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ON THE DOMINATING SETS OF THE COMPLEMENT OF THE ANNIHILATING IDEAL GRAPH OF A COMMUTATIVE RING

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Abstract. Let R be a commutative ring with identity which is not an integral domain. Let $A(R)^*$ denote the collection of all nonzero annihilating ideals of R and $AG(R)$ denote the annihilating ideal graph of R. In this article, we consider the dominating sets of $(AG(R))^c$(where, $(AG(R))^c$ is the complement of $AG(R)$) and study the influence of the dominating sets of $(AG(R))^c$ on the ring structure of R and vice-versa.

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1. Introduction

The rings considered in this article are commutative with identity and which are not integral domains. The concept of associating a graph with a ring and investigating the structure of the ring under consideration, possessing some graph theoretic properties of the graph associated with them, was initiated by I. Beck in [8]. In [8], I. Beck was mainly interested in colorings. Subsequently, several researchers associated graphs with algebraic structures and studied the interplay between the algebraic structure of the objects under consideration and their graph theoretic properties. The concept of zero-divisor graph of a ring was introduced and studied by D..F. Anderson and P.S. Livingston in [5]. Recall from [5] that the zero-divisor graph of a ring R, denoted by $\Gamma(R)$, is an undirected graph whose vertex set is $Z(R)^*$ (where, $Z(R)$ is the set of all zero-divisors of R and $Z(R)^* = Z(R)\backslash\{0\}$) and two distinct vertices x, y are joined by an edge in this graph if and only if $xy = 0$. A lot of research work was carried out by several mathematicians in the last two decades on the zero-divisor graphs of commutative rings and for an inspiring and excellent survey on the research work done in the area of zero-divisor graphs of rings, the reader is referred to [4].

Let R be a ring. Recall from [9] that an ideal I of R is said to be an annihilating ideal if $Ir = (0)$ for some $r \in R\backslash\{0\}$. As in [9], we denote by $A(R)$, the set of

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all annihilating ideals of R. The concept of the annihilating ideal graph of a ring R, denoted by $AG(R)$, was introduced and studied in [9] by M. Behboodi and Z. Rakeei. Recall from [9] that the annihilating ideal graph of a ring R, denoted by $AG(R)$, is an undirected graph whose vertex set is $A(R)^* = A(R)\backslash\{(0)\}$ and two distinct vertices I, J are joined by an edge in this graph if and only if $IJ = (0)$. The interplay between the ring theoretic properties of a ring R and the graph theoretic properties of $AG(R)$ was very well investigated by M. Behboodi and Z. Rakeei in [9, 10]. The annihilating ideal graph of a commutative ring was also considered by several others, to mention a few, refer [1, 2, 14].

The graphs considered in this article are undirected. Let $G = (V, E)$ be a simple graph. Recall from [7, Definition 1.1.13] that the complement of G, denoted by G^c, is a graph whose vertex set is V, and two distinct vertices u, v are joined by an edge in G^c if and only if u and v are nonadjacent in G. Let $G = (V, E)$ be a graph. Recall from [11, p.172] that a subset A of V is called a dominating set in G if A dominates every vertex v in G in the following sense: Either $v \in A$ or v is adjacent in G to one or more members of A. Recall from [11, p.172] that a minimal dominating set is a dominating set from which no vertex can be removed without destroying its dominance property. Suppose that a graph $G = (V, E)$ admits a finite dominating set. The domination number of G, denoted by $\gamma(G)$, is the number of vertices in the smallest minimal dominating set of G. A good amount of work has been done on the dominating sets of the zero-divisor graphs of rings, to mention a few, refer [17, 19]. Motivated by the work of R. Nikandish and H.R. Maimani in [20] on the dominating sets of the annihilating ideal graph of a ring, in this article, we focus our study on the dominating sets of $(AG(R))^c$.

It is useful to recall the following definitions from commutative ring theory. Let I be an ideal of a ring R with $I \neq R$. Recall from [16] that a prime ideal P of R is said to be a maximal N-prime of I, if P is maximal with respect to the property of being contained in $Z_R(R/I) = \{r \in R | rx \in I \text{ for some } x \in R\backslash I\}$. Thus a prime ideal P of R is a maximal N-prime of (0), if P is maximal with respect to the property of being contained in $Z(R)$. Note that $S = R\backslash Z(R)$ is a multiplicatively closed subset of R. Let $x \in Z(R)$. Observe that $Rx \cap S = \emptyset$. Hence it follows from Zorn's Lemma and [18, Theorem 1] that there exists a maximal N-prime P of (0) in R such that $x \in P$. This implies that $Z(R) = \cup_{\alpha\in\Lambda} P_\alpha$, where $\{P_\alpha\}_{\alpha\in\Lambda}$ is the set of all maximal N-primes of (0) in R.

Let I be an ideal of a ring R with $I \neq R$. Recall from [15] that a prime ideal P of R is an associated prime of I in the sense of Bourbaki, if $P = (I :_R x)$ for some $x \in R$. In such a case, we simply say that P is a B-prime of I. Recall that a principal ideal ring is called a special principal ring(SPIR), if R has a unique prime ideal. If M is the unique prime ideal of a SPIR R, then we denote this situation using the notation (R, M) is a SPIR.

