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SOME CHARACTERIZATIONS OF CLEAR IDEALS

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Abstract. Let $\mathcal{T}$ be a ring with identity. This paper introduces the notion of a clear ideal, extending the concept of clear rings to arbitrary ideals. An ideal P is called clear if every element of P is the sum of a unit and a unit-regular element of $\mathcal{T}$. We investigate the properties and characterizations of clear ideals. Every clean ideal is clear, but the converse does not hold. We also show that the Jacobson radical, which is clean, is clear. Furthermore, we study the preservation of clear ideals under homomorphisms, quotient rings, direct products, matrix rings, Morita rings, formal power series rings, localizations, and Nagata's extensions.

Keywords. Clear ideal, clear ring, unit-regular element, Jacobson radical
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1. Introduction

One of the fundamental objects studied in algebra is the notion of a ring. A ring is a nonempty set together with two binary operations, addition "+" and multiplication "·", that satisfy certain axioms. The study of rings is not only of theoretical interest but is also closely connected to various other branches of mathematics. For a more detailed about ring theory, see [7]. Consequently, research on rings continues to develop, both in terms of deepening the theory and exploring new conceptual directions. Moreover, ring theory plays an important role in practical areas such as cryptography, where it underlies the construction of secure information systems, and in coding theory, where it contributes to the design and analysis of error-correcting codes for digital communication.

In 1977, Nicholson [9] introduced the concept of a clean element. Let $(\mathcal{T}, +, \cdot)$ be a ring with unity $1_{\mathcal{T}}$. An element $t \in \mathcal{T}$ is called clean if it can be written as $t = u + e$, where $u \in U(\mathcal{T})$ and $e \in Id(\mathcal{T})$. In the same paper, Nicholson also defined a clean ring, that is, a ring in which every element is clean. This concept initiated a systematic study of cleanness conditions and inspired several natural generalizations.

Among these generalizations, Ashrafi and Nasibi introduced the notion of r-clean rings by replacing units with regular elements [1]. In a different direction,

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Diesl [5] defined nil clean rings, in which each element can be expressed as the sum of an idempotent and a nilpotent element. Further weakening this condition, weakly clean rings and weakly nil clean rings were introduced, allowing elements to be written as either the sum or the difference of the corresponding components [3, 2]. These notions illustrate how varying the types of elements involved in such decompositions leads to distinct classes of rings with different structural properties.

Earlier, in 1968, Ehrlich [6] introduced the concept of a unit-regular element. An element $y \in \mathcal{T}$ is called unit-regular if there exists some $y \in U(\mathcal{T})$ such that $t = tyt$. In particular, each idempotent element is unit-regular. By replacing idempotent elements with unit-regular elements, Zabavsky [13] introduced the notion of clear rings, in which each element can be expressed as the sum of a unit and a unit-regular element. Since unit-regular elements generalize idempotent elements, clear rings are a generalization of clean rings. Several structural properties of clear rings were further investigated in [13].

Furthermore, the study of cleanness conditions has been extended from rings to their ideals. In 2003, Chen [4] introduced the concept of a clean ideal. An ideal $P \subseteq \mathcal{T}$ is called clean if every element of P can be expressed as the sum of a unit and an idempotent of $\mathcal{T}$. Following this idea, r-clean ideals were studied by replacing units with regular elements [12]. In a parallel development, Sharma and Basnet introduced nil clean ideals and weakly nil clean ideals by replacing units with nilpotent elements and allowing elements to be written as either a sum or a difference of the corresponding components [11, 10]. These concepts form natural ideal-theoretic analogues of their corresponding ring notions.

Motivated by the notion of clear rings and the development of cleanness conditions on ideals, it is natural to extend the concept of clear rings to ideals. Therefore, in this paper, we define the notion of a clear ideal and examine its basic properties.

Throughout this paper, let $\mathcal{T}$ be a ring with unity $1_{\mathcal{T}}$. The notations $U(\mathcal{T})$, $Id(\mathcal{T})$, $U_{\text{reg}}(\mathcal{T})$, $Cln(\mathcal{T})$, and $Clr(\mathcal{T})$ denote the sets of all units, idempotents, unit-regular elements, clean elements, and clear elements of $\mathcal{T}$, respectively.

2. Result and Discussion

Inspired by the notion of clean ideals, we extend this idea by introducing the concept of a clear ideal, which serves as a natural generalization of clear rings, noting that any ring $(\mathcal{T}, +, \cdot)$ can be viewed as an ideal of itself. In this paper, the notion of a clear ring is generalized to an arbitrary ideal P of $\mathcal{T}$.

Definition 2.1. Let $(\mathcal{T}, +, \cdot)$ be a ring and P an ideal of $\mathcal{T}$. The ideal P is said to be clear if for every element $x \in P$, there exist $u \in U(\mathcal{T})$ and $v \in U_{\text{reg}}(\mathcal{T})$ such that $x = u + v$.

Remark 2.2. It follows from Definition 2.1 that an ideal P of $\mathcal{T}$ is a clear ideal if $P \subseteq Clr(\mathcal{T})$.

To facilitate the understanding of Definition 2.1, the following are examples of clear ideals.

Example 2.3. Any ideal P of a division ring $(\mathcal{T}, +, \cdot)$ is a clear ideal.

Proposition 2.4. *Any ideal P of a clear ring $(\mathcal{T}, +, \cdot)$ is a clear ideal.*

Proof. Let $x \in P$. Since $P \subseteq \mathcal{T}$ and $\mathcal{T}$ is a clear ring, there exist elements $u \in U(\mathcal{T})$ and $v \in U_{\text{reg}}(\mathcal{T})$ such that

$$x = u + v.$$

Thus, every element of P can be expressed as the sum of a unit and a unit-regular element of $\mathcal{T}$. By definition, P is a clear ideal of $\mathcal{T}$. □

Example 2.5. Let $(\mathcal{T}, +, \cdot)$ be a clear ring. If P_1 and P_2 are ideals of $\mathcal{T}$, then

(1) $P_1 \cap P_2$ is a clear ideal of $\mathcal{T}$.
(2) $P_1 + P_2$ is a clear ideal of $\mathcal{T}$.
(3) $P_1 P_2$ is a clear ideal of $\mathcal{T}$.
(4) $P_1 \cup P_2$ is a clear ideal of $\mathcal{T}$ where $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$.

As shown in Proposition 2.4, every ideal of a clear ring is a clear ideal. However, there exist rings that are not clear rings but still contain clear ideals.

Example 2.6. Let $(\mathcal{T}_1, +_1, \cdot_1)$ and $(\mathcal{T}_2, +_2, \cdot_2)$ be rings such that $\mathcal{T}_1$ is not a clear ring, whereas $\mathcal{T}_2$ is a clear ring. Consequently, the direct product $\mathcal{T}_1 \times \mathcal{T}_2$ is not a clear ring. Consider the ideal $\{0_{\mathcal{T}_1}\} \times \mathcal{T}_2$ of $\mathcal{T}_1 \times \mathcal{T}_2$. We show that $\{0_{\mathcal{T}_1}\} \times \mathcal{T}_2$ is a clear ideal of $\mathcal{T}_1 \times \mathcal{T}_2$.

