha-\theta+1)}\Big) + M \right\} \|x_1 - x_2\|_E \\ =& (N + M)\|x_1 - x_2\|_E. \end{aligned}$$

From (3.2), T is a contraction map. Hence, using the Theorem 2.4, we conclude that problem (1.1) has a unique solution on $[0, 1]$. □

4. Generalized Ulam Stabilities

In this section, we present a theorem showing that problem (1.1) admits both Ulam-Hyers and generalized Ulam-Hyers stabilities. Using the integral representation

$$\begin{aligned} & I_q^{\alpha+\beta} f\Big(t, y(t), {}^cD_q^\theta y(t), I_q^\varphi y(t)\Big) - \lambda I_q^\alpha y(t) \\ & - \frac{t^\alpha}{\Delta\,\Gamma_q(\alpha+1)} \Bigg\{ \sum_{i=1}^{\infty} (A_3\mu_i - A_1\sigma_i)\Big(I_q^{\alpha+\beta+\gamma} f(\eta_i, y(\eta_i), {}^cD_q^\theta y(\eta_i), I_q^\varphi y(\eta_i)) - \lambda I_q^{\alpha+\gamma} y(\eta_i)\Big) \\ & - A_3\delta_3\Big(I_q^{\alpha+\beta} f(1, y(1), {}^cD_q^\theta y(1), I_q^\varphi y(1)) - \lambda I_q^\alpha y(1)\Big) \\ & + A_3\delta_4\Big(I_q^\beta f(1, y(1), {}^cD_q^\theta y(1), I_q^\varphi y(1)) - \lambda y(1)\Big) \Bigg\} \end{aligned}$$

$$-\frac{1}{\Delta}\Bigg\{\sum_{i=1}^{\infty}(A_4\sigma_i - A_2\mu_i)\Big(I_q^{\alpha+\beta+\gamma}f(\eta_i, y(\eta_i), {}^cD_q^{\theta}y(\eta_i), I_q^{\varphi}y(\eta_i)) - \lambda I_q^{\alpha+\gamma}y(\eta_i)\Big)$$
$$+ A_2\delta_3\Big(I_q^{\alpha+\beta}f(1, y(1), {}^cD_q^{\theta}y(1), I_q^{\varphi}y(1)) - \lambda I_q^{\alpha}y(1)\Big)$$
$$- A_2\delta_4\Big(I_q^{\beta}f(1, y(1), {}^cD_q^{\theta}y(1), I_q^{\varphi}y(1)) - \lambda y(1)\Big)\Bigg\} := y(t),$$

where Δ and A_i , $(i = 1, 2, 3, 4)$ are given in Lemma 2.1, we discuss the Ulam-Hyers and generalized Ulam-Hyers stability of problem (1.1), here $y(t) \in \mathcal{C}([0,1],\mathbb{R})$ and $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ are continuous functions. Then we define the nonlinear continuous operator

$$A : \mathcal{C}([0,1],\mathbb{R}) \to \mathcal{C}([0,1],\mathbb{R})$$

as follows

$$Ay(t) = {}^cD_q^{\beta}\left({}^cD_q^{\alpha} + \lambda\right)y(t) - f\left(t, y(t), {}^cD_q^{\theta}y(t),\ I_q^{\varphi}y(t)\right). \tag{4.1}$$

Definition 4.1. For each $\varepsilon > 0$ and for each solution $y(t)$ of (4.1), such that $\|Ay\| \leq \varepsilon$, the problem (1.1) is said to be Ulam-Hyers stability if we can find a positive real number v and a solution $x(t) \in \mathcal{C}([0,1],\mathbb{R})$ of (1.1) satisfying the inequality.

$$|x - y| \leq v\varepsilon^*$$

where ε^* is a positive real number depending on ε.

Definition 4.2. Let $m \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$|x(t) - y(t)| \leq m(\varepsilon), \qquad t \in [0,1]$$

then the problem (1.1) is said to be generalized Ulam-Hyers stable.

Theorem 4.3. *Under assumptions $(H_1) - (H_2)$, with the problem (1.1) is both Ulam-Hyers and generalized Ulam-Hyers stable, if $N + M < 1$ such that N and M are given in Theorem 3.2.*

Proof. Let $x(t) \in \mathcal{C}([0,1],\mathbb{R})$ be a solution of (1.1) in the sense of Theorem 3.1. Let y be any solution that satisfies (4.1). Corollary 2.2 implies the equivalence between the operators A and T for every solution $y \in \mathcal{C}([0,1],\mathbb{R})$ of (1.1) satisfying.

Therefore, we deduce by the fixed point property of the operator T that

$$\begin{aligned}
|x(t)-y(t)| &= |x(t)-Ty(t)+Ty(t)-y(t)| \\
&\leq |Tx(t)-Ty(t)|+|Ty(t)-y(t)| \\
&\leq \|Tx-Ty\|+\|(T-I)\,y\| \\
&\leq \|Tx-Ty\|+\|Ay\| \\
&\leq \left\{ C\left(1+\frac{1}{\Gamma_q(\varphi+1)}\right)\left(\frac{1}{\Gamma_q(\alpha+\beta+1)}+\frac{1}{\Gamma_q(\alpha+\beta-\theta+1)}\right)\right. \\
&\qquad \left. +|\lambda|\left(\frac{1}{\Gamma_q(\alpha+1)}+\frac{1}{\Gamma_q(\alpha-\theta+1)}\right)+M\right\}\|x-y\|_E+\varepsilon.
\end{aligned}$$

