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REGULARIZATION OF THE BOUNDARY VALUE PROBLEM FOR SMALL RADIAL OSCILLATIONS

ASAL V. YULDASHEVA[1]

Abstract. This paper studies the ill-posed Dirichlet problem for the wave equation. A regularization method based on the spectral decomposition of the underlying operator is proposed. Convergence estimates for the regularized solutions are obtained under appropriate assumptions on the parameters. For parameters related to algebraic numbers, Diophantine estimates are used to control small denominators. Stability with respect to data perturbations is established, and uniform convergence of the regularized solutions in both spatial and temporal variables is proved.

Keywords. Dirichlet problem, wave equation, regularization, ill-posed problems, algebraic numbers.
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1. Introduction and preliminaries

Boundary value problems for hyperbolic equations, in particular the Dirichlet problem for the wave equation, play an important role in the theory of partial differential equations and mathematical physics. They arise in the modeling of oscillatory processes in continuous media, acoustics, elasticity theory, and other applied fields. The classical foundations of well-posedness, including the existence, uniqueness, and stability of solutions, were established in the works of Jacques Hadamard [6].

It is well known that the Dirichlet problem for the wave equation with data prescribed on the entire boundary of the domain may be ill-posed. This is related to the phenomenon of small denominators and the high sensitivity of the solution to perturbations in the given data, which leads to a loss of stability. Such effects have been extensively studied within the framework of spectral theory and the analysis of Fourier series [3], [5].

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* Corresponding author
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To stabilize solutions of ill-posed problems, regularization methods are employed, allowing the construction of stable approximate solutions. The foundations of regularization theory were laid by Andrey Tikhonov [13], and further developed in the works of Mikhail Lavrentiev and his school [11].

In recent years, considerable attention has been devoted to the development of regularization methods based on spectral expansions, functional analysis, and numerical approaches. Current research addresses both theoretical questions of stability and the construction of efficient algorithms for solving ill-posed problems (see, for example, [2], [4], [7]–[10], [14]). In particular, studies focus on the regularization of hyperbolic problems, the stability of solutions in the presence of noisy data, and the convergence of regularized approximations.

Of particular interest is the analysis of problems in which small denominators arise from the Diophantine properties of the parameters. The use of results from the theory of transcendental numbers, including estimates of Liouville type [12], makes it possible to obtain the conditions required for the convergence of spectral expansions.

In this paper, we consider the Dirichlet problem for the wave equation, propose a regularization method for its solution, and prove the convergence of the regularized solutions to the exact solution. Stability estimates for perturbed data are obtained, and conditions for uniform convergence with respect to both the spatial and temporal variables are established.

2. Statement of the Problem

We consider the radial oscillations of a sphere in which each point of the sphere is displaced only in the radial direction. In this case, the oscillatory process starting from the state of rest is described by the wave equation [1]

$$u_{tt} = \Delta u, \qquad 0 < t < T, \quad |x| = r < R, \tag{2.1}$$

subject to the conditions

$$u(r,0) = 0, \qquad u(r,T) = \psi(r), \qquad 0 \le r \le R, \tag{2.2}$$

and

$$u(R,t) = 0, \qquad 0 \le t \le T. \tag{2.3}$$

Solving the corresponding spectral problem, we obtain that the sequence of eigenvalues is given by

$$\lambda_k = \left(\frac{\pi k}{R}\right)^2, \qquad k = 1, 2, \dots,$$

and the complete orthonormal system of eigenfunctions has the form

$$v_k(r) = \frac{1}{r\sqrt{2\pi R}} \sin\left(\frac{\pi k r}{R}\right), \qquad k = 1, 2, \dots. \tag{2.4}$$

Then the solution of problem (2.1)–(2.3) can be represented as

$$u(r,t) = \sum_{k=1}^{\infty} \frac{\sin\left(\frac{\pi k}{R}t\right)}{\sin\left(\frac{\pi k}{R}T\right)} \psi_k \, v_k(r), \tag{2.5}$$

where ψ_k are the Fourier coefficients of the function $\psi(r)$ with respect to the system of eigenfunctions (2.4).

3. Solvability of the Problem

Expanding the solution in terms of the eigenfunctions (2.4), we obtain the representation (2.5). The following theorem holds.

Theorem 3.1. *Let the number* $\dfrac{T}{R}$ *be an algebraic number of degree two. If* $\psi \in H^5(|x| < R)$ *then the function represented by formula* (2.5) *is a solution of problem* (2.1)–(2.3).