Let $(0_{\mathcal{T}_1}, t_2) \in \{0_{\mathcal{T}_1}\} \times \mathcal{T}_2$. Then $t_2 \in \mathcal{T}_2$. Since $\mathcal{T}_2$ is a clear ring, there exist elements $u \in U(\mathcal{T}_2)$ and $v \in U_{\text{reg}}(\mathcal{T}_2)$ such that

$$t_2 = u +_2 v.$$

Since $v \in U_{\text{reg}}(\mathcal{T}_2)$, there exists $y \in U(\mathcal{T}_2)$ satisfying

$$v \cdot_2 y \cdot_2 v = v.$$

Consider the following decomposition:

$$\begin{aligned}(0_{\mathcal{T}_1}, t_2) &= (1_{\mathcal{T}_1} +_1 (-1_{\mathcal{T}_1}),\ u +_2 v)\\ &= (1_{\mathcal{T}_1}, u) + (-1_{\mathcal{T}_1}, v).\end{aligned}$$

The element $(1_{\mathcal{T}_1}, u) \in U(\mathcal{T}_1 \times \mathcal{T}_2)$ because there exists $(1_{\mathcal{T}_1}, u^{-1}) \in \mathcal{T}_1 \times \mathcal{T}_2$ such that

$$(1_{\mathcal{T}_1}, u)(1_{\mathcal{T}_1}, u^{-1}) = (1_{\mathcal{T}_1}, 1_{\mathcal{T}_2}) = (1_{\mathcal{T}_1}, u^{-1})(1_{\mathcal{T}_1}, u).$$

Moreover, $(-1_{\mathcal{T}_1}, v) \in U_{\text{reg}}(\mathcal{T}_1 \times \mathcal{T}_2)$ since there exists $(-1_{\mathcal{T}_1}, y) \in U(\mathcal{T}_1 \times \mathcal{T}_2)$ such that

$$(-1_{\mathcal{T}_1}, v)(-1_{\mathcal{T}_1}, y)(-1_{\mathcal{T}_1}, v) = (-1_{\mathcal{T}_1}, v \cdot_2 y \cdot_2 v) = (-1_{\mathcal{T}_1}, v).$$

Therefore, $\{0_{\mathcal{T}_1}\} \times \mathcal{T}_2$ is a clear ideal of $\mathcal{T}_1 \times \mathcal{T}_2$.

The next proposition establishes an equivalent condition for an ideal to be clear by expressing each of its elements as a combination of two units and an idempotent.

Proposition 2.7. *Let $(\mathcal{T}, +, \cdot)$ be a ring and P an ideal of $\mathcal{T}$. Then P is a clear ideal if and only if for every element $x \in P$, there exist units $u_1, u_2 \in U(\mathcal{T})$ and an idempotent $e \in Id(\mathcal{T})$ such that $x = u_1 + e \cdot u_2$.*

Proof. ($\Longrightarrow$) Suppose P is a clear ideal of $\mathcal{T}$. Let $x \in P$. By definition, there exist $u_1 \in U(\mathcal{T})$ and $v \in U_{\text{reg}}(\mathcal{T})$ such that $x = u_1 + v$. Since $v \in U_{\text{reg}}(\mathcal{T})$, there exists a unit $y \in U(\mathcal{T})$ satisfying $v \cdot y \cdot v = v$.

Let us define $e = v \cdot y$. We first show that e is an idempotent element. Observe that

$$e^2 = (v \cdot y) \cdot (v \cdot y) = (v \cdot y \cdot v) \cdot y.$$

Since $v \cdot y \cdot v = v$, we substitute this back into the equation to get $e^2 = v \cdot y = e$. Thus, $e \in Id(\mathcal{T})$.

Next, since $y \in U(\mathcal{T})$, it has a multiplicative inverse $y^{-1} \in U(\mathcal{T})$. We can manipulate the expression for v as follows:

$$v = v \cdot (y \cdot y^{-1}) = (v \cdot y) \cdot y^{-1} = e \cdot y^{-1}.$$

Let $u_2 = y^{-1}$. Since y^{-1} is the inverse of a unit, $u_2 \in U(\mathcal{T})$. Therefore, we can express the original element as

$$x = u_1 + v = u_1 + e \cdot u_2,$$

where $u_1, u_2 \in U(\mathcal{T})$ and $e \in Id(\mathcal{T})$.

($\Longleftarrow$) Conversely, suppose that for every $x \in P$, there exist $u_1, u_2 \in U(\mathcal{T})$ and $e \in Id(\mathcal{T})$ such that $x = u_1 + e \cdot u_2$. We must show that P is a clear ideal.

Let $v = e \cdot u_2$. We claim that $v \in U_{\text{reg}}(\mathcal{T})$. Let $y = u_2^{-1} \in U(\mathcal{T})$. We evaluate the unit-regular condition:

$$v \cdot y \cdot v = (e \cdot u_2) \cdot u_2^{-1} \cdot (e \cdot u_2) = e \cdot (u_2 \cdot u_2^{-1}) \cdot e \cdot u_2.$$

Since $u_2 \cdot u_2^{-1} = 1_{\mathcal{T}}$, this simplifies to

$$v \cdot y \cdot v = e \cdot 1_{\mathcal{T}} \cdot e \cdot u_2 = e^2 \cdot u_2.$$

Because $e \in Id(\mathcal{T})$, we know $e^2 = e$. Thus, the expression becomes $e \cdot u_2 = v$. This proves that $v \cdot y \cdot v = v$, meaning $v \in U_{\text{reg}}(\mathcal{T})$.

Since $x = u_1 + v$ with $u_1 \in U(\mathcal{T})$ and $v \in U_{\text{reg}}(\mathcal{T})$, we conclude that P is a clear ideal.

□

This characterization shows that clear ideals are a generalization of clean ideals.

Corollary 2.8. *Let $(\mathcal{T}, +, \cdot)$ be a ring and P an ideal of $\mathcal{T}$. If P is a clean ideal, then P is a clear ideal.*

It is clear that every clean ideal is a clear ideal, but not conversely. We show this by constructing a proper ideal that is clear but not clean.

Example 2.9. Let $\mathcal{T} = M_2(\mathbb{Z}) \times \mathbb{Z}$ be a direct product of rings. Consider the proper ideal $P = M_2(\mathbb{Z}) \times \{0\}$ of $\mathcal{T}$. We will show that P is a clear ideal but not a clean ideal.

First, we prove that P is a clear ideal. Let $x \in P$, so that $x = (M, 0)$ for some $M \in M_2(\mathbb{Z})$. We will show that M can be expressed as the sum of two units in $M_2(\mathbb{Z})$.

Since the ring $\mathbb{Z}$ is a Principal Ideal Domain, every matrix $A \in M_2(\mathbb{Z})$ admits a diagonalization via the Smith Normal Form. That is, there exist invertible matrices $P, Q \in GL_2(\mathbb{Z})$ such that $A = PDQ$, where D is a diagonal matrix. Let

$$D = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} x & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & y \end{bmatrix}.$$

It is clear that $\begin{bmatrix} x & 1 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & y \end{bmatrix} \in U(M_2(\mathbb{Z}))$. Consequently,

$$A = PDQ = P\left(\begin{bmatrix} x & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & y \end{bmatrix}\right) Q = P\begin{bmatrix} x & 1 \\ -1 & 0 \end{bmatrix} Q + P\begin{bmatrix} 0 & -1 \\ 1 & y \end{bmatrix} Q.$$

Since $P, Q \in GL_2(\mathbb{Z})$ and both summands are units, it follows that each product PU_1Q and PU_2Q is also a unit in $M_2(\mathbb{Z})$. Hence, every matrix $A \in M_2(\mathbb{Z})$ can be expressed as the sum of two units. Thus, $M = U_1 + U_2$ for some units $U_1, U_2 \in M_2(\mathbb{Z})$. Therefore,

$$x = (M, 0) = (U_1 + U_2, 1 + (-1)) = (U_1, 1) + (U_2, -1).$$

It is straightforward to verify that $(U_1, 1) \in U(\mathcal{T})$, since its inverse is $(U_1^{-1}, 1)$. Similarly, $(U_2, -1) \in U(\mathcal{T})$ with inverse $(U_2^{-1}, -1)$. Since every unit in $\mathcal{T}$ is unit-regular, it follows that $(U_2, -1) \in U_{\text{reg}}(\mathcal{T})$. Hence, x can be written as the sum of a unit and a unit-regular element in $\mathcal{T}$. Therefore, P is a clear ideal.

