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STABILITY AND CONTROLLABILITY OF CAPUTO-FABRIZIO FRACTIONAL SYSTEMS

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Abstract. This paper is dedicated to studying the stability and controllability of a fractional differential system with delay. Using the Laplace transform, we derive an integral representation of the solution, and by applying the Banach fixed point theorem, we prove the existence and uniqueness of the solution. We establish the Ulam-Hyers and Ulam-Hyers-Rassias stability of the fractional system. Regarding controllability, we prove the associated linear system is controllable if and only if its scalar controllability Gramian is strictly positive, and use the Schauder theorem to establish the controllability of the nonlinear case. Finally, we provide examples to illustrate the results.

Keywords. Ulam-Hyers stability, Ulam-Hyers-Rassias stability, Controllability, Fractional Dynamical Systems, Caputo-Fabrizio Fractional Derivative.
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1. Introduction

Fractional differential theory began in 1695. It is a generalization of integer-order derivatives and is characterized by its memory property. This property enables it to describe physical and biological phenomena that ordinary derivatives cannot explain. Therefore, fractional derivatives have found wide use in practical applications, such as bioengineering, physics, and viscoelasticity. Classical fractional derivatives have a singular kernel that is not always convenient. So, new derivatives without a singular kernel were proposed [6, 4, 10, 12].

Over the past few years, the stability of fractional differential systems has received increasing attention. The analysis of the stability of dynamical systems (integer and fractional orders) is a crucial qualitative principle in numerical computation and optimization problems. Over the past decades, various types of stability have been studied, such as Mittag-Leffler stability, exponential stability, and Lyapunov stability. In this context, Ulam and Hyers introduced a new stability known as Ulam-Hyers stability, which has broad applications in chemistry,

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physics, biology, and engineering. Researchers have demonstrated the stability for differential equations as well as ordinary and fractional difference equations [15, 20, 19] and the references therein.

Controllability is a fundamental and pivotal concept in modern mathematical control theory, first introduced by Kalman in 1960. It quickly developed into a vibrant field of research and a cornerstone of control theory for dynamical systems. The crucial importance of controllability lies in its direct impact on the design and analysis of various control systems, whether deterministic or stochastic. In general, controllability is demonstrated by the ability of an appropriate control function to steer any arbitrary initial state to any targeted final state. The principle of controllability has been extended to encompass control problems formulated in terms of abstract differential equations. It has been studied using various methods operating in both finite- and infinite-dimensional spaces. Many mathematicians have investigated the controllability results of linear and nonlinear dynamical systems [18, 13, 14, 2].

Based on the above and given the importance of controllability and stability of fractional derivative systems, many researchers have presented valuable studies in this direction, including, but not limited to, those presented below. In this context, Ali et al. [1] presented a study on the controllability and stability of Ulam-Hyers linear differential equations with variable coefficients and fractional derivatives as defined by Riemann-Liouville in Banach spaces. Among the related studies, the contribution of Almarri et al. [3] stands out; by using Krasnoselskii's fixed point theory, they were able to deduce sufficient conditions that guarantee both controllability and Ulam-Hyers stability in the context of nonlinear partial delay systems. In [17], Selvam and Govindaraj turned their attention to a class of fractional dynamical systems formulated in terms of the Ψ-Caputo derivative, where they analyzed the properties of controllability and stability according to the Ulam-Hyers criterion, taking into account the presence of a time delay in the control. In [5], Bharath et al. investigated the controllability and Ulam-Hyers-Rassias stability of semilinear stochastic fractional functional differential equations with impulsive effects and piecewise Caputo derivatives. In [9], Guendouz et al. proved the existence and Ulam-Hyers stability of the solutions of a coupled system of impulsive Caputo fractional systems. In [11], Hettadj and Djourdem studied the existence and uniqueness of solutions for a Caputo fractional system.