Let R be a ring which is not an integral domain. In this article, we study on the dominating sets of $(AG(R))^c$. We also study the relatioship between $\gamma((AG(R))^c)$ and $\gamma((\Gamma(R))^c)$. In Proposition 2.1, we determine rings R such that $\gamma((\Gamma(R))^c) = 1$ (respectively, $\gamma((AG(R))^c) = 1$). Let R be a ring such that $Z(R)$ is not an ideal of R. It is proved in Lemma 2.3 that $\gamma((\Gamma(R))^c) = \gamma((AG(R))^c) = 2$. Let R be a reduced ring. Then it is shown in Lemma 2.5(irrespective of whether or

not $Z(R)$ is an ideal) that $\gamma((AG(R))^c) = 2$. Let R be a ring such that $Z(R)$ is an ideal of R. We provide examples (see, Examples 2.4, 2.14) to illustrate that $(\Gamma(R))^c$ (respectively, $(AG(R))^c$) need not admit any finite dominating set. Let R be a ring such that $Z(R) = P$ is an ideal of R. We determine necessary and sufficient conditions in order that $(\Gamma(R))^c$) (respectively, $(AG(R))^c$)) admits a finite dominating set (see, Remark 2.7, Propositions 2.8 and 2.17) under the hypothesis that P is finitely generated. Also, we provide examples to illustrate the results proved in this article (see, Examples 2.9 and 2.12).

Whenever a set A is a subset of a set B and $A \neq B$, we denote it symbolically using the notation $A \subset B$. A ring R with a unique maximal ideal is referred to as a quasilocal ring. A ring R which admits only a finite number of maximal ideals is referred to as a semiquasilocal ring. A Noetherian quasilocal (respectively, semiquasilocal) ring is referred to as a local (respectively,, a semilocal) ring. For a ring R, we denote the polynomial ring in one variable X over R by $R[X]$. For any positive integer $n \geq 2$, we denote the ring of integers modulo n by $\mathbf{Z}_n$.

2. Main Results

As mentioned in the introduction, the rings considered here are commutative with identity and which are not integral domains.

Proposition 2.1. *Let R be a ring. Then the following hold.*
(i) $\gamma((\Gamma(R))^c) = 1$ if and only if either $R \cong \mathbf{Z}_4$ or $R \cong \mathbf{Z}_2[X]/X^2\mathbf{Z}_2[X]$ as rings.
(ii) $\gamma((AG(R))^c) = 1$ if and only if $(R, Z(R))$ is a SPIR with $(Z(R))^2 = (0)$.

Proof. (i) Suppose that $\gamma((\Gamma(R))^c) = 1$. Let $x \in Z(R)^*$ be such that $\{x\}$ is a dominating set of $(\Gamma(R))^c$. Since $x \in Z(R)$, there exists $y \in R\backslash\{0\}$ such that $xy = 0$. As $\{x\}$ is a dominating set of $(\Gamma(R))^c$, it follows that $y = x$. Thus $x^2 = 0$. For any $r \in R$, $rx \in Z(R)$. It follows from $rx^2 = 0$ that $rx \in \{0, x\}$. This proves that $Rx = \{0, x\}$ is a simple R-module. Hence $M = ((0) :_R x)$ is a maximal ideal of R. From $Mx = (0)$, we obtain that $M = Rx$ is a maximal ideal of R with $M^2 = (0)$. Therefore, $M = Z(R) = \{0, x\}$. This shows that $|Z(R)^*| = 1$. In such a case, it is well-known that either $R \cong \mathbf{Z}_4$ or $R \cong \mathbf{Z}_2[X]/X^2\mathbf{Z}_2[X]$ as rings [5, Example 2.1].

Conversely, if either $R \cong \mathbf{Z}_4$ or $R \cong \mathbf{Z}_2[X]/X^2\mathbf{Z}_2[X]$ as rings, then $(\Gamma(R))^c$ is a graph on a single vertex. Hence $\gamma((\Gamma(R))^c) = 1$.

(ii) Suppose that $\gamma((AG(R))^c) = 1$. Let $I \in A(R)^*$ be such that $\{I\}$ is a dominating set of $(AG(R))^c$. Since $I \in A(R)$, there exists $r \in R\backslash\{0\}$ such that $Ir = (0)$. It follows from $\{I\}$ is a dominating set of $(AG(R))^c$ that $I = Rr$. Hence $I^2 = (0)$. Let $a \in I\backslash\{0\}$. From $Ia = (0)$, it follows that $I = Ra$. This proves that I is a simple R-module. As in (i), it follows that $((0) :_R I) = I = M$ is a maximal ideal of R with $M^2 = (0)$. It is now clear that $M = Z(R)$ and $(R, Z(R))$ is a SPIR with $(Z(R))^2 = (0)$.

Conversely, if $(R, Z(R))$ is a SPIR with $(Z(R))^2 = (0)$, then $(AG(R))^c$ is a graph on the single vertex $Z(R)$ and so $\gamma((AG(R))^c) = 1$. □

Let R be a ring. In Lemma 2.3, we determine, $\gamma((\Gamma(R))^c)$ and $\gamma((AG(R)^c)$ under the assumption that $Z(R)$ is not an ideal of R. We make use of Lemma 2.2 in the proof of Lemma 2.3.

Lemma 2.2. *Let R be a ring. If there exist $I, J \in A(R)^*$ such that $I+J \notin A(R)$, then $\gamma((AG(R))^c) = 2$.*

Proof. From $I + J \notin A(R)$, it is clear that $I \neq J$. Let K be any nonzero ideal of R. It follows from $I + J \notin A(R)$ that either $KI \neq (0)$ or $KJ \neq (0)$. Thus given $K \in A(R)^*$, either $K \in \{I, J\}$ or K is adjacent to at least one between I and J in $(AG(R))^c$. This proves that $\{I, J\}$ is a dominating set of $(AG(R))^c$. Hence $\gamma((AG(R))^c) \leq 2$. It now follows immediately from Proposition 2.1(ii) that $\gamma((AG(R))^c) = 2$. □