Next, we show that P is **not** a clean ideal. Consider the matrix

$$A = \begin{pmatrix} 12 & 5 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{Z}),$$

and let $y = (A, 0) \in P$. Suppose, for the sake of contradiction, that P is a clean ideal. Then there exist an idempotent $E \in Id(\mathcal{T})$ and a unit $V \in U(\mathcal{T})$ such that $y = E + V$.

By the structure of the direct product, we may write $E = (E_1, E_2)$ with $E_1 \in Id(M_2(\mathbb{Z}))$ and $E_2 \in Id(\mathbb{Z})$, and $V = (V_1, V_2)$ with $V_1 \in U(M_2(\mathbb{Z}))$ and $V_2 \in U(\mathbb{Z})$. Hence,

$$(A, 0) = (E_1 + V_1, E_2 + V_2).$$

By comparing components, we obtain $A = E_1 + V_1$. This implies that A is a clean element in $M_2(\mathbb{Z})$, which contradicts the fact that A is not clean [8].

Therefore, the assumption is false. The ideal $P = M_2(\mathbb{Z}) \times \{0\}$ is not a clean ideal. This provides an example of a clear ideal that is not clean.

Furthermore, to investigate the structural properties of clear ideals, we examine their relationship with the Jacobson radical. Recall that the Jacobson radical of a ring $\mathcal{T}$, denoted by $J(\mathcal{T})$, is the intersection of all maximal ideals of $\mathcal{T}$. A well-known property of $J(\mathcal{T})$ is that for any $x \in J(\mathcal{T})$, the element $1_{\mathcal{T}} - x$ is a unit in $\mathcal{T}$. Using this property, we show that any ideal contained in the Jacobson radical is a clear ideal.

Proposition 2.10. *Let $(\mathcal{T}, +, \cdot)$ be a ring. If P is an ideal of $\mathcal{T}$ such that $P \subseteq J(\mathcal{T})$, then P is a clear ideal of $\mathcal{T}$.*

Proof. Let P be an ideal of $\mathcal{T}$ such that $P \subseteq J(\mathcal{T})$. Let $x \in P$ be an arbitrary element. We must show that x can be expressed as the sum of a unit and a unit-regular element of $\mathcal{T}$. Since $x \in P$ and $P \subseteq J(\mathcal{T})$, we have $x \in J(\mathcal{T})$. By the fundamental characterization of the Jacobson radical, $1_{\mathcal{T}} - rx \in U(\mathcal{T})$ for all $r \in \mathcal{T}$. By taking $r = 1_{\mathcal{T}}$, we obtain that $1_{\mathcal{T}} - x \in U(\mathcal{T})$. Then, we have $-(1_{\mathcal{T}} - x) = x - 1_{\mathcal{T}} \in U(\mathcal{T})$. We can construct the following decomposition for x:

$$x = (x - 1_{\mathcal{T}}) + 1_{\mathcal{T}}.$$

Let $u = x - 1_{\mathcal{T}}$. As shown above, $u \in U(\mathcal{T})$. Next, let $v = 1_{\mathcal{T}}$. We observe that v is a unit-regular element because there exists $1_{\mathcal{T}} \in U(\mathcal{T})$ such that

$$v \cdot 1_{\mathcal{T}} \cdot v = 1_{\mathcal{T}} \cdot 1_{\mathcal{T}} \cdot 1_{\mathcal{T}} = 1_{\mathcal{T}} = v.$$

Thus, $v \in U_{\text{reg}}(\mathcal{T})$. Since every element $x \in P$ can be written as $x = u + v$ where $u \in U(\mathcal{T})$ and $v \in U_{\text{reg}}(\mathcal{T})$, we conclude that P is a clear ideal of $\mathcal{T}$. □

As a direct consequence of Proposition 2.10, we obtain the following corollary.

Corollary 2.11. *Let $(\mathcal{T}, +, \cdot)$ be a ring. The Jacobson radical $J(\mathcal{T})$ and the nilradical $Nil(\mathcal{T})$ are all clear ideals of $\mathcal{T}$.*

Proof. By definition, the nilradical $Nil(\mathcal{T})$ is always contained in the Jacobson radical $J(\mathcal{T})$. Applying Proposition 2.10, since $J(\mathcal{T}) \subseteq J(\mathcal{T})$ and $Nil(\mathcal{T}) \subseteq J(\mathcal{T})$, it immediately follows that $J(\mathcal{T})$ and $Nil(\mathcal{T})$ are clear ideals of $\mathcal{T}$. □

Next, we present a proposition showing that the image of a clear ideal under a ring homomorphism is also a clear ideal.

Proposition 2.12. *Let $(\mathcal{T}_1, +_1, \cdot_1)$ and $(\mathcal{T}_2, +_2, \cdot_2)$ be rings, $\zeta : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ a ring homomorphism, and P an ideal of $\mathcal{T}_1$. If P is a clear ideal of $\mathcal{T}_1$, then $\zeta(P) = \{\zeta(p) \mid p \in P\}$ is a clear ideal of $\zeta(\mathcal{T}_1)$.*

Proof. Let $(\mathcal{T}_1, +_1, \cdot_1)$ and $(\mathcal{T}_2, +_2, \cdot_2)$ be rings and $\zeta : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ be a ring homomorphism. Suppose P is an ideal of $\mathcal{T}_1$. We will show that $\zeta(P)$ is a clear ideal of $\zeta(\mathcal{T}_1)$. Let any $\zeta(x) \in \zeta(P)$, where $x \in P$. Since P is a clear ideal of $\mathcal{T}_1$, there exist $u \in U(\mathcal{T}_1)$ and $v \in U_{\text{reg}}(\mathcal{T}_1)$ such that $x = u +_1 v$. Hence,

$$\zeta(x) = \zeta(u +_1 v) = \zeta(u) +_2 \zeta(v).$$

Clearly, $\zeta(u) \in U(\zeta(\mathcal{T}_1))$ since $\zeta(u) \cdot_2 \zeta(t) = \zeta(u \cdot_1 t) = \zeta(1_{\mathcal{T}_1}) = 1_{\zeta(\mathcal{T}_1)}$ for some $t \in \mathcal{T}_1$ and $\zeta(v) \in U_{\text{reg}}(\zeta(\mathcal{T}_1))$ since $\zeta(v) \cdot_2 \zeta(y) \cdot_2 \zeta(v) = \zeta(v \cdot_1 y \cdot_1 v) = \zeta(v)$. Therefore, $\zeta(P)$ is a clear ideal of $\zeta(\mathcal{T}_1)$. □