This paper aims to investigate the controllability, Ulam-Hyers stability, and Ulam-Hyers-Rassias stability of the following Caputo-Fabrizio fractional nonlinear system with delay:

$$\begin{cases} \dot{x}(t) + {}^{CF}D^{\gamma}x(t) = f\left(t, x_t\right), & t \in [0, b] \\ x(t) = \varphi(t), & t \in [-a, 0] \end{cases} \tag{1.1}$$

${}^{CF}D^{\gamma}$ refers to the fractional derivative of the Caputo-Fabrizio type, whose order is γ, $0 < \gamma < 1$, $x(t) \in \mathbb{R}$ is a scalar state, x_t is the delay function given by $x_t(\tau) = x(t+\tau)$ where $\tau \in [-a, 0]$, $f : [0, b] \times C\left([-a, 0], \mathbb{R}\right) \to \mathbb{R}$ is a continuous nonlinear function, $\varphi \in C([-a, 0], \mathbb{R})$ is the initial function.

The rest of this work proceeds as outlined below. The second section was devoted to presenting the basic definitions and theories on which the research is based, while the third section dealt with deriving the conditions sufficient to prove the existence and uniqueness of the solution to the Caputo-Fabrizio fractional-order differential system (1.1) together with a set of conditions ensuring Ulam-Hyers stability. The fourth section was concerned with verifying the controllability of linear and nonlinear fractional systems. Finally, the fifth section offered three examples illustrating the theorems established throughout the paper.

2. Preliminaries

In this section, we present basic definitions and lemmas relevant to our work. Let $\mathcal{X} = C([-a, b], \mathbb{R})$ be the space of all continuous functions from $[-a, b]$ into $\mathbb{R}$. If this space is equipped with the norm $\|x\|_{\mathcal{X}} = \sup_{t\in[-a,b]} |x|$, then the resulting pair $(\mathcal{X}, \|.\|_{\mathcal{X}})$ is a Banach space.

Definition 2.1. [6] The Caputo-Fabrizio fractional derivative is given by

$$\left({}^{CF}D^{\alpha}_{\widehat{a}}x\right)(t) = \frac{1}{1-\alpha}\int_{\widehat{a}}^{t} \exp\left[-\frac{\alpha}{1-\alpha}(t-s)\right]\left(\frac{d}{ds}x\right)(s)\, ds.$$

Definition 2.2. [7] The Caputo-Fabrizio fractional integral is given by

$$\left(\mathcal{I}^{\alpha}_{\widehat{a}}x\right)(t) = \frac{1}{\alpha}\int_{\widehat{a}}^{t} \exp\left[-\frac{1-\alpha}{\alpha}(t-s)\right]x(s)\, ds.$$

Lemma 2.3. [7] *The Laplace transform of the Caputo-Fabrizio fractional derivative is given by*

$$\mathcal{L}\left\{\left({}^{CF}D^{\alpha}_{\widehat{a}}x\right)(t)\right\}(\lambda) = \left[\frac{\lambda\mathcal{L}\left\{x\left(t\right)\right\}(\lambda) - x(0)}{(1-\alpha)\lambda + \alpha}\right]$$

.

Definition 2.4. [20] We say that the system (1.1) has Ulam-Hyers stability if there is a positive real number k such that if $y \in C^1([-a, b], \mathbb{R})$, $\epsilon > 0$,

$$\left\|\dot{y}(t) + {}^{CF}D^{\gamma}y(t) - f\left(t, y_t\right)\right\| < \epsilon,$$

then the system (1.1) admits a solution $x \in \mathcal{X}$ satisfying

$$\|x(t) - y(t)\| \leq k\epsilon, t \in [-a, b].$$

Definition 2.5. [16] The system (1.1) is said to be Ulam-Hyers-Rassias stable relative to $\psi \in C\left([-a, b], R_+\right)$ if a positive constant $K_{f,\psi}$ satisfying for every $\epsilon > 0$ and every $y \in C^1([-a, b], \mathbb{R})$ the inequality

$$\left\|\dot{y}(t) + {}^{CF}D^{\gamma}y(t) - f\left(t, y_t\right)\right\| < \epsilon\psi(t), t \in [-a, b],$$

then there exists a solution $x \in \mathcal{X}$ of the system (1.1) such that

$$\|x(t) - y(t)\| \leq K_{f,\psi}\epsilon\psi(t), t \in [-a, b].$$

Now, we prove the following lemma, which plays a key role in providing the solution to the system (1.1):