Lemma 2.3. *Let R be a ring. If $Z(R)$ is not an ideal of R, then $\gamma((\Gamma(R))^c) = \gamma((AG(R))^c) = 2$.*

Proof. By hypothesis, $Z(R)$ is not an ideal of R. Hence there exist $a, b \in Z(R)$ such that $a+b \notin Z(R)$. Let $x \in R\backslash\{0\}$. Since $a+b \notin Z(R)$, it follows that either $ax \neq 0$ or $bx \neq 0$. Thus for any $x \in Z(R)^*$, either $x \in \{a, b\}$ or x is adjacent to at least one between a and b in $(\Gamma(R))^c$. This shows that $\{a, b\}$ is a dominating set of $(\Gamma(R))^c$. Hence $\gamma((\Gamma(R))^c) \leq 2$. We know from Proposition 2.1(i) that $\gamma((\Gamma(R))^c) \geq 2$. Therefore, $\gamma((\Gamma(R))^c) = 2$.

It is clear that $Ra, Rb \in A(R)^*$. Since $a+b \notin Z(R)$, we obtain that $Ra+Rb \notin A(R)$. Therefore, it follows from Lemma 2.2 that $\gamma((AG(R))^c) = 2$. □

Let R be a ring such that $Z(R)$ is an ideal. Then $Z(R)$ is necessarily a prime ideal of R. If $(Z(R))^2 = (0)$, then $Z(R)^*$ is the only dominating set of $(\Gamma(R))^c$. Moreover, in such a case, $A(R)^*$ is the only dominating set of $(AG(R))^c$. Our aim is to investigate on the dominating sets of $(\Gamma(R))^c$ (respectively, $(AG(R))^c$) under the assumption that $Z(R)$ is an ideal of R. In Example 2.4, we provide an example from [13] which illustrates that $(\Gamma(R))^c$ may not admit any finite dominating set.

Example 2.4. Let $\{X_i\}_{i=1}^{\infty}$ be a set of indeterminates over a field K. Let $D = \cup_{n=1}^{\infty} K[[X_1, \ldots, X_n]]$. Let I be the ideal of D generated by $\{X_iX_j | i, j \in \mathbf{N}, i \neq j\}$. Let $R = D/I$. It was already noted in [13, Example p.16] that R is reduced and is quasilocal with $M =$ the ideal of R generated by $\{X_i + I | i \in \mathbf{N}\}$ as its unique maximal ideal. We denote $X_i + I$ by x_i for each $i \in \mathbf{N}$. We assert that any finitely generated ideal contained in M is an annihilating ideal. Let J be any finitely generated ideal of R such that $J \subseteq M$. Then there exists $n \in \mathbf{N}$ such that $J \subseteq Rx_1 + \cdots + Rx_n$. Since $x_{n+1}x_i = 0 + I$ for each $i \in \{1, \ldots, n\}$, we obtain that $Jx_{n+1} = (0 + I)$. The above assertion clearly implies that $M = Z(R)$ and so $Z(R)$ is an ideal of R. Let $A = \{a_1, \ldots, a_k\} \subseteq Z(R)^*$ be any finite set. Observe that there exists $m \in \mathbf{N}$ such that $Ra_1 + \cdots + Ra_k \subseteq Rx_1 + \cdots + Rx_m$. Hence we get that $a_ix_{m+1} = 0 + I$ for each $i \in \{1, \ldots, k\}$. Since R is reduced, it follows that $x_{m+1} \notin A$. This shows that A is not a dominating set of $(\Gamma(R))^c$.

The ring R mentioned in Example 2.4 is reduced. We prove in Lemma 2.5 that for any reduced ring R, $\gamma((AG(R))^c) = 2$.

Lemma 2.5. *If R is a reduced ring, then $\gamma((AG(R))^c) = 2$.*

Proof. Since R is not an integral domain, there exist $a, b \in R\backslash\{0\}$ such that $ab = 0$. It is clear that $Ra, Rb \in A(R)^*$ and as R is reduced, $Ra \neq Rb$. Let $I \in A(R)^*$. Let $J = ((0) :_R I)$. Observe that $J \in A(R)^*$. We claim that $I + J \notin A(R)$. Let $r \in R$ be such that $(I + J)r = (0)$. Then $r \in ((0) :_R I) = J$. It now follows from $Jr = (0)$ that $r^2 = 0$ and so $r = 0$. This proves that $I + J \notin A(R)$. Hence we obtain from Lemma 2.2 that $\gamma((AG(R))^c) = 2$. □

We make use of Lemma 2.6 in the verification of Remark 2.7.

Lemma 2.6. *Let R be a ring which admits P as its unique maximal N-prime of (0). Then the following hold.*
(i) If I is any proper ideal of R with $P \subset I$, then for any $a \in I\backslash P$ and $p \in P\backslash\{0\}$, $Rpa^n \neq Rpa^m$ for all distinct $n, m \in \mathbf{N}$.
(ii) If P is a maximal ideal of R and if $x, y \in P$ are such that $\{x + P^2, y + P^2\}$ is linearly independent over R/P, then for any $r, s \in R$ with $r - s \notin P$, $R(x + ry) \neq R(x + sy)$.