In general, the converse of Proposition 2.12 does not hold. As a counterexample, consider the rings $(\mathbb{Z}, +, \cdot)$ and $(\mathbb{Z}_6, +_6, \cdot_6)$. Define a ring homomorphism

$$\begin{aligned} \zeta : \mathbb{Z} &\longrightarrow \mathbb{Z}_6 \times \mathbb{Z} \\ z &\longmapsto (z \bmod 6, 0), \quad \text{for every } z \in \mathbb{Z}. \end{aligned}$$

Consider the ideal $3\mathbb{Z}$ of $\mathbb{Z}$. By the definition of ζ, we have

$$\zeta(3\mathbb{Z}) = \{(\overline{0}, 0), (\overline{3}, 0)\} \quad \text{and} \quad \zeta(\mathbb{Z}) = \mathbb{Z}_6 \times \mathbb{Z}.$$

Observe that

$$(\overline{0},0) = (\overline{1},0) \oplus (\overline{5},0) \quad \text{and} \quad (\overline{3},0) = (\overline{1},0) \oplus (\overline{2},0),$$

where $(\overline{1},0) \in U(\zeta(\mathbb{Z}))$ and $(\overline{2},0), (\overline{5},0) \in U_{\text{reg}}(\zeta(\mathbb{Z}))$. This shows that $\zeta(3\mathbb{Z})$ is a clear ideal of $\zeta(\mathbb{Z})$, whereas $3\mathbb{Z}$ is **not** a clear ideal of $\mathbb{Z}$.

Corollary 2.13. *Let $(\mathcal{T}_1, +_1, \cdot_1)$ and $(\mathcal{T}_2, +_2, \cdot_2)$ be rings, and let $\zeta : \mathcal{T}_1 \to \mathcal{T}_2$ be a ring epimorphism. Suppose P is an ideal of $\mathcal{T}_1$. If P is clear in $\mathcal{T}_1$, then the image $\zeta(P)$ is a clear ideal of $\mathcal{T}_2$.*

Proof. Let $(\mathcal{T}_1, +_1, \cdot_1)$ and $(\mathcal{T}_2, +_2, \cdot_2)$ be rings, $\zeta : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ be an epimorphism, and P is a clear ideal of $\mathcal{T}_1$. Consequently, $\zeta(\mathcal{T}_1) = \mathcal{T}_2$. Applying Proposition 2.12, we conclude that $\zeta(P)$ is a clear ideal of $\mathcal{T}_2$. □

However, the converse of Corollary 2.13 does not hold. As a counterexample, consider the rings $(\mathbb{Z}, +, \cdot)$ and $(\mathbb{Z}_6, +_6, \cdot_6)$. Define the ring epimorphism

$$\begin{aligned} \zeta : \mathbb{Z} &\longrightarrow \mathbb{Z}_6 \\ z &\longmapsto z \bmod 6, \quad \text{for every } z \in \mathbb{Z}. \end{aligned}$$

Consider the ideal $3\mathbb{Z}$ of $\mathbb{Z}$. By the definition of ζ, we obtain $\zeta(3\mathbb{Z}) = \{\overline{0}, \overline{3}\}$. Observe that

$$\overline{0} = \overline{1} +_6 \overline{5} \quad \text{and} \quad \overline{3} = \overline{1} +_6 \overline{2},$$

where $\overline{1} \in U(\mathbb{Z}_6)$ and $\overline{2}, \overline{5} \in U_{\text{reg}}(\mathbb{Z}_6)$. Therefore, $\zeta(3\mathbb{Z})$ is a clear ideal of $\mathbb{Z}_6$, while $3\mathbb{Z}$ is **not** a clear ideal of $\mathbb{Z}$.

Proposition 2.14. *Let $(\mathcal{T}, +, \cdot)$ be a ring, and let P and Q be ideals of $\mathcal{T}$ with $P \subseteq Q$. If Q is a clear ideal of $\mathcal{T}$, then the quotient Q/P is a clear ideal of $\mathcal{T}/P$.*

Proof. Let $(\mathcal{T}, +, \cdot)$ be a ring, and let P and Q be ideals of $\mathcal{T}$ with $P \subseteq Q$. Suppose that Q is a clear ideal of $\mathcal{T}$. Define the canonical ring homomorphism

$$\begin{aligned} \zeta : \mathcal{T} &\longrightarrow \mathcal{T}/P, \\ t &\longmapsto t + P, \quad \text{for every } t \in \mathcal{T}. \end{aligned}$$

The map ζ is a ring epimorphism. Moreover, since Q is a clear ideal of $\mathcal{T}$, it follows from Proposition 2.12 that $\zeta(Q)$ is a clear ideal of $\mathcal{T}/P$. Moreover, we have $\zeta(Q) = Q/P$. Therefore, Q/P is a clear ideal of $\mathcal{T}/P$. □

However, the converse of Proposition 2.14 does not hold. As a counterexample, consider the ring $(\mathbb{Z}, +, \cdot)$ and the ideals $P = 6\mathbb{Z}$ and $Q = 3\mathbb{Z}$. Clearly, $P \subseteq Q$. Consequently,

$$Q/P = 3\mathbb{Z}/6\mathbb{Z} = \{\overline{0}, \overline{3}\} = \{0 + 6\mathbb{Z}, 3 + 6\mathbb{Z}\},$$

which is a clear ideal in $\mathbb{Z}/6\mathbb{Z}$. However, $Q = 3\mathbb{Z}$ is **not** a clear ideal in $\mathbb{Z}$.

In the following result, we show that the intersection of any nonempty family of clear ideals is also a clear ideal.

Proposition 2.15. *Let $(\mathcal{T}, +, \cdot)$ be a ring and let $\{P_\gamma\}_{\gamma \in \Gamma}$ be a family of ideals of $\mathcal{T}$, where $\Gamma \neq \emptyset$. If every P_γ is a clear ideal of $\mathcal{T}$, then $\bigcap_{\gamma \in \Gamma} P_\gamma$ is a clear ideal of the ring $\mathcal{T}$.*

Proof. By assumption and Remark 2.2, we have $P_\gamma \subseteq Clr(\mathcal{T})$ for every $\gamma \in \Gamma$. It is clear that the intersection of any non-empty family of ideals is an ideal of $\mathcal{T}$. Moreover, since $P_\gamma \subseteq Clr(\mathcal{T})$ for every $\gamma \in \Gamma$, we have

$$\bigcap_{\gamma\in\Gamma} P_\gamma \subseteq Clr(\mathcal{T}).$$

Therefore, by Remark 2.2, $\bigcap_{\gamma\in\Gamma} P_\gamma$ is a clear ideal of $\mathcal{T}$. □

Since the union of ideals is not always an ideal, the union of clear ideals may not be clear. The next result establishes a condition ensuring that the union of clear ideals is also a clear ideal.

Proposition 2.16. *Let $(\mathcal{T}, +, \cdot)$ be a ring and $\{P_\gamma\}_{\gamma\in\Gamma}$ be a family of ideals of $\mathcal{T}$ satisfying the ascending chain condition. If every P_γ is a clear ideal of $\mathcal{T}$, then $\bigcup_{\gamma\in\Gamma} P_\gamma$ is a clear ideal of the ring $\mathcal{T}$.*

Proof. Let $P = \bigcup_{\gamma\in\Gamma} P_\gamma$. It is clear that P is an ideal of $\mathcal{T}$. Next, we show that P is a clear ideal of $\mathcal{T}$. Let $x \in P$. By the definition of the union, there exists some index $\lambda \in \Gamma$ such that $x \in P_\lambda$. Because P_λ is a clear ideal of $\mathcal{T}$ by assumption, there exist $u \in U(\mathcal{T})$ and $v \in U_{\text{reg}}(\mathcal{T})$ such that $x = u + v$. Since this holds for any $x \in P$, the ideal $P = \bigcup_{\gamma\in\Gamma} P_\gamma$ is a clear ideal of $\mathcal{T}$. □

Next, the following proposition presents a property of clear ideals in an opposite ring.