Lemma 2.6. *Let* $f : [0,\ b] \times C([-a,\ 0],\ \mathbb{R}) \to \mathbb{R}$ *be continuous. If* $x \in C([-a,\ b],\ \mathbb{R})$ *satisfies the integral equation*

$$x(t) = \varphi(0) + \int_0^t K(t,\ \xi)\ f(\xi,\ x_\xi)\ d\xi, t \in [0,\ b],$$

then $x \in C([-a,\ b],\ \mathbb{R})$, *and*

$$x'(t) = f(t,\ x_t) - \left(\frac{c}{1+\gamma}\right) \int_0^t e^{-c(t-\xi)} f(\xi,\ x_\xi)\ d\xi,$$

where

$$c = \frac{\gamma+1}{1-\gamma}.$$

Proof. Since x is continuous on the compact interval $[-a,\ b]$, it is uniformly continuous by the Heine-Cantor Theorem. Hence

$$\xi_n \to \xi \Rightarrow \sup_{\tau \in [-a,0]} |x(\xi_n + \tau) - x(\xi + \tau)| \to 0,$$

so the map $\xi \mapsto x_\xi$ is continuous from $[0,\ b]$ into $C([-a,\ 0],\ \mathbb{R})$.

Define $g(\xi) := f(\xi,\ x_\xi)$. As a composition of the continuous map $\xi \mapsto x_\xi$ with the continuous function f, g is continuous on $[0,\ b]$.

Writing

$$K(t,\ \xi) = \frac{\gamma}{(1+\gamma)} + \frac{1}{(1+\gamma)}\ e^{-ct}\ e^{c\xi},$$

we obtain

$$x(t) = \varphi(0) + \frac{\gamma}{(1+\gamma)} \int_0^t g(\xi)d\xi + \frac{e^{-ct}}{1+\gamma}\ h(t),$$

$$h(t) := \int_0^t e^{c\xi} g(\xi)d\xi$$

Since g is continuous, both terms above are differentiable by the Fundamental Theorem of Calculus:

$$\frac{d}{dt} \int_0^t g(\xi)d\xi = g(t), \quad h'(t) = e^{ct}\ g(t).$$

Differentiating $x(t)$ and using

$$\frac{\gamma}{(1+\gamma)} + \frac{1}{(1+\gamma)} = 1$$

gives

$$x'(t) = \frac{\gamma}{(1+\gamma)}\, g(t) + \frac{1}{(1+\gamma)}[\, -c\, e^{-ct}\, h(t) + g(t)]$$

$$= f(t, x_t) - \frac{c}{(1+\gamma)} \int_0^t e^{-c(t-\xi)} f\,(\xi,\ x_\xi) d\xi.$$

Since g is continuous and the exponential convolution of a continuous function is continuous, x' is continuous. Hence $x \in C^1([-a,\ b],\ \mathbb{R})$. □

3. Stability of Fractional Nonlinear System

This section investigates the Ulam-Hyers stability and Ulam-Hyers-Rassias stability for system (1.1).

The solution of the fractional nonlinear system (1.1) is given in the following result.

Theorem 3.1. *The solution of system (1.1) is given by*

$$\begin{cases} x(t) = \varphi(0) + \displaystyle\int_0^t \left[\frac{\gamma}{\gamma+1} + \frac{1}{1+\gamma} e^{-\left(\frac{\gamma+1}{1-\gamma}\right)(t-\xi)}\right] f\,(\xi, x_\xi)\, d\xi, & t \in [0, b] \\ x(t) = \varphi(t), & t \in [-a, 0] \end{cases}$$

Proof. Taking the Laplace transform on both sides of system (1.1), we obtain

$$\lambda X(\lambda) - \varphi(0) + \frac{\lambda X(\lambda) - \varphi(0)}{(1-\gamma)\lambda + \gamma} = F(\lambda),$$

where

$$X(\lambda) = \mathcal{L}\,\{x(t)\}\,(\lambda),$$

$$F(\lambda) = \mathcal{L}\,\{f\,(t, x_t)\}\,(\lambda)$$

$$\lambda X(\lambda) = \varphi(0) + \frac{(1-\gamma)\lambda + \gamma}{(1-\gamma)\lambda + \gamma + 1} F(\lambda)$$

we have

$$X(\lambda) = \frac{\varphi(0)}{\lambda} + \frac{(1-\gamma)\lambda + \gamma}{\lambda\,[(1-\gamma)\lambda + \gamma + 1]} F(\lambda)$$

$$X(\lambda) = \frac{\varphi(0)}{\lambda} + \left[\frac{\frac{\gamma}{\gamma+1}}{\lambda} + \frac{1}{1+\gamma}\left(\frac{1}{\lambda + \frac{\gamma+1}{1-\gamma}}\right)\right] F(\lambda).$$