Proof. First, recall that

$$H^{\alpha}(\Omega) = \left\{ f \in L_2(\Omega) : \sum_{k=1}^{\infty} |(f, v_k)|^2 k^{2\alpha} < \infty \right\}.$$

It remains to prove that the series (2.5), as well as the derivatives involved in equation (2.1), converge uniformly. The second derivative with respect to time of the solution (2.5) has the form

$$u_{tt}(r,t) = -(\frac{\pi}{R})^2 \sum_{k=1}^{\infty} \frac{k^2 \psi_k \sin(\frac{\pi k}{R} t)}{\sin(\frac{\pi k}{R} T)} v_k(r) \tag{3.1}$$

It is sufficient to prove the convergence of the series (3.1). The corresponding majorant series is

$$|u_{tt}(r,t)| \le C \sum_{k=1}^{\infty} \frac{k^3 |\psi_k|}{|\sin(\frac{\pi k}{R} T)|}. \tag{3.2}$$

Indeed, the $\sin(\frac{\pi k}{R} r)$ behaves as $\frac{\pi k}{R} r$ as $r \to 0$.

Next, we estimate the denominator in (3.2). Recall that $\sin\left(\frac{\pi k}{R} T\right) = 0$, if $\frac{\pi k}{R} T = \pi m$. Assume that $\sin\left(\frac{\pi k}{R} T\right) \neq 0$ and introduce the notation

$$\delta_{mk} = \frac{\pi k}{R} T - \pi m$$

or

$$\frac{\pi k}{R} T = \delta_{mk} + \pi m \tag{3.3}$$

We may assume that $|\delta_{mk}| < \frac{\pi}{2}$. From (3.3) we obtain

$$\frac{T}{R} = \frac{\delta_{mk}}{\pi k} + \frac{m}{k}. \tag{3.4}$$

By the assumption of the theorem, $\dfrac{T}{R}$ is an algebraic number of degree two. Therefore, by Liouville's theorem (see [10]), there exists a constant $C > 0$ such that for all natural numbers m and k,

$$\left| \frac{T}{R} - \frac{m}{k} \right| \ge \frac{C}{k^2}.$$

Hence, we obtain the following estimate,

$$|\delta_{mk}| = \pi k \left| \frac{T}{R} - \frac{m}{k} \right| \geq \frac{C}{k}. \tag{3.5}$$

Then, from the fact that $|\delta_{mk}| < \frac{\pi}{2}$ and estimate (3.5), it follows that

$$|\sin(\frac{\pi k}{R}T)| \geq \frac{C_1}{k} \tag{3.6}$$

for some constant $C_1 > 0$. Consequently, estimate (3.2) can be rewritten as

$$|u_{tt}| \leq C_1 \sum_{k=1}^{\infty} k^4 |\psi_k| \leq C_1 \sqrt{\sum_{k=1}^{\infty} k^{10} |\psi_k|^2} \sqrt{\sum_{k=1}^{\infty} \frac{1}{k^2}}.$$

By the assumptions imposed on the function ψ, the latter series converges. Therefore, the series (3.1) converges uniformly, and the same is true for the derivatives involved in equation (2.1). □

4. Regularization and Convergence to the Solution of the Problem

For any $\alpha > 0$ and $p > 0$, introduce the function

$$u_\alpha(r,t) = \sum_{k=1}^{\infty} \frac{\sin(\frac{\pi k}{R}T)}{\sin^2(\frac{\pi k}{R}T) + \alpha \lambda_k^p} \psi_k \sin(\frac{\pi k}{R}t) v_k(r). \tag{4.1}$$

The function (4.1) possesses the following property.

Theorem 4.1. *Let $\alpha > 0$ and $s > 0$. If $\psi \in H^{2s}(|x| < R)$ then function $u_\alpha(r,t)$ belongs to the Sobolev class $W_2^{p+2s}(|x| < R, [0,T])$.*

Proof. To prove that the function (4.1) belongs to the required Sobolev class, it is sufficient to prove the convergence of the series

$$\sum_{k=1}^{\infty} k^{2p+4s} |u_{\alpha,k}|^2,$$

where $u_{\alpha,k}$ are the Fourier coefficients of function (4.1) with respect to the system of functions (2.4). Using the estimate

$$\sin^2(\frac{\pi k}{R}T) + \alpha \lambda_k^p \geq 2|\sin(\frac{\pi k}{R}T)| \sqrt{\alpha \lambda_k^p},$$

we obtain

$$\sum_{k=1}^{\infty} k^{2p+4s} |u_{\alpha,k}|^2 \leq \frac{C_2}{\alpha} \sum_{k=1}^{\infty} k^{2p+4s} \frac{|\psi_k|^2}{k^{2p}} = \frac{C_2}{\alpha} \|\psi\|^2_{W_2^{p+2s}} < \infty$$

□

Now let us investigate the limit of the function (4.1) as $\alpha \to 0^+$. The following theorem holds.