Proposition 2.17. *Let $(\mathcal{T}, +, \cdot)$ be a ring and P an ideal of $\mathcal{T}$. Then P is a clear ideal of $\mathcal{T}$ if and only if P is a clear ideal of $\mathcal{T}^{\text{op}}$.*

Proof. Recall that the opposite ring $\mathcal{T}^{\text{op}}$ has the same underlying additive group as $\mathcal{T}$, but with multiplication reversed; that is, for all $t_1, t_2 \in \mathcal{T}$,

$$t_1 \cdot_{\text{op}} t_2 = t_2 \cdot t_1.$$

Consequently, the set of two-sided ideals of $\mathcal{T}$ coincides with that of $\mathcal{T}^{\text{op}}$, since both share the same additive structure and the definition of a two-sided ideal is symmetric with respect to multiplication.

Before proving the main result, we first note two basic facts regarding units and unit-regular elements:

(1) $U(\mathcal{T}) = U(\mathcal{T}^{\text{op}})$. Indeed, if $u \in U(\mathcal{T})$ has inverse t (i.e., $u \cdot t = t \cdot u = 1_\mathcal{T}$), then in $\mathcal{T}^{\text{op}}$ we have

$$u \cdot_{\text{op}} t = t \cdot u = 1_\mathcal{T}, \quad t \cdot_{\text{op}} u = u \cdot t = 1_\mathcal{T},$$

showing that u is also invertible in $\mathcal{T}^{\text{op}}$. The converse follows similarly.

(2) $U_{\text{reg}}(\mathcal{T}) = U_{\text{reg}}(\mathcal{T}^{\text{op}})$. Recall that $v \in U_{\text{reg}}(\mathcal{T})$ if there exists $y \in U(\mathcal{T})$ such that $v \cdot y \cdot v = v$. In $\mathcal{T}^{\text{op}}$, this condition reads

$$v \cdot_{\text{op}} y \cdot_{\text{op}} v = v \cdot y \cdot v = v,$$

which is equivalent. Hence, an element is unit-regular in $\mathcal{T}$ if and only if it is unit-regular in $\mathcal{T}^{\mathrm{op}}$.

We now prove the equivalence of clarity for P in $\mathcal{T}$ and $\mathcal{T}^{\mathrm{op}}$.

($\Longrightarrow$) Suppose that P is a clear ideal of $\mathcal{T}$. By definition, for each $x \in P$, there exist elements $u \in U(\mathcal{T})$ and $v \in U_{\mathrm{reg}}(\mathcal{T})$ such that

$$x = u + v.$$

From the facts established above, u is also a unit in $\mathcal{T}^{\mathrm{op}}$ and v is unit-regular in $\mathcal{T}^{\mathrm{op}}$. Therefore, the same decomposition shows that x can be written as a sum of a unit and a unit-regular element in $\mathcal{T}^{\mathrm{op}}$, implying that P is a clear ideal of $\mathcal{T}^{\mathrm{op}}$.

($\Longleftarrow$) Conversely, suppose that P is a clear ideal of $\mathcal{T}^{\mathrm{op}}$. Then for each $x \in P$, there exist $u \in U(\mathcal{T}^{\mathrm{op}})$ and $v \in U_{\mathrm{reg}}(\mathcal{T}^{\mathrm{op}})$ such that

$$x = u + v.$$

Using the same reasoning as above, u is a unit in $\mathcal{T}$ and v is unit-regular in $\mathcal{T}$. Hence, every element $x \in P$ decomposes as a sum of a unit and a unit-regular element in $\mathcal{T}$, showing that P is a clear ideal of $\mathcal{T}$.

Hence, we conclude that P is a clear ideal of $\mathcal{T}$ if and only if it is a clear ideal of $\mathcal{T}^{\mathrm{op}}$. □

Next, we present a proposition showing that the direct product of clear ideals is also a clear ideal.

Proposition 2.18. *Let $\{(\mathcal{T}_\gamma, +_\gamma, \cdot_\gamma)\}_{\gamma\in\Gamma}$ be a family of rings, and let P_γ be an ideal of $\mathcal{T}_\gamma$ for each $\gamma \in \Gamma$. Then the direct product $\prod_{\gamma\in\Gamma} P_\gamma$ is a clear ideal of $\prod_{\gamma\in\Gamma} \mathcal{T}_\gamma$ if and only if each P_γ is a clear ideal of $\mathcal{T}_\gamma$.*

Proof. Let $(\mathcal{T}_\gamma, +_\gamma, \cdot_\gamma)$ be a ring and P_γ an ideal of $\mathcal{T}_\gamma$ for every $\gamma \in \Gamma$.

($\Longrightarrow$) Suppose that $\prod_{\gamma\in\Gamma} P_\gamma$ is a clear ideal of $\prod_{\gamma\in\Gamma} \mathcal{T}_\gamma$. We show that every P_γ is a clear ideal of $\mathcal{T}_\gamma$.

For every $\gamma \in \Gamma$, define the projection map

$$\zeta_\gamma : \prod_{\gamma\in\Gamma} \mathcal{T}_\gamma \longrightarrow \mathcal{T}_\gamma$$
$$(r_\gamma)_{\gamma\in\Gamma} \longmapsto r_\gamma, \quad \text{for every } (r_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} \mathcal{T}_\gamma.$$

Clearly, the canonical projection $\zeta_\gamma : \prod_{\gamma\in\Gamma} \mathcal{T}_\gamma \to \mathcal{T}_\gamma$ is a ring epimorphism. Furthermore, since $\prod_{\gamma\in\Gamma} P_\gamma$ is a clear ideal of $\prod_{\gamma\in\Gamma} \mathcal{T}_\gamma$ and satisfies $\zeta_\gamma\left(\prod_{\gamma\in\Gamma} P_\gamma\right) = P_\gamma$, Proposition 2.12 implies that P_γ is a clear ideal of $\mathcal{T}_\gamma$.

($\Longleftarrow$) Suppose that every P_γ is a clear ideal of $\mathcal{T}_\gamma$. We show that $\prod_{\gamma\in\Gamma} P_\gamma$ is a clear ideal of $\prod_{\gamma\in\Gamma} \mathcal{T}_\gamma$.