Using the inverse of the Laplace transform on both sides, we have

$$x(t) = \varphi(0) + \int_0^t \frac{\gamma}{\gamma+1} f\,(\xi, x_\xi)\, d\xi + \int_0^t \frac{1}{1+\gamma} e^{-\left(\frac{\gamma+1}{1-\gamma}\right)(t-\xi)} f\,(\xi, x_\xi)\, d\xi.$$

Therefore

$$\begin{cases} x(t) = \varphi(0) + \displaystyle\int_0^t \left[\frac{\gamma}{\gamma+1} + \frac{1}{1+\gamma} e^{-\left(\frac{\gamma+1}{1-\gamma}\right)(t-\xi)}\right] f(\xi, x_\xi)\, d\xi, & t \in [0,b] \\ x(t) = \varphi(t), & t \in [-a,0] \end{cases}$$

□

Before establishing the existence and uniqueness theorem for the system's solution, we need to estimate the norm of the delay function x_t.

For any $t \in [0,b]$, the norm of the delay function satisfies:

$$\begin{aligned} \|x_t\| &= \sup_{\tau\in[-a,0]} |x(t+\tau)| = \sup_{s\in[t-a,t]} |x(s)| \\ &\leq \sup_{s\in[-a,b]} |x(s)| = \|x\|_{\mathcal{X}}, \end{aligned}$$

and, for all $\gamma \in (0,1)$,

$$\sup_{t\geq\xi} \left| e^{-\left(\frac{\gamma+1}{1-\gamma}\right)(t-\xi)} \right| = 1.$$

Theorem 3.2. *Assume that the continuous function $f : [0,b]\times C([-a,0],\mathbb{R}) \to \mathbb{R}$ satisfies the Lipschitz condition with respect to the second variable*

$$|f(t,m) - f(t,n)| \leq L\|m-n\|,$$

for all $t \in [0,b]$ and $m,n \in \mathcal{X}$, where L is a Lipschitz constant. Then provided that $bL < 1$, the system (1.1) has a unique solution in $\mathcal{X}$.

Proof. Define the operator $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ such that

$$\begin{cases} (\mathcal{T}x)(t) = \varphi(0) + \displaystyle\int_0^t \left[\frac{\gamma}{\gamma+1} + \frac{1}{1+\gamma} e^{-\left(\frac{\gamma+1}{1-\gamma}\right)(t-\xi)}\right] f(\xi, x_\xi)\, d\xi, & t \in [0,b] \\ x(t) = \varphi(t), & t \in [-a,0] \end{cases}$$

Let $x, y \in \mathcal{X}$

$$|(\mathcal{T}x)(t) - (\mathcal{T}y)(t)| = 0\ , \quad t \in [-a,0]$$

$$|(\mathcal{T}x)(t) - (\mathcal{T}y)(t)| = \left| \int_0^t \left[\frac{\gamma}{\gamma+1} + \frac{1}{1+\gamma} e^{-\left(\frac{\gamma+1}{1-\gamma}\right)(t-\xi)}\right] (f(\xi, x_\xi) - f(\xi, y_\xi))\, d\xi \right|, \ t \in [0,b]$$

$$\leq \int_0^t \left[\frac{\gamma}{\gamma+1} + \frac{1}{1+\gamma}\right] |f(\xi, x_\xi) - f(\xi, y_\xi)|\, d\xi$$

$$\leq bL\|x-y\|.$$

Since $bL < 1$, the operator $\mathcal{T}$ is a contraction, and we conclude that the system (1.1) has a unique solution $x \in \mathcal{X}$ according to the Banach fixed point theorem. □

In the following theorems, we present the stability results for system (1.1) according to the Ulam-Hyers and Ulam-Hyers-Rassias concepts.