Proof. Suppose that $Rpa^n = Rpa^m$ for some distinct $n, m \in \mathbf{N}$. We may assume without loss of generality that $n < m$. Note that $pa^n = xpa^m$ for some $x \in R$. This implies that $pa^n(1 - xa^{m-n}) = 0$. Since $Z(R) = P$, it follows that $a \notin Z(R)$ and so $pa^n \neq 0$. Therefore, $1 - xa^{m-n} \in Z(R) = P \subset I$. Hence $1 = 1 - xa^{m-n} + xa^{m-n} \in I$. This is impossible, since $I \neq R$. Therefore, $Rpa^n \neq Rpa^m$ for all distinct $n, m \in \mathbf{N}$.
(ii) Suppose that $R(x + ry) = R(x + sy)$ for some $r, s \in R$ with $r - s \notin P$. Then $(r - s)y \in R(x + ry)$. Hence there exists $\lambda \in R$ such that $(r - s)y = \lambda(x + ry)$. Thus $\lambda x + (\lambda r - (r - s))y = 0$. Since $\{x + P^2, y + P^2\}$ is linearly independent over R/P, we obtain that $\lambda, \lambda r - (r - s) \in P$. Hence it follows that $r - s \in P$ and this is a contradiction. Therefore, $R(x + ry) \neq R(x + sy)$ for any $r, s \in R$ with $r - s \notin P$. □

Remark 2.7. Let R be a ring such that R has a unique maximal N-prime of (0) (that is, equivalently $Z(R)$ is an ideal of R). Let P be the unique maximal N-prime of (0) in R. We focus our attempt on characterising rings R in order that $(\Gamma(R))^c$ (respectively,$(AG(R))^c$) admits a finite dominating set. We consider two cases.
(i) Suppose that $P^2 = (0)$. In such a case, it is clear that $Z(R)^*$ is the only dominating set of $(\Gamma(R))^c$. Hence $(\Gamma(R))^c$ admits a finite dominating set if and only if $Z(R)^*$ is finite. This implies by [12, Theorem 1] that R is finite. Moreover, in this case, $\gamma((\Gamma(R))^c) = |P| - 1 = |R/P|^k - 1$, where $k = dim_{R/P}(P)$, that is, k is the dimension of P as a vector space over R/P.

When $P^2 = (0)$, the collection $A(R)^*$ is the only dominating set of $(AG(R))^c$. Thus $(AG(R))^c$ admits a finite dominating set if and only if $A(R)^*$ is finite. Thus the number of ideals I of R such that $I \subseteq P$ must be finite. Hence we obtain from Lemma 2.6(i) that P is a maximal ideal of R. From $P^2 = (0)$, it follows that P is the unique maximal ideal of R. Thus the number of ideals of R is finite and so (R, P) is a local Artinian ring. Note that $\gamma((AG(R))^c)$ is equal to the

number of nonzero proper ideals of R. If P is principal, then (R, P) is a SPIR and in this case, $\gamma((AG(R))^c) = 1$. Suppose that P is not principal. Then it is clear that $dim_{R/P}(P) \geq 2$. Since the number of ideals of R is finite, we obtain from Lemma 2.6(ii) that R/P is finite and so P is finite. Hence R is finite.

(ii) Suppose that $P^2 \neq (0)$. We next proceed to determine rings R in order that $(\Gamma(R))^c$ (respectively, $(AG(R))^c$) admits a finite dominating set. In Proposition 2.8, we investigate this problem under the assumption that P is finitely generated.

Proposition 2.8. *Let R be a ring such that R has P as its unique maximal N-prime of (0). Suppose that P is finitely generated and $P^2 \neq (0)$. Then the following hold.*
(i) *$(\Gamma(R))^c$ admits a finite dominating set if and only if there exists only a finite number of elements $x \in R\backslash\{0\}$ such that $P = ((0) :_R x)$.*
(ii) *$(AG(R))^c$ admits a finite dominating set if and only if there exists only a finite number of nonzero ideals I of R such that $P = ((0) :_R I)$.*

Proof. (i) Let $A \subseteq Z(R)^* = P\backslash\{0\}$ be a finite dominating set of $(\Gamma(R))^c$. Let $x \in R\backslash\{0\}$ be such that $P = ((0) :_R x)$. As $px = 0$ for each $p \in P$ and A is a dominating set of $(\Gamma(R))^c$, we obtain that $x \in A$. Since A is a finite set, we get that $\{x \in R\backslash\{0\} | P = ((0) :_R x)\}$ is a finite set.

Conversely, assume that there exists only a finite number of elements $x \in R\backslash\{0\}$ such that $P = ((0) :_R x)$. Let $B = \{x \in R\backslash\{0\} | P = ((0) :_R x)\}$. It may happen that B may be an empty set. Let $n \geq 0$ be the number of elements of B. By hypothesis, P is finitely generated. Let $a_1, \ldots, a_t \in P$ be such that $P = Ra_1 + \cdots + Ra_t$. From $P^2 \neq (0)$, it follows that there exist $a, b \in P$ such that $ab \neq 0$. Hence $P \neq ((0) :_R a)$ and $P \neq ((0) :_R b)$. Let $p \in P\backslash(B \cup \{0\})$. Hence $P \neq ((0) :_R p)$. Therefore, $pa_i \neq 0$ for some $i \in \{1, \ldots, t\}$. This proves that $B \cup \{a_1, \ldots, a_t\}$ is a dominating set of $(\Gamma(R))^c$. Hence $(\Gamma(R))^c$ admits a finite dominating set.

(ii) Let $\mathbf{A} = \{I_1, I_2, \ldots, I_k\} \subseteq A(R)^*$ be a finite dominating set of $(AG(R))^c$. Let I be a nonzero ideal of R such that $P = ((0) :_R I)$. As $I_i \subseteq P$ for each $i \in \{1, 2, \ldots, k\}$ and $\mathbf{A}$ is a dominating set of $(AG(R))^c$, we obtain that $I \in \mathbf{A}$. Hence it follows that there can be only a finite number of nonzero ideals I of R such that $P = ((0) :_R I)$.