Let any $(x_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} P_\gamma$. Since every P_γ is clear, for every $\gamma \in \Gamma$ there exist $u_\gamma \in U(\mathcal{T}_\gamma)$ and $v_\gamma \in U_{\text{reg}}(\mathcal{T}_\gamma)$ such that

$$x_\gamma = u_\gamma +_\gamma v_\gamma.$$

Then

$$(x_\gamma)_{\gamma\in\Gamma} = (u_\gamma +_\gamma v_\gamma)_{\gamma\in\Gamma} = (u_\gamma)_{\gamma\in\Gamma} + (v_\gamma)_{\gamma\in\Gamma}.$$

Since $u_\gamma \in U(\mathcal{T}_\gamma)$, there exists $t_\gamma \in \mathcal{T}_\gamma$ such that $u_\gamma \cdot_\gamma t_\gamma = t_\gamma \cdot_\gamma u_\gamma = 1_{\mathcal{T}_\gamma}$. Hence,

$$(u_\gamma)_{\gamma\in\Gamma} \cdot (t_\gamma)_{\gamma\in\Gamma} = (u_\gamma \cdot_\gamma t_\gamma)_{\gamma\in\Gamma} = (1_{\mathcal{T}_\gamma})_{\gamma\in\Gamma},$$

showing that $(u_\gamma)_{\gamma\in\Gamma} \in U\left(\prod_{\gamma\in\Gamma} \mathcal{T}_\gamma\right)$. Similarly, since $v_\gamma \in U_{\text{reg}}(\mathcal{T}_\gamma)$, there exists $y_\gamma \in U(\mathcal{T}_\gamma)$ such that $v_\gamma \cdot_\gamma y_\gamma \cdot_\gamma v_\gamma = v_\gamma$. Therefore,

$$(v_\gamma)_{\gamma\in\Gamma} \cdot (y_\gamma)_{\gamma\in\Gamma} \cdot (v_\gamma)_{\gamma\in\Gamma} = (v_\gamma \cdot_\gamma y_\gamma \cdot_\gamma v_\gamma)_{\gamma\in\Gamma} = (v_\gamma)_{\gamma\in\Gamma},$$

which shows that $(v_\gamma)_{\gamma\in\Gamma} \in U_{\text{reg}}\left(\prod_{\gamma\in\Gamma} \mathcal{T}_\gamma\right)$. Hence, $(x_\gamma)_{\gamma\in\Gamma} = (u_\gamma)_{\gamma\in\Gamma} + (v_\gamma)_{\gamma\in\Gamma}$ with $(u_\gamma)_{\gamma\in\Gamma} \in U\left(\prod_{\gamma\in\Gamma} \mathcal{T}_\gamma\right)$ and $(v_\gamma)_{\gamma\in\Gamma} \in U_{\text{reg}}\left(\prod_{\gamma\in\Gamma} \mathcal{T}_\gamma\right)$, proving that $\prod_{\gamma\in\Gamma} P_\gamma$ is a clear ideal of $\prod_{\gamma\in\Gamma} \mathcal{T}_\gamma$.

Therefore, $\prod_{\gamma\in\Gamma} P_\gamma$ is a clear ideal of $\prod_{\gamma\in\Gamma} \mathcal{T}_\gamma$ if and only if every P_γ is a clear ideal of $\mathcal{T}_\gamma$ for every $\gamma \in \Gamma$. □

We next present a necessary and sufficient condition for the ideal $D_{n\times n}(P)$ in $D_{n\times n}(\mathcal{T})$ to be clear, assuming that P is a clear ideal of $\mathcal{T}$.

Proposition 2.19. *Let $(\mathcal{T}, +, \cdot)$ be a ring and P an ideal of $\mathcal{T}$. Then the ideal $D_{n\times n}(P)$ of the diagonal matrix ring $D_{n\times n}(\mathcal{T})$ is clear if and only if P is a clear ideal of $\mathcal{T}$.*

Proof. Let $(\mathcal{T}, +, \cdot)$ be a ring and P an ideal of $\mathcal{T}$.

($\Longrightarrow$) Assume that $D_{n\times n}(P)$ is a clear ideal of $D_{n\times n}(\mathcal{T})$. We show that P is a clear ideal of $\mathcal{T}$. Define the map

$$\begin{aligned} \zeta : D_{n\times n}(\mathcal{T}) &\longrightarrow \mathcal{T} \\ \text{diag}(t_1, \ldots, t_n) &\longmapsto t_1, \quad \text{for every } \text{diag}(t_1, \ldots, t_n) \in D_{n\times n}(\mathcal{T}). \end{aligned}$$

It is clear that ζ is a ring epimorphism. Furthermore, $\zeta(D_{n\times n}(P)) = P$. Since $D_{n\times n}(P)$ is a clear ideal of $D_{n\times n}(\mathcal{T})$, Proposition 2.12 implies that P is a clear ideal of $\mathcal{T}$.

($\Longleftarrow$) Assume that P is a clear ideal of $\mathcal{T}$. We show that $D_{n\times n}(P)$ is a clear ideal of $D_{n\times n}(\mathcal{T})$.

Let $\operatorname{diag}(x_1,\dots,x_n) \in D_{n\times n}(P)$, so that $x_i \in P$ for every $i = 1,\dots,n$. Since P is clear, $x_i = u_i + v_i$ with $u_i \in U(\mathcal{T})$ and $v_i \in U_{\mathrm{reg}}(\mathcal{T})$. Since $u_i \in U(\mathcal{T})$, then there exist $t_i \in U(\mathcal{T})$ such that $u_i t_i = t_i u_i = 1_{\mathcal{T}}$ and since $v_i \in U_{\mathrm{reg}}(\mathcal{T})$, then there exist $y_i \in U(\mathcal{T})$ such that $v_i y_i v_i = v_i$. Consider the decomposition:

$$\operatorname{diag}(x_1,\dots,x_n) = \operatorname{diag}(u_1,\dots,u_n) + \operatorname{diag}(v_1,\dots,v_n).$$

It is clear that $\operatorname{diag}(u_1,\dots,u_n) \in U(D_{n\times n}(\mathcal{T}))$ because

$$\operatorname{diag}(u_1,\dots,u_n)\operatorname{diag}(t_1,\dots,t_n) = \operatorname{diag}(1_{\mathcal{T}},\dots,1_{\mathcal{T}}) = \operatorname{diag}(t_1,\dots,t_n)\operatorname{diag}(u_1,\dots,u_n),$$

and $\operatorname{diag}(v_1,\dots,v_n) \in U_{\mathrm{reg}}(D_{n\times n}(\mathcal{T}))$ because

$$\operatorname{diag}(v_1,\dots,v_n)\operatorname{diag}(y_1,\dots,y_n)\operatorname{diag}(v_1,\dots,v_n) = \operatorname{diag}(v_1y_1v_1,\dots,v_ny_nv_n) = \operatorname{diag}(v_1,\dots,v_n).$$

Hence, $D_{n\times n}(P)$ is a clear ideal of $D_{n\times n}(\mathcal{T})$.

□

According to the result in [4], the cleanness of an ideal P in $\mathcal{T}$ guarantees that $M_{n\times n}(P)$ is a clean ideal of the matrix ring $M_{n\times n}(\mathcal{T})$. Based on this, we can establish a sufficient condition for $M_{n\times n}(P)$ to be a clear ideal of $M_{n\times n}(\mathcal{T})$.

Proposition 2.20. *Let $(\mathcal{T},+,\cdot)$ be a ring, and let P be an ideal of $\mathcal{T}$. If P is clean in $\mathcal{T}$, then the ideal $M_{n\times n}(P)$ of the full matrix ring $M_{n\times n}(\mathcal{T})$ is a clear ideal.*

Proof. Given that P is clean in $\mathcal{T}$, we deduce that $M_{n\times n}(P)$ is a clean ideal of $M_{n\times n}(\mathcal{T})$, which implies that it is clear. □

In the following result, we establish a characterization for an ideal of a Morita ring to be clear, based on the clearness of the ideals in its component rings.