Theorem 4.2. *Let $\psi \in H^{2s}(|x| < R)$. If $s > p + 4$ and the number $\frac{T}{R}$ is an algebraic number of degree 2, then*

$$u_\alpha(r,t) \to u(r,t) \text{ if } \alpha \to 0^+$$

uniformly with respect to $|x| < R$ and $t \in [0,T]$.

Proof. Consider the difference

$$u(r,t) - u_\alpha(r,t) = \alpha \sum_{k=1}^{\infty} \frac{\lambda_k^p}{\sin(\frac{\pi k}{R}T)(\sin^2(\frac{\pi k}{R}T) + \alpha\lambda_k^p)} \psi_k \sin(\frac{\pi k}{R}t)v_k(r). \tag{4.2}$$

We estimate the denominator of the series in (4.2). Using estimate (3.6), we obtain

$$|\sin(\frac{\pi k}{R}T)|(\sin^2(\frac{\pi k}{R}T) + \alpha\lambda_k^p) \geq 2\sqrt{\alpha\lambda_k^p}\sin^2(\frac{\pi k}{R}T \geq 2\sqrt{\alpha}(\frac{\pi k}{R})^p(\frac{2C}{\pi k})^2$$

or

$$|\sin(\frac{\pi k}{R}T)|(\sin^2(\frac{\pi k}{R}T) + \alpha\lambda_k^p) \geq C_3\sqrt{\alpha}k^{p-2}.$$

Then from (4.2) we obtain

$$|u(r,t) - u_\alpha(r,t)| \leq C_4\sqrt{\alpha}\sum_{k=1}^{\infty} \frac{\lambda_k^{p+1/2}}{k^{p-2}}|\psi_k| = C_5\sqrt{\alpha}\sum_{k=1}^{\infty} k^{p+3}|\psi_k|.$$

Using the Cauchy–Bunyakovsky inequality, from the last inequality we have

$$|u(r,t) - u_\alpha(r,t)| \leq C_5\sqrt{\alpha}\sqrt{\sum_{k=1}^{\infty} k^{2p+8}|\psi_k|^2}\sqrt{\sum_{k=1}^{\infty}\frac{1}{k^2}}.$$

Taking into account the assumptions of the theorem, we conclude that $|u(r,t) - u_\alpha(r,t)|$ tends to 0 as $\alpha \to 0^+$, uniformly with respect to $|x| < R$ and $t \in [0,T]$. □

5. Regularization of the Problem with Inaccurate Data

The results obtained in the previous section establish the convergence of the regularized solution to the exact solution for exact input data. We now turn to a more realistic situation in which the terminal data are known only approximately and investigate the stability and convergence of the regularized solution under data perturbations.

Assume that instead of $\psi(r)$, a function $\psi^\delta(r)$ is given such that

$$\|\psi - \psi^\delta\|_{H^3(|x|<R)} < \delta \tag{5.1}$$

Introduce the function

$$u_\alpha^\delta(r,t) = \sum_{k=1}^{\infty} \frac{\sin(\frac{\pi k}{R}T)}{\sin^2(\frac{\pi k}{R}T) + \alpha\lambda_k^p} \psi_k^\delta \sin(\frac{\pi k}{R}t)v_k(r). \tag{5.2}$$

If the function ψ^δ satisfies the assumptions of Theorem 4.2, then function (5.2) converges, as $\alpha \to 0^+$, uniformly with respect to $|x| < R$ and $t \in [0,T]$, to the function $u_\alpha(r,t)$ defined by formula (4.1), in which ψ is replaced by ψ_δ.

It is of interest to investigate whether function (5.2) converges to solution (2.5) as α and δ tend to zero, and to determine the character of this convergence.