Conversely, assume that the collection $\mathbf{B} = \{I \in A(R)^* | P = ((0) :_R I)\}$ is finite. It may happen that $\mathbf{B}$ may be an empty collection. Suppose that $P = Ra_1 + \cdots + Ra_t$. From $P^2 \neq (0)$, it follows that there exist $a, b \in P$ such that $ab \neq 0$. Therefore, $P \neq ((0) :_R Ra)$ and $P \neq ((0) :_R Rb)$. Let $A \in A(R)^*$ be such that $A \notin \mathbf{B}$. Then $A(Ra_i) \neq (0)$ for some $i \in \{1, \ldots, t\}$. It is clear from the above arguments that $\mathbf{B} \cup \{Ra_i | i \in \{1, \ldots, t\}\}$ is a dominating set of $(AG(R))^c$ and this collection is finite. □

We illustrate Proposition 2.8 with the help of some examples in Example 2.9.

Example 2.9. Let $T = K[X, Y]$ be the polynomial ring in two variables X, Y over a field K. Let $I = TX^2 + TXY$ and $R = T/I$. Let $M = TX + TY$. It is clear that $Z(R) = M/I$. We claim that if $f(X, Y) \in M$ is such that $M/I = ((0 + I) :_R f(X, Y) + I)$, then $f(X, Y) + I = \alpha X + I$ for some $\alpha \in K\backslash\{0\}$. Note

that $f(X,Y) = t_1X + t_2Y$ for some $t_1, t_2 \in T$. Now it follows from $f(X,Y)Y \in I$ that $t_2Y^2 \in I \subseteq TX$. Hence $t_2 = t_3X$ for some $t_3 \in T$ and so $t_2Y = t_3XY \in I$. Therefore, $f(X,Y) + I = t_1X + I$. Observe that $t_1 = \alpha + tX + sY$ for some $\alpha \in K$, $t, s \in T$. Thus $f(X,Y) + I = \alpha X + tX^2 + sXY + I = \alpha X + I$. As $f(X,Y) \notin I$, it is clear that $\alpha \neq 0$. Conversely, it is obvious that for any $\alpha \in K\backslash\{0\}$, $M/I = ((0+I) :_R \alpha X + I)$. Thus $\{r \in R\backslash\{0+I\} | Z(R) = M/I = ((0);_R r)\} = \{\alpha X + I | \alpha \in K\backslash\{0\}\}$. Note that if α, β are any distinct elements of K, then $\alpha X + I \neq \beta X + I$. Thus if K is infinite, then there exists an infinite number of elements $r \in R\backslash\{0+I\}$ such that $M/I = ((0+I) :_R r)$. Therefore, we obtain from Proposition 2.8(i) that $(\Gamma(R))^c$ admits a finite dominating set if and only if K is finite.

Let R be as above. We next assert that if $g(X,Y) \in M\backslash I$ is such that $M/I \neq ((0+I) :_R g(X,Y) + I)$, then $g(X,Y)Y \notin I$. The above assertion follows as $M/I = R(X+I) + R(Y+I)$ and $g(X,Y)X \in I$.

For any prime number p and $n \geq 1$, we denote the finite field containing exactly p^n elements by $\mathbf{F_{p^n}}$.

(i) Let $K = \mathbf{F_{p^n}}$ in the example mentioned above. We now verify that $\gamma((\Gamma(R))^c) = p^n$. It follows from the proof of Proposition 2.8(i) and from the arguments given above that $\{\alpha X + I | \alpha \in K\backslash\{0\}\} \cup \{Y + I\}$ is a dominating set of $(\Gamma(R))^c$. Hence $\gamma((\Gamma(R))^c) \leq p^n$. Moreover, we know from the proof of Proposition 2.8(i) that any dominating set of $(\Gamma(R))^c$ must contain $B = \{\alpha X + I | \alpha \in K\backslash\{0\}\}$. It is clear that B is not a dominating set of $(\Gamma(R))^c$ and so $\gamma((\Gamma(R))^c) \geq p^n$. This proves that $\gamma((\Gamma(R))^c) = p^n$.

(ii) Let K be any field and R be as above. We assert that $\gamma((AG(R))^c) = 2$. Let $A \in A(R)^*$ be such that $M/I = ((0+I) :_R A)$. Then for any $a \in A\backslash\{0+I\}$, $M/I = ((0+I) :_R a)$ and so we obtain from the arguments given in the first paragraph of this example that $a = \alpha X + I$ for some $\alpha \in K\backslash\{0\}$. Therefore, $A = R(X+I)$. If $B \in A(R)^*$ is such that $M/I \neq ((0+I) :_R B)$, then as $B \subseteq M/I$ and $B(X+I) = (0+I)$, we obtain that $B(Y+I) \neq (0+I)$. It follows from the proof of Proposition 2.8(ii) that $\{R(X+I), R(Y+I)\}$ is a dominating set of $(AG(R))^c$ and so $\gamma((AG(R))^c) \leq 2$. It is now clear from Proposition 2.1(ii) that $\gamma((AG(R))^c) = 2$.

Remark 2.10. In Example 2.4, an example is provided to illustrate that if a ring R has a unique maximal N-prime of (0), then $(\Gamma(R))^c$ may not admit any finite dominating set. In the notation of Example 2.4, (R, M) is a quasilocal reduced ring with $Z(R) = M$. Since R is reduced, it is clear that M is not a B-prime of (0).

Let R be a ring such that R admits P as its unique maximal N-prime of (0). It is natural to ask whether $(\Gamma(R))^c$ admits a finite dominating set implies that P is necessarily a B-prime of (0). We provide an example in Example 2.12 to illustrate that P need not be a B-prime of (0). We make use of Lemma 2.11 in the verification of Example 2.12.