Proposition 2.21. *Let $\Lambda_{(0,0)} = \begin{bmatrix} \mathcal{T} & B \\ C & \mathcal{S} \end{bmatrix}$ be a Morita ring and P be an ideal of $\mathcal{T}$ and Q an ideal of $\mathcal{S}$. Then P and Q are clear ideals of $\mathcal{T}$ and $\mathcal{S}$, respectively, if and only if $\begin{bmatrix} P & B \\ C & Q \end{bmatrix}$ is a clear ideal of $\Lambda_{(0,0)}$.*

Proof. Let $\Lambda_{(0,0)} = \begin{bmatrix} \mathcal{T} & B \\ C & \mathcal{S} \end{bmatrix}$ be a Morita ring, and let P be an ideal of $\mathcal{T}$ and Q an ideal of S.

($\Longrightarrow$) Suppose P and Q are clear ideals of $\mathcal{T}$ and $\mathcal{S}$, respectively. We show that $\begin{bmatrix} P & B \\ C & Q \end{bmatrix}$ is a clear ideal of $\Lambda_{(0,0)}$. Let an arbitrary $\begin{bmatrix} r & b \\ c & s \end{bmatrix} \in \begin{bmatrix} P & B \\ C & Q \end{bmatrix}$, so that $r \in P$, $b \in B$, $c \in C$, and $s \in Q$. Since P and Q are clear ideals,

we have $r = u_1 + v_1$ and $s = u_2 + v_2$ for some $u_1 \in U(\mathcal{T})$, $v_1 \in U_{\text{reg}}(\mathcal{T})$, $u_2 \in U(\mathcal{S})$, and $v_2 \in U_{\text{reg}}(\mathcal{S})$. Consider the decomposition:

$$\begin{bmatrix} r & b \\ c & s \end{bmatrix} = \begin{bmatrix} u_1 & b \\ c & u_2 \end{bmatrix} + \begin{bmatrix} v_1 & 0 \\ 0 & v_2 \end{bmatrix}.$$

It is straightforward to verify that

$$\begin{bmatrix} u_1 & b \\ c & u_2 \end{bmatrix} \begin{bmatrix} t_1 & -t_1 b t_2 \\ -t_2 c t_1 & t_2 \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{T}} & 0 \\ 0 & 1_{\mathcal{S}} \end{bmatrix} = \begin{bmatrix} t_1 & -t_1 b t_2 \\ -t_2 c t_1 & t_2 \end{bmatrix} \begin{bmatrix} u_1 & b \\ c & u_2 \end{bmatrix},$$

and

$$\begin{bmatrix} v_1 & 0 \\ 0 & v_2 \end{bmatrix} \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix} \begin{bmatrix} v_1 & 0 \\ 0 & v_2 \end{bmatrix} = \begin{bmatrix} v_1 & 0 \\ 0 & v_2 \end{bmatrix}.$$

Therefore, $\begin{bmatrix} u_1 & b \\ c & u_2 \end{bmatrix} \in U(\Lambda_{(0,0)})$ and $\begin{bmatrix} v_1 & 0 \\ 0 & v_2 \end{bmatrix} \in U_{\text{reg}}(\Lambda_{(0,0)})$, showing that $\begin{bmatrix} P & B \\ C & Q \end{bmatrix}$ is a clear ideal of $\Lambda_{(0,0)}$.

($\Longleftarrow$) Suppose $\begin{bmatrix} P & B \\ C & Q \end{bmatrix}$ is a clear ideal of $\Lambda_{(0,0)}$. Define the projections

$$\zeta : \Lambda_{(0,0)} \longrightarrow \mathcal{T},$$
$$\begin{bmatrix} r & b \\ c & s \end{bmatrix} \longmapsto r, \quad \text{for every } \begin{bmatrix} r & b \\ c & s \end{bmatrix} \in \Lambda_{(0,0)}$$

and

$$\eta : \Lambda_{(0,0)} \longrightarrow \mathcal{S},$$
$$\begin{bmatrix} r & b \\ c & s \end{bmatrix} \longmapsto s, \quad \text{for every } \begin{bmatrix} r & b \\ c & s \end{bmatrix} \in \Lambda_{(0,0)}.$$

Clearly, ζ and η are ring epimorphisms. Moreover,

$$\zeta\left(\begin{bmatrix} P & B \\ C & Q \end{bmatrix}\right) = P \text{ and } \eta\left(\begin{bmatrix} P & B \\ C & Q \end{bmatrix}\right) = Q.$$

Since $\begin{bmatrix} P & B \\ C & Q \end{bmatrix}$ is a clear ideal of $\Lambda_{(0,0)}$, it follows from Proposition 2.12 that P and Q are clear ideals of $\mathcal{T}$ and $\mathcal{S}$, respectively.

□

Assume that P is an ideal of a ring $\mathcal{T}$ and e a central idempotent in $\mathcal{T}$. The ideal $e\mathcal{T}e$ can then be constructed. We now present a necessary condition for P to be a clear ideal of $\mathcal{T}$.

Proposition 2.22. *Let $(\mathcal{T}, +, \cdot)$ be a ring and let P be an ideal of $\mathcal{T}$. If P is a clear ideal of $\mathcal{T}$ and e is a central idempotent in $\mathcal{T}$, then ePe is a clear ideal of $e\mathcal{T}e$.*

Proof. Let $(\mathcal{T}, +, \cdot)$ be a ring, let P be a clear ideal of $\mathcal{T}$, and let $e \in \mathcal{T}$ be a central idempotent. Define the map $\zeta : \mathcal{T} \longrightarrow e\mathcal{T}e$ by $\zeta(t) = ete$ for all $t \in \mathcal{T}$. Then ζ is a ring epimorphism. Since P is a clear ideal of $\mathcal{T}$ and $\zeta(P) = ePe$, it follows from Proposition 2.12 that ePe is a clear ideal of $e\mathcal{T}e$. □

In the following result, we present a characterization that provides a necessary and sufficient condition for the ideal $P[[x]]$ of the formal power series ring $\mathcal{T}[[x]]$ to be clear, in terms of the ideal P of $\mathcal{T}$.

Proposition 2.23. *Let $(\mathcal{T}, +, \cdot)$ be a ring and P an ideal of $\mathcal{T}$. Then P is clear in $\mathcal{T}$ if and only if the ideal $P[[x]]$ is clear in the formal power series ring $\mathcal{T}[[x]]$.*

Proof. Let $(\mathcal{T}, +, \cdot)$ be a ring and P an ideal of $\mathcal{T}$.

($\Longrightarrow$) Suppose P is a clear ideal of $\mathcal{T}$. We show that $P[[x]]$ is a clear ideal of $\mathcal{T}[[x]]$. Let an arbitrary element $f(x) = i_0 + i_1x + i_2x^2 + \cdots \in P[[x]]$. Then $i_0, i_1, i_2, \ldots \in P$. Since P is clear in $\mathcal{T}$, we can write $i_0 = u_0 + v_0$ with $u_0 \in U(\mathcal{T})$ and $v_0 \in U_{\text{reg}}(\mathcal{T})$. It follows that

$$f(x) = v_0 + (u_0 + i_1x + i_2x^2 + \cdots).$$

Clearly, $v_0 \in U_{\text{reg}}(\mathcal{T}[[x]])$ and $(u_0 + i_1x + i_2x^2 + \cdots) \in U(\mathcal{T}[[x]])$. Hence, $P[[x]]$ is a clear ideal of $\mathcal{T}[[x]]$.