Theorem 3.3. *Assume that the continuous function* $f : [0,b]\times C([-a,0],\mathbb{R}) \to \mathbb{R}$ *satisfies the Lipschitz condition with respect to the second variable*

$$|f(t,m) - f(t,n)| \leq L\|m-n\|,$$

for all $t \in [0,b]$ *and* $m,n \in \mathcal{X}$, *where* L *is a Lipschitz constant; moreover, if* $y \in C^1([-a,b],\mathbb{R})$ *satisfies*

$$\left\|\dot{y}(t) +^{CF} D^{\gamma}y(t) - f(t,y_t)\right\| < \epsilon$$

and

$$bL < 1,$$

then the system (1.1) *has a solution* $x \in \mathcal{X}$ *satisfying*

$$\|x(t) - y(t)\| \leq \frac{b\epsilon}{1-bL}.$$

This means the system (1.1) *has Ulam-Hyers stability.*

Proof. Define $\sigma_y : [-a,b] \to \mathbb{R}$ such that

$$\sigma_y(t) := \dot{y}(t) +^{CF} D^{\gamma}y(t) - f(t,y_t)$$

so that

$$\dot{y}(t) +^{CF} D^{\gamma}y(t) = f(t,y_t) + \sigma_y(t), \tag{3.1}$$

so that $\|\sigma_y\| < \epsilon$. By the same technique as in Theorem 3.1, we can get the solution of (3.1) as

$$y(t) = \varphi(0) + \int_0^t \frac{\gamma}{\gamma+1}\left[f(\xi,y_\xi) + \sigma_y(\xi)\right]d\xi + \int_0^t \frac{1}{1+\gamma}e^{-\left(\frac{\gamma+1}{1-\gamma}\right)(t-\xi)}\left[f(\xi,y_\xi) + \sigma_y(\xi)\right]d\xi$$

$$= \varphi(0) + \int_0^t \left[\frac{\gamma}{\gamma+1} + \frac{1}{1+\gamma}e^{-\left(\frac{\gamma+1}{1-\gamma}\right)(t-\xi)}\right] f(\xi,y_\xi)\,d\xi + \mathcal{R}(t)$$

where

$$\mathcal{R}(t) = \int_0^t \left[\frac{\gamma}{\gamma+1} + \frac{1}{1+\gamma}e^{-\left(\frac{\gamma+1}{1-\gamma}\right)(t-\xi)}\right]\sigma_y(\xi)d\xi.$$

Then

$$|\mathcal{R}(t)| \leq b\,|\sigma_y(\xi)| \leq b\epsilon.$$

Now

$$x(t)-y(t)=\int_0^t\left(\frac{\gamma}{\gamma+1}+\frac{1}{1+\gamma}e^{-\left(\frac{\gamma+1}{1-\gamma}\right)(t-\xi)}\right)f\left(\xi,x_\xi\right)d\xi$$
$$-\int_0^t\left(\frac{\gamma}{\gamma+1}+\frac{1}{1+\gamma}e^{-\left(\frac{\gamma+1}{1-\gamma}\right)(t-\xi)}\right)f\left(\xi,y_\xi\right)d\xi-\mathcal{R}(t)$$

$$=\int_0^t\left(\frac{\gamma}{\gamma+1}+\frac{1}{1+\gamma}e^{-\left(\frac{\gamma+1}{1-\gamma}\right)(t-\xi)}\right)\left(f\left(\xi,x_\xi\right)-f\left(\xi,y_\xi\right)\right)d\xi-\mathcal{R}(t)$$

then

$$|x(t)-y(t)|\leq b\left|f\left(\xi,x_\xi\right)-f\left(\xi,y_\xi\right)\right|+|\mathcal{R}(t)|$$

$$\leq bL\left\|x-y\right\|+b\epsilon$$

therefore

$$\left\|x-y\right\|(1-bL)\leq b\epsilon$$

$$\left\|x-y\right\|\leq\frac{b\epsilon}{1-bL}.$$

□

Theorem 3.4. *The system (1.1) has Ulam-Hyers-Rassias stability with a stability constant*

$$C_\psi=\frac{b}{1-bL}$$

where

$$\int_0^t\left[\frac{\gamma}{1+\gamma}+\frac{1}{\gamma+1}e^{-\frac{(\gamma+1)}{1-\gamma}(t-\xi)}\right]\psi(\xi)d\xi\leq C_\psi\psi(t),t\in[0,b].$$