For this purpose, let us estimate the difference

$$|u_\alpha^\delta(r,t) - u(r,t)| \le |u_\alpha^\delta(r,t) - u_\alpha(r,t)| + |u_\alpha(r,t) - u(r,t)|$$

From formulas (4.1) and (5.2), we obtain

$$u_\alpha^\delta(r,t) - u_\alpha(r,t) = \sum_{k=1}^{\infty} \frac{\sin(\frac{\pi k}{R}T)}{\sin^2(\frac{\pi k}{R}T) + \alpha\lambda_k^p} (\psi_k^\delta - \psi_k) \sin(\frac{\pi k}{R}t) v_k(r)$$

Taking into account the estimates obtained earlier and the fact that the number T/R is an algebraic number of degree 2, we obtain

$$|u_\alpha^\delta(r,t) - u(r,t)| \le C(\sqrt{\alpha} \sum_{k=1}^{\infty} k^{p+3}|\psi_k| + \frac{1}{\sqrt{\alpha}} \sum_{k=1}^{\infty} k^2|\psi_k - \psi_k^\delta|) \tag{5.3}$$

From estimate (5.3) we obtain the following statement.

Theorem 5.1. *Let $\psi_\delta \in H^{2s}(\{|x| < R\})$ satisfy inequality (5.1). Assume that $s > p + 4$, and that the number $\frac{T}{R}$ is an algebraic number of degree 2. Then the regularized solution $u_\alpha^\delta(r,t)$ converges to the exact solution $u(r,t)$ as $\delta \to 0$, uniformly with respect to $|x| < R$ and $t \in [0,T]$, provided that the regularization parameter is chosen in the form $\alpha \asymp \delta$.*

Proof. Using the Cauchy–Bunyakovsky inequality, we can rewrite inequality (5.3) in the form

$$|u_\alpha^\delta(r,t) - u(r,t)| \le C(\sqrt{\alpha} \sqrt{\sum_{k=1}^{\infty} k^{2p+4}|\psi_k|^2} \sqrt{\sum_{k=1}^{\infty} \frac{1}{k^2}} + \frac{1}{\sqrt{\alpha}} \sqrt{\sum_{k=1}^{\infty} k^6|\psi_k - \psi_k^\delta|^2} \sqrt{\sum_{k=1}^{\infty} \frac{1}{k^2}})$$

Taking into account the assumptions on the given function, we obtain

$$|u_\alpha^\delta(r,t) - u(r,t)| \le C(\sqrt{\alpha} + \frac{\delta}{\sqrt{\alpha}}). \tag{5.4}$$

Minimizing the right-hand side of (5.4) with respect to α , we obtain

$$C(\frac{1}{2\sqrt{\alpha}} - \frac{\delta}{2\sqrt{\alpha^3}}) = 0.$$

From the last equality it follows that if we choose $\alpha \asymp \delta$, then

$$|u_\alpha^\delta(r,t) - u(r,t)| = O(\sqrt{\delta}).$$

□

The obtained estimates show that the proposed regularization method is stable with respect to perturbations of the input data. Thus, an appropriate choice of the regularization parameter ensures the convergence of the regularized solution to the exact solution and provides a quantitative characterization of the convergence rate.

Conclusion

In this paper, the Dirichlet problem for the wave equation describing small radial oscillations of a sphere has been investigated. A representation of the solution in the form of a spectral expansion with respect to the system of eigenfunctions of the corresponding boundary value problem has been obtained.

Conditions for the solvability of the problem have been established under which the series representation of the solution converges uniformly together with the required derivatives. It has been shown that, under appropriate Diophantine conditions on the ratio $\frac{T}{R}$, the phenomenon of small denominators can be avoided, which ensures the correctness of the constructed solution.

A regularization method based on a modification of the spectral expansion with the use of a regularization parameter has been proposed. It has been proved that the regularized solution possesses higher smoothness and belongs to the corresponding Sobolev spaces.

The convergence of the regularized solution to the exact solution as the regularization parameter tends to zero has been studied. Uniform convergence with respect to both spatial and temporal variables has been established under suitable smoothness assumptions on the initial data.

The case of inaccurate input data has also been considered. Stability estimates for the solution have been obtained, and it has been proved that the regularized solution converges to the exact one when the regularization parameter and the error level tend to zero in a coordinated manner. An estimate for the convergence rate has been established.

The obtained results can be used in the study of ill-posed boundary value problems for hyperbolic equations, as well as in the development of stable numerical algorithms for solving inverse and ill-posed problems of mathematical physics.

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[1] Branch of Lomonosov Moscow State University in Tashkent, Tashkent, Uzbekistan.

Email address: a_v_yuldasheva@mail.ru