Because $N+M<1$ and $\varepsilon>0$, we find

$$\|x-y\|_E \leq \frac{\varepsilon}{1-N-M}.$$

Fixing $\varepsilon^*=\dfrac{\varepsilon}{1-N-M}$ and $v=1$, we obtain Ulam-Hyers stability. In addition, the generalized Ulam-Hyers stability follows taking $m(\varepsilon)=\dfrac{\varepsilon}{1-N-M}$. □

5. Numerical example

Consider the following Langevin-type generalized fractional q-difference equation with infinite-point boundary conditions of the form

$$\begin{aligned}
{}^cD_{0.5}^{1/5}\big({}^cD_{0.5}^{1/3}+\frac{1}{10^6}\big)x(t) &= f\Big(t,x(t),{}^cD_{0.5}^{1/10}x(t),I_{0.5}^3x(t)\Big), \qquad t\in[0,1], \\
\frac{1}{4}x(0)+2\ {}^cD_{0.5}^{\alpha}x(0) &= \sum_{i=1}^{\infty}\frac{1}{i^2}\,I_{0.5}^2x(\frac{1}{i^3}), \\
x(1)-3\ {}^cD_{0.5}^{\alpha}x(1) &= \sum_{i=1}^{\infty}\frac{1}{i^4}\,I_{0.5}^2x(\frac{1}{i^3}),
\end{aligned}$$

that $q=1/2$, $\alpha=1/3$, $\beta=1/5$, $\theta=1/10$, $\varphi=3$, $\gamma=2$, $\delta_1=1/4$, $\delta_2=2$, $\delta_3=1$, $\delta_4=3$, $\lambda=10^{-6}$, $\sigma_i=i^{-2}$, $\mu_i=i^{-4}$, $\eta_i=i^{-3}$ where $x\in C([0,1],\mathbb{R})$ and $f:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ is a function given by

$$f(t,u(t),v(t),w(t))=\frac{1}{\sqrt{t^2+10^{10}}}\left(\frac{|u(t)|}{10^2(1+|u(t)|)}+\frac{1}{2}\arctan v(t)+\frac{1}{5}\sin w(t)\right).$$

First, we see that conditions

(H_1) $|f(t,u_1,v_1,w_1)-f(t,u_2,v_2,w_2)|$

$$\begin{aligned}
&=\left|\frac{1}{\sqrt{t^2+10^{10}}}\left(\frac{|u_1|}{10^2(1+|u_1|)}-\frac{|u_2|}{10^2(1+|u_2|)}\right.\right.\\
&\quad\left.\left.+\frac{1}{2}\arctan(v_1)-\frac{1}{2}\arctan(v_2)+\frac{1}{5}\sin(w_1)-\frac{1}{5}\sin(w_2)\right)\right|\\
&\le\frac{1}{10^2}\left\{\frac{|u_1-u_2|}{10^5}+\frac{1}{2}|v_1-v_2|+\frac{1}{5}|w_1-w_2|\right\}\\
&\le\frac{1}{10^5}\big(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|\big)
\end{aligned}$$

and

(H_2) $|f(t,u,v,w)|\le\dfrac{1}{\sqrt{t^2+10^{10}}}\left(\dfrac{1}{10^2}+\dfrac{\pi}{4}+\dfrac{1}{5}\right)\le\dfrac{3}{\sqrt{t^2+10^{10}}}:=y(t)$

are satisfied with $C=\dfrac{1}{10^5}$ and $\tilde{y}=\sup_{t\in[0,1]}|y(t)|=\dfrac{3}{10^5}$.

In addition, we can easily calculate that $|\Delta|\approx 0.58$, $1-|\lambda|H(0)\approx 1-2.10^{-5}>0$, $M\approx 7.10^{-4}<1$. It follows from Theorem 3 that this problem has at least one solution.

Also, since $N\approx 3.10^{-5}$, we have $M+N\approx 7,3.10^{-4}<1$. Thus, this problem has a unique solution from Theorem 3.2 and both Ulam-Hyers and the generalized Ulam-Hyers stable from Theorem 4.3.

6. Conclusions

In this work, we have established new existence, uniqueness, and Ulam–Hyers stability results for a Langevin-type generalized q-difference fractional boundary value problem with infinite-point boundary conditions. By employing fixed point theory, it has been shown that the considered problem admits well-posed solutions under suitable assumptions. These results not only extend and unify several existing findings in the literature but also strengthen the theoretical foundation of discrete fractional systems.

Beyond their theoretical value, the obtained results contribute to the advancement of q-fractional calculus and hold significant potential for modeling complex systems characterized by memory effects and nonlocal interactions. By integrating the richness of fractional Langevin-type dynamics with the flexibility of generalized q-difference operators, the present study provides a comprehensive framework for exploring both pure and applied aspects of fractional dynamics.

Furthermore, these findings establish a solid basis for future studies on the stability analysis of discrete fractional systems, including theoretical generalizations and computational implementations. They also underline the effectiveness of fixed point techniques in addressing boundary value problems of increasing mathematical complexity.

Overall, this study enhances the theoretical landscape of fractional difference equations and reinforces the connection between pure mathematics, applied analysis, and dynamical modeling, paving the way for further interdisciplinary research in this evolving field.

Acknowledgment. The authors sincerely thank the anonymous reviewers for their careful reading, constructive comments, and valuable suggestions, which have significantly contributed to improving the quality and presentation of this manuscript.

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