Lemma 2.11. *Let $G = (V, E)$ be a simple graph. If there exist $u, v \in V$ such that $d_{G^c}(u, v) \geq 3$, then $\{u, v\}$ is a dominating set of G.*

Proof. Let $w \in V\backslash\{u,v\}$. Suppose that u and w are not adjacent in G. Then u and w are adjacent in G^c. Since $d_{G^c}(u,v) \geq 3$ by assumption, we get that v and w are not adjacent in G^c. Therefore, w and v are adjacent in G. This proves that $\{u,v\}$ is a dominating set of G. □

Example 2.12 presented here is from [18, Exercises 6 and 7, pp.62-63].

Example 2.12. Let $S = K[X,Y]$ be the polynomial ring in two variables X, Y over a field K. Let $M = SX + SY$. Let $T = S_M$. Let $\mathbf{P}$ denote the set of all pairwise nonassociate prime elements of T. Let W be the T-module given by $W = \oplus_{p\in\mathbf{P}} T/Tp$. Let $R = T \oplus W$ be the ring obtained by using Nagata's principle of idealization. It was shown in [22, Example 2.8] that $P = MT \oplus W$ is the unique maximal N-prime of the zero ideal in R and P is not a B-prime of the zero ideal in R. Indeed, it has been observed in [3, Example 2.6] that if p and q are distinct elements of $\mathbf{P}$, then there is no nonzero element of R which annihilates both $(p,0)$ and $(q,0)$. Thus $d_{\Gamma(R)}((p,0),(q,0)) \geq 3$. Therefore, we obtain from Lemma 2.11 that for any distinct elements $p, q \in \mathbf{P}$, $\{(p,0),(q,0)\}$ is a dominating set of $(\Gamma(R))^c$. Let $p, q \in \mathbf{P}$, $p \neq q$. Then $R(p,0), R(q,0)$ are distinct members of $A(R)^*$. Note that there is no $I \in A(R)^*$ which annihilates both $R(p,0)$ and $R(q,0)$. Hence $d_{AG(R)}(R(p,0), R(q,0)) \geq 3$. Therefore, it follows from Lemma 2.11 that $\{R(p,0), R(q,0)\}$ is a dominating set of $(AG(R))^c$.

It follows from the above discussion and Proposition 2.1 that $\gamma((\Gamma(R))^c) = \gamma((AG(R))^c) = 2$.

Let R be a ring which admits exactly one maximal N-prime of (0). Note that all the rings R of this type exhibited so far in this article, satisfies the property that $\gamma((AG(R))^c) \leq 2$. With the help of Lemma 2.13, in Example 2.14, we provide an example of a ring R such that $(AG(R))^c$ does not admit any finite dominating set. Recall that a ring R is a chained ring, if the ideals of R are comparable under the inclusion relation. If R is a chained ring, then it is clear that $Z(R)$ is an ideal of R.

Lemma 2.13. *Let R be a chained ring with P as its unique maximal N-prime of (0). Then the following statements hold.*
(i) If $(\Gamma(R))^c$ admits a finite dominating set , then P is necessarily a B-prime of (0) in R.
(ii) If $(AG(R))^c$ admits a finite dominating set, then P is necessarily a B-prime of (0) in R.

Proof. (i) Let $A = \{a_1, \ldots, a_n\}$ be a finite dominating set of $(\Gamma(R))^c$. Since R is a chained ring, we may assume without loss of generality that $Ra_1 \subseteq \cdots \subseteq Ra_n$. Suppose that P is not a B-prime of (0) in R. Let $b \in R\backslash\{0\}$ be such that $a_n b = 0$. Let us denote b by b_1. Since we are assuming that P is not a B-prime of (0) in R, $Pb_1 \neq (0)$. So there exists $b_2 \in P$ such that $b_1 b_2 \neq 0$. Assume that we have found elements $b_1, \ldots, b_m \in P$ such that $\prod_{i=1}^m b_i \neq 0$. Note that $P(\prod_{i=1}^m b_i) \neq (0)$. Hence there exists $b_{m+1} \in P$ such that $(\prod_{i=1}^m b_i)b_{m+1} \neq 0$. Thus there exist elements $b_k \in P$ for each $k \in \mathbf{N}$ such that $a_n b_1 = 0$, whereas $\prod_{i=1}^k b_i \neq 0$ for

each $k \in \mathbf{N}$. It is easy to verify that the ideals $R(\prod_{i=1}^{k} b_i)$ $(k \in \mathbf{N})$ are distinct. Therefore, we can find $k \in \mathbf{N}$ such that $\prod_{i=1}^{k} b_i \notin A$. From $a_n \prod_{i=1}^{k} b_i = 0$ and $Ra_1 \subseteq \cdots \subseteq Ra_n$, it follows that $\prod_{i=1}^{k} b_i$ is not adjacent to any element of A in $(\Gamma(R))^c$. This is in contradiction to the assumption that A is a dominating set of $(\Gamma(R))^c$. Therefore, P is a B-prime of (0) in R.