($\Longleftarrow$) Suppose $P[[x]]$ is a clear ideal of $\mathcal{T}[[x]]$. Define the map

$$\zeta : \mathcal{T}[[x]] \longrightarrow \mathcal{T},$$
$$\sum_{n=0}^{\infty} a_n x^n \longmapsto a_0, \quad \text{for every } \sum_{n=0}^{\infty} a_n x^n \in \mathcal{T}[[x]].$$

It is clear that ζ is a ring epimorphism. Moreover, we have $\zeta(P[[x]]) = P$. Since $P[[x]]$ is a clear ideal of $\mathcal{T}[[x]]$, by using Proposition 2.12, we conclude that P is a clear ideal of $\mathcal{T}$.

□

In commutative algebra, localization is a standard construction that introduces denominators into a ring. It is therefore natural to investigate whether the property of being a clear ideal is preserved under localization. Let $\mathcal{T}$ be a commutative ring and let S be a multiplicatively closed subset of $\mathcal{T}$ containing $1_{\mathcal{T}}$ but not $0_{\mathcal{T}}$. The localization of $\mathcal{T}$ with respect to S is denoted by $S^{-1}\mathcal{T}$, while the localization of an ideal P is denoted by $S^{-1}P$.

Proposition 2.24. *Let $(\mathcal{T}, +, \cdot)$ be a commutative ring and S be a multiplicatively closed subset of $\mathcal{T}$. If P is a clear ideal of $\mathcal{T}$, then the localization $S^{-1}P$ is a clear ideal of the localized ring $S^{-1}\mathcal{T}$.*

Proof. Let $\frac{x}{s} \in S^{-1}P$, where $x \in P$ and $s \in S$. Since P is a clear ideal of $\mathcal{T}$, there exist $u \in U(\mathcal{T})$ and $v \in U_{\text{reg}}(\mathcal{T})$ such that $x = u + v$. Hence, in $S^{-1}\mathcal{T}$, $\frac{x}{s} = \frac{u}{s} + \frac{v}{s}$. Since $u \in U(\mathcal{T})$, there exists $w \in \mathcal{T}$ such that $uw = 1_{\mathcal{T}}$. Thus, $\frac{u}{s} \cdot \frac{ws}{1} = \frac{uws}{s} = \frac{1_{\mathcal{T}}s}{s} = 1_{S^{-1}\mathcal{T}}$. Therefore, $\frac{u}{s}$ is a unit in $S^{-1}\mathcal{T}$. Similarly, since $v \in U_{\text{reg}}(\mathcal{T})$, there exists $y \in U(\mathcal{T})$ such that $vyv = v$. Since $y \in U(\mathcal{T})$, we have $\frac{ys}{1} \in U(S^{-1}\mathcal{T})$. Moreover, $\frac{v}{s} \cdot \frac{ys}{1} \cdot \frac{v}{s} = \frac{vysv}{s^2} = \frac{v}{s}$. Thus, $\frac{v}{s} \in U_{\text{reg}}(S^{-1}\mathcal{T})$. Therefore, $\frac{x}{s}$ is the sum of a unit and a unit-regular element in $S^{-1}\mathcal{T}$. Hence, $S^{-1}P$ is a clear ideal of $S^{-1}\mathcal{T}$. □

Next, we consider the concept of idealization, also known as Nagata's trivial extension. Let $\mathcal{T}$ be a ring and M a $\mathcal{T}$-bimodule. The trivial extension of $\mathcal{T}$ by M, denoted by $\mathcal{T} \propto M$, is the set $\mathcal{T} \times M$ equipped with componentwise addition and multiplication defined by

$$(t_1, m_1)(t_2, m_2) = (t_1 t_2, t_1 m_2 + m_1 t_2)$$

for all $t_1, t_2 \in \mathcal{T}$ and $m_1, m_2 \in M$. The identity element of this ring is $(1_\mathcal{T}, 0_M)$. We show that the trivial extension of a clear ideal is again a clear ideal in the extended ring.

Proposition 2.25. *Let $(\mathcal{T}, +, \cdot)$ be a ring, M be a $\mathcal{T}$-bimodule, and P be an ideal of $\mathcal{T}$. If P is a clear ideal of $\mathcal{T}$, then the ideal $P \propto M$ is a clear ideal of the trivial extension ring $\mathcal{T} \propto M$.*

Proof. Let (p, m) be an arbitrary element in $P \propto M$. Since $p \in P$ and P is a clear ideal, there exist elements $u \in U(\mathcal{T})$ and $v \in U_{\text{reg}}(\mathcal{T})$ such that $p = u + v$. We can decompose the element (p, m) in the trivial extension ring as:

$$(p, m) = (u + v, m) = (u, m) + (v, 0_M).$$

First, we prove that $(u, m) \in U(\mathcal{T} \propto M)$. Since $u \in U(\mathcal{T})$, it has a multiplicative inverse $u^{-1} \in \mathcal{T}$. We construct the candidate inverse $(u^{-1}, -u^{-1}mu^{-1})$ and verify it:

$$\begin{aligned}(u, m)(u^{-1}, -u^{-1}mu^{-1}) &= (uu^{-1}, u(-u^{-1}mu^{-1}) + mu^{-1}) \\ &= (1_\mathcal{T}, -mu^{-1} + mu^{-1}) \\ &= (1_\mathcal{T}, 0_M).\end{aligned}$$

a similar calculation holds for left multiplication. Therefore, (u, m) is a unit in $\mathcal{T} \propto M$. Second, we prove that $(v, 0_M) \in U_{\text{reg}}(\mathcal{T} \propto M)$. Since $v \in U_{\text{reg}}(\mathcal{T})$, there exists a unit $y \in U(\mathcal{T})$ such that $vyv = v$. Let us consider the element $(y, 0_M) \in \mathcal{T} \propto M$. As established previously, because $y \in U(\mathcal{T})$, the element $(y, 0_M)$ is a unit in $\mathcal{T} \propto M$ with inverse $(y^{-1}, 0_M)$. We evaluate the unit-regular condition:

$$(v, 0_M)(y, 0_M)(v, 0_M) = (vy, v \cdot 0_M + 0_M \cdot y)(v, 0_M) = (vy, 0_M)(v, 0_M) = (vyv, 0_M).$$

Since $vyv = v$, this simplifies to:

$$(v, 0_M)(y, 0_M)(v, 0_M) = (v, 0_M).$$

This demonstrates that $(v, 0_M)$ is a unit-regular element in $\mathcal{T} \propto M$. Because every element $(p, m) \in P \propto M$ can be written as the sum of a unit (u, m) and a unit-regular element $(v, 0_M)$, the ideal $P \propto M$ is a clear ideal of $\mathcal{T} \propto M$. □

3. Conclusions

In this paper, we introduced and systematically investigated the concept of a clear ideal, establishing it as a natural generalization of clear rings and a related extension of clean ideals. We provide an alternative characterization of clear ideals and clarified the relationship between cleanliness and clearness by showing that all clean ideals are clear, while the converse does not hold in general.

Furthermore, we established a connection with radical theory by proving that any ideal contained in the Jacobson radical (and hence the nilradical) is clear. We also studied the behavior of clear ideals under various algebraic constructions. In particular, we showed that the clear ideal property is preserved under homomorphic images, quotient structures, direct products, formal power series, localizations, and Nagata's extensions.

Additionally, we characterized clear ideals in matrix-related structures, including diagonal matrix rings and Morita rings. These results provide a systematic framework for understanding the behavior of clear ideals in ring theory.

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