Proof. Assume

$$\sigma_y(t)=\dot{y}(t)+\ {}^{CF}D^\gamma y(t)-f\left(t,y_t\right)$$

$$|\sigma_y(t)|\leq\varepsilon\psi(t)$$

$$y(t)=\varphi(0)+\int_0^t\left[\frac{\gamma}{1+\gamma}+\frac{1}{\gamma+1}e^{-\left(\frac{\gamma+1}{1-\gamma}\right)}(t-\xi)\right]\left[f\left(\xi,y_\xi\right)+\sigma(\xi)\right]d\xi$$

$$=\varphi(0)+\int_0^t\left[\frac{\gamma}{1+\gamma}+\frac{1}{\gamma+1}e^{-\left(\frac{\gamma+1}{1-\gamma}\right)(t-\xi)}\right]f\left(\xi,y_\xi\right)d\xi$$
$$+\int_0^t\left[\frac{\gamma}{1+\gamma}+\frac{1}{\gamma+1}e^{-\left(\frac{\gamma+1}{1-\gamma}\right)(t-\xi)}\right]\sigma(\xi)d\xi$$

$$= \varphi(0) + \int_0^t \left[\frac{\gamma}{1+\gamma} + \frac{1}{\gamma+1} e^{-\left(\frac{\gamma+1}{1-\gamma}\right)(t-\xi)}\right] f(\xi, y_\xi)\, d\xi + R(t)$$

then

$$x(t) - y(t) = \int_0^t \left[\frac{\gamma}{1+\gamma} + \frac{1}{\gamma+1} e^{-\left(\frac{\gamma+1}{1-\gamma}\right)(t-\xi)}\right] [f(\xi, x_\xi) - f(\xi, y_\xi)]\, d\xi - R(t)$$

.

Since

$$|R(t)| \leq b\psi(t)\varepsilon$$

,

therefore

$$|x(t) - y(t)| \leq b\,|f(\xi, x_\xi) - f(\xi, y_\xi)| + |R(t)|$$

$$\leq bL\|x - y\| + b\psi(t)\varepsilon$$

$$|x(t) - y(t)| - bL\|x - y\| \leq b\varepsilon\psi(t)$$

$$\|x - y\| \leq \frac{b\psi(t)\varepsilon}{1 - b\ L}.$$

□

4. Controllability of Fractional Dynamical Systems

4.1. **Controllability of a Fractional Linear System.** Here, we investigate the controllability of a fractional linear system with time delay in the initial condition

$$\begin{cases} \dot{x}(t) + {}^{CF}D^{\gamma}x(t) = u(t), & t \in J = [0, b], \\ x(t) = \varphi(t), & t \in [-a, 0]. \end{cases} \tag{4.1}$$

where the components of the system are as defined in system (1.1), and $u(t) \in \mathbb{R}$ is the control function. By the same technique as in Theorem 3.1, we can find the solution of system (4.1) as

$$x(t) = \varphi(0) + \int_0^t K(t, \xi) u(\xi) d\xi \tag{4.2}$$

where

$$K(t, \xi) = \frac{\gamma}{1+\gamma} + \frac{1}{1+\gamma} e^{-\left(\frac{\gamma+1}{1-\gamma}\right)(t-\xi)}.$$

Definition 4.1. The system (4.1) is said to be controllable on the interval J if every initial state $\varphi(0)$ can be steered to any final state x_1 by an appropriate control u so that the corresponding solution x of the system 4.1 satisfies $x(b) = x_1$.

Theorem 4.2. *The linear system (4.1) with kernel $K(t,\xi)$ is controllable if and only if the scalar Gramian*

$$W = \int_0^b K(b,\xi)^2 d\xi > 0.$$

Proof. Assume that $W > 0$. Define the control function in the form

$$u(\xi) = K(b,\xi)W^{-1}\left[x_1 - \varphi(0)\right]$$

By substituting t in (4.2) by the final time b, we get

$$\begin{aligned} x(b) &= \varphi(0) + \int_0^b K(b,\xi)K(b,\xi)W^{-1}\left[x_1 - \varphi(0)\right] d\xi \\ &= \varphi(0) + WW^{-1}\left[x_1 - \varphi(0)\right] \\ &= x_1 \end{aligned}$$