(ii) Let $\mathbf{A} = \{I_1, \ldots, I_n\}$ be a finite dominating set of $(AG(R))^c$. Since R is a chained ring, we may assume without loss of generality that $I_1 \subseteq \cdots \subseteq I_n$. As $I_n \in A(R)^*$, there exists $b \in R\backslash\{0\}$ such that $I_n b = (0)$. Suppose that P is not a B-prime of (0) in R. Then it is shown in the previous paragraph that there exist elements $b_k \in P$ for each $k \in \mathbf{N}$ with $b_1 = b$ such that $\prod_{i=1}^{k} b_i \neq 0$ for each $k \in \mathbf{N}$. Let $A_k = R(\prod_{i=1}^{k} b_i)$ for each $k \in \mathbf{N}$. Observe that $A_i \in A(R)^*$ for each $i \in \mathbf{N}$ and for any distinct $r, s \in \mathbf{N}$, $A_r \neq A_s$. Hence there exists $k \in \mathbf{N}$ such that $A_k \notin \mathbf{A}$. It follows from $I_n b = (0)$ that $I_n A_k = (0)$. As $I_i \subseteq I_n$, we obtain that $I_i A_k = (0)$ for each $i \in \{1, \ldots, n\}$. This contradicts the assumption that $\mathbf{A}$ is a dominating set of $(AG(R))^c$. Therefore, P is a B-prime of (0) in R. □

Example 2.14. Let (V, M) be a rank one valuation domain which is not discrete. Let $m \in M\backslash\{0\}$. Let $R = V/mV$. It is clear that R is a chained ring with $P = M/mV$ as its unique maximal ideal and indeed, P is the only prime ideal of R. Note that P is the unique maximal N-prime of the zero ideal in R and it was verified in [21, Example 3.1(ii)] that P is not a B-prime of the zero ideal in R. Hence we obtain from Lemma 2.13 that neither $(\Gamma(R))^c$ nor $(AG(R))^c$ admits a finite dominating set.

Our goal is to characterize chained rings R in order that $(\Gamma(R))^c$ (respectively, $(AG(R))^c$) admits a finite dominating set. To achieve that aim, in Lemma 2.15, we provide another necessary condition.

Lemma 2.15. *Let R be a ring such that R admits P as its only maximal N-prime of (0). Suppose that P is a B-prime of (0) in R. Then the following hold.*
(i) If $(\Gamma(R))^c$ admits a finite dominating set, then R/P must be finite and hence P is a maximal ideal of R.
(ii) If $(AG(R))^c$ admits a finite dominating set, then P is a maximal ideal of R.

Proof. By assumption, $P = ((0) :_R r) = ((0) :_R Rr)$ for some $r \in R\backslash\{0\}$.
(i) Suppose that $(\Gamma(R))^c$ admits a finite dominating set. Then we know from the proof of Proposition 2.8(i) that there exists only a finite number of nonzero elements $x \in R$ such that $P = ((0) :_R x)$ (for this part of the proof, we do not need the assumption that P is finitely generated). Observe that if a is any element of $R\backslash P$, $ra \neq 0$ and $P = ((0) :_R r) \subseteq ((0) :_R ra) \subseteq Z(R) = P$. Therefore, $P = ((0) :_R ra)$. Moreover, if $a, b \in R\backslash P$ are such that $a - b \notin P$, then $(a - b)r \neq 0$. That is, $ar \neq br$. Since the number of elements $x \in R\backslash\{0\}$ such that $P = ((0) :_R x)$ is finite, the above arguments imply that R/P is finite. As a finite integral domain is a field, we obtain that P is a maximal ideal of R.
(ii) Suppose that $(AG(R))^c$ admits a finite dominating set. Then we know from the proof of Proposition 2.8(ii) that there exists only a finite number of $I \in A(R)^*$

such that $P = ((0) :_R I)$ (for this part of the proof, we do not need the assumption that P is finitely generated). By hypothesis, $P = ((0) :_R Rr)$. We assert that P is a maximal ideal of R. Suppose that P is not a maximal ideal of R. Then there exists a maximal ideal M of R such that $P \subset M$. Let $m \in M\backslash P$. Observe that for any $k \in \mathbf{N}$, $P = ((0) :_R Rr) \subseteq ((0) :_R Rrm^k) \subseteq P$ and so $P = ((0) :_R Rrm^k)$ for all $k \in \mathbf{N}$. We know from Lemma 2.6(i) that $Rrm^k \neq Rrm^t$ for all distinct $k, t \in \mathbf{N}$. Hence there exists an infinite number of $I \in A(R)^*$ such that $P = ((0) :_R I)$. This is impossible. Therefore, P is a maximal ideal of R. □

Let R be a chained ring with P as its unique maximal N-prime of (0) such that $P^2 \neq (0)$. In Proposition 2.17, we determie $\gamma((\Gamma(R))^c)$ and $\gamma((AG(R))^c)$ under the assumption that P is finitely generated. We make use of Lemma 2.16 in the proof of Proposition 2.17.

Lemma 2.16. *Let R be a chained ring with P as its unique maximal N-prime of (0).*
(i) If R/P is finite, then there exists only a finite number of elements $r \in P\backslash\{0\}$ such that $P = ((0) :_R r)$.
(ii) If P is the maximal ideal of R, then there exists at most one $I \in A(R)^*$ such that $P = ((0) :_R I)$.