Conversely, if $W = 0$ since $K(b,\xi)$ is continuous and nonnegative on $[0,b]$, then

$$W = \int_0^b K(b,\xi)^2 d\xi = 0$$

means $K(b,\xi) = 0$ for every $\xi \in [0,b]$, but

$$K(b,\xi) = \left[\frac{\gamma}{1+\gamma} + \frac{1}{1+\gamma}e^{-\left(\frac{\gamma+1}{1-\gamma}\right)(b-\xi)}\right] \geq \frac{\gamma}{1+\gamma} > 0 \quad \text{for all} \quad \xi \in [0,b]$$

which means a contradiction. Hence $W > 0$. □

4.2. **Controllability of a Fractional Nonlinear Dynamical System.** This section investigates the controllability of a fractional nonlinear system with time delay

$$\begin{cases} \dot{x}(t) + {}^{CF}D^{\gamma}x(t) = u(t) + f\left(t, x_t\right), \ t \in [0,b] \\ x(t) = \varphi(t), \quad t \in [-a,0] \end{cases} \tag{4.3}$$

where the components of the system are as defined in system (1.1), and $u(t) \in \mathbb{R}$ is the control function, For each $y \in \mathcal{X}$, consider the linear system

$$\begin{cases} \dot{x}(t) + {}^{CF}D^{\gamma}x(t) = u(t) + f\left(t, y_t\right), & t \in [0,b] \\ x(t) = \varphi(t), & t \in [-a,0]\,. \end{cases} \tag{4.4}$$

Then, the solution of system (4.4) is

$$x(t) = \varphi(0) + \int_0^t K(t,s)u(s)ds + \int_0^t K(t,s)f\left(s, y_s\right) ds$$

where

$$K(t,\xi) = \frac{\gamma}{1+\gamma} + \frac{1}{1+\gamma}e^{-\left(\frac{\gamma+1}{1-\gamma}\right)(t-\xi)}.$$

The controllability scalar Gramian and control function are described by

$$W = \int_0^b K(b,s)^2 ds$$

$$u(t) = K(b,t)W^{-1}\left(x_1 - \varphi(0) - \int_0^b K(b,s) f\left(s, y_s\right) ds\right).$$

In the following theorem, we prove the controllability of system (4.3).

Theorem 4.3. *Assume that the continuous function f satisfies the condition*

$$\lim_{\|x_t\|\to\infty} \frac{|f\left(t, x_t\right)|}{\|x_t\|} = 0$$

uniformly in J, where $\|x_t\| = \sup_{\tau\in[-a,0]} |x(t+\tau)|$, and suppose the scalar Gramian $W > 0$. Then the system (4.3) is controllable on J.

Proof. We now define an operator $P : \mathcal{X} \to \mathcal{X}$ given by

$$P(y) = x$$

where

$$u(t) = K(b,t)W^{-1}\left(x_1 - \varphi(0) - \int_0^b K(b,s) f\left(s, y_s\right) ds\right), t \in J$$

and

$$x(t) = \varphi(0) + \int_0^t K(t,s)u(s)ds + \int_0^t K(t,s) f\left(s, y_s\right) ds.$$

Set $c_1 = max\{bW^{-1}, b\}, d_1 = max\left\{ W^{-1}\left(|x_1| + |\varphi(0)|\right), |\varphi(0)|\right\}$

$$\|f\| = \sup_{t\in[0,b]} |f\left(t, y_t\right)|, \ d = d_1(1+b), c = b\left(c_1 + 1\right)$$

since $0 < K(b,s) \leq 1$,

$$|u_y(t)| \leq W^{-1}|x_1 - \varphi(0)| + W^{-1}b\|f\|$$

$$|u_y(t)| \leq \left(d_1 + c_1\|f\|\right), t \in [0,b].$$

$$|x(t)| \leq |\varphi(0)| + \int_0^b |u(s)|ds + \int_0^b |f\left(s, y_s\right)| ds$$

$$\leq d_1 + b\left(d_1 + c_1\|f\|\right) + b\|f\|$$

$$= d_1 + d_1 b + bc_1\|f\| + b\|f\|$$

$$= d_1(1+b) + \|f\|b\left(c_1 + 1\right)$$

, therefore

$$|x(t)| \leq d + c \|f\| .$$

Define the growth function:

$$M(r) = \sup \{|f(t,\xi)| : t \in [0,b], \| \xi \| \leq r\} .$$

By the growth hypothesis on f, $M(r)$ grows slower than r:

$$\frac{M(r)}{r} \to 0, \text{where } r \to \infty$$

, hence there exists $r_0 > 0$ such that: $d + c \cdot M(r_0) \leq r_0