Proof. (i) If there exists no $r \in P\backslash\{0\}$ such that $P = ((0) :_R r)$, then we are done. Hence we may assume that there exists $r \in P\backslash\{0\}$ such that $P = ((0) :_R r)$. Let $s \in P\backslash\{0\}$ be such that $P = ((0) :_R s)$. We claim that $s = ru$ for some unit $u \in R$. As R is a chained ring, we obtain that either $Rr \subseteq Rs$ or $Rs \subseteq Rr$. Hence either $r = as$ for some $a \in R$ or $s = br$ for some $b \in R$. Without loss of generality, we may assume that $r = as$. Since $Ps = (0)$ and $r \neq 0$, it follows that $a \notin P$. We are assuming that R/P is finite. Hence P is necessarily the unique maximal ideal of R. Therefore, a is a unit in R and so $s = ru$ for some unit $u \in R$ (in this part of the proof, we make use only the fact that P is the unique maximal ideal of R). Let $R/P = \{0 + P, r_1 + P, \ldots, r_{m-1} + P\}$ for some $r_1, \ldots r_{m-1} \in R\backslash P$. If $s \in P\backslash\{0\}$ is such that $P = ((0) :_R s)$, then $s = ru$ for some unit $u \in R$. Hence $u = r_i + p$ for some $i \in \{1, \ldots, m-1\}$ and $p \in P$. It follows from $Pr = (0)$ that $s = r(r_i + p) = r_i r$. Moreover, observe that $P = ((0) :_R rr_i)$ for each $i \in \{1, \ldots, m-1\}$. We may assume without loss of generality that $r_1 = 1$. This shows that $\{s \in P\backslash\{0\} | P = ((0) :_R s)\} = \{rr_i | i \in \{1, \ldots, m-1\}\}$.
(ii) If there exists no $I \in A(R)^*$ such that $P = ((0) :_R I)$, then we are done. So we may assume that there exists $I \in A(R)^*$ such that $P = ((0) :_R I)$. Let $J \in A(R)^*$ be such that $P = ((0) :_R J)$. Observe that for any $x \in I\backslash\{0\}$ and $y \in J\backslash\{0\}$, $P = ((0) :_R x) = ((0) :_R y)$. Since P is the unique maximal ideal of R, it follows as in (i) that $y = ux$ for some unit $u \in R$. This proves that $I = J$. Moreover, for any $a_1, a_2 \in I\backslash\{0\}$, $P = ((0) :_R a_1) = ((0) :_R a_2)$. Hence it follows as in (i) that $a_2 = ua_1$ for some unit $u \in R$ and so $I = Ra$ for any nonzero $a \in I$. □

Proposition 2.17. *Let R be a chained ring with P as its unique maximal N-prime of (0) such that $P^2 \neq (0)$. Suppose that P is finitely generated. Then the following hold.*

(i) $(\Gamma(R))^c$ *admits a finite dominating set if and only if* R/P *is finite and moreover, in this case,* $\gamma((\Gamma(R))^c) = |R/P|$.
(ii) $(AG(R))^c$ *admits a finite dominating set if and only if* P *is the unique maximal ideal of* R *and moreover, in this case,* $\gamma((AG(R))^c) = 2$.

Proof. By hypothesis, P is finitely generated. Since R is a chained ring, it follows that there exists $p \in P$ such that $P = Rp$. As $P = Z(R)$, there exists $r \in R\backslash\{0\}$ such that $pr = 0$ (note that $r \in P$). Therefore, $P = ((0) :_R r)$.
(i) Assume that $(\Gamma(R))^c$ admits a finite dominating set. Then we obtain from Lemma 2.15*(i)* that R/P is finite. Conversely, assume that R/P is finite. Then it follows from Lemma 2.16*(i)* and Proposition 2.8*(i)* that $(\Gamma(R))^c$ admits a finite dominating set.

Assume that R/P is finite. We verify that $\gamma((\Gamma(R))^c) = |R/P|$. Let $r_1 = 1, \ldots, r_{m-1} \in R\backslash P$ be such that $R/P = \{0 + P, r_1 + P, \ldots, r_{m-1} + P\}$ (m is necessarily a power of a prime). We know from the proof of Lemma 2.16*(i)* that $\{s \in P\backslash\{0\} | P = ((0) :_R s)\} = \{rr_i | i \in \{1, \ldots, m-1\}\}$. Moreover, it is shown in the proof of Proposition 2.8*(i)* that if A is any dominating set of $(\Gamma(R))^c$, then A must contain $\{rr_i | i \in \{1, \ldots, m-1\}\}$ and furthermore, $\{rr_i | i \in \{1, \ldots, m-1\}\} \cup \{p\}$ is a dominating set of $(\Gamma(R))^c$. As $\{rr_i | i \in \{1, \ldots, m-1\}\}$ is not a dominating set of $(\Gamma(R))^c$ it follows that $\gamma((\Gamma(R))^c) = m = |R/P|$.
(ii) Assume that $(AG(R))^c$ admits a finite dominating set. Then we obtain from Lemma 2.15*(ii)* that P is necessarily the unique maximal ideal of R. Conversely, assume that P is the unique maximal ideal of R. Then it follows from Lemma 2.16*(ii)* and Proposition 2.8*(ii)* that $(AG(R))^c$ admits a finite dominating set.

Assume that P is the unique maximal ideal of R. Note that $P = ((0) :_R Rr)$. We know from the proof of Lemma 2.16*(ii)* that $\{I \in A(R)^* | P = ((0) :_R I)\} = \{Rr\}$. If $\mathbf{A}$ is any dominating set of $(AG(R))^c$, then it is shown in the proof of Proposition 2.8*(ii)* that $Rr \in \mathbf{A}$ and furthermore, $\{Rr, Rp\}$ is a dominating set of $(AG(R))^c$. Since $\{Rr\}$ is not a dominating set of $(AG(R))^c$, it follows that $\gamma((AG(R))^c) = 2$. □

Let (R, M) be a SPIR with $M^2 \neq (0)$. Let $n \geq 3$ be least with the property that $M^n = (0)$. We know from the proof of *(iii)* $\Rightarrow$ *(i)* of [6, Proposition 8.8] that $\{M, M^2, \ldots, M^{n-1}\}$ is the set of all nonzero proper ideals of R. Thus R is a chained ring with M as its unique maximal N-prime of (0). Hence Corollary 2.18 follows immediately from Proposition 2.17.

Corollary 2.18. *Let* (R, M) *be a SPIR with* $M^2 \neq (0)$. *Then the following hold.*
(i) $(\Gamma(R))^c$ *admits a finite dominating set if and only if* $|R/M|$ *is finite and moreover, in this case,* $\gamma((\Gamma(R))^c) = |R/M|$.
(ii) $\gamma((AG(R))^c) = 2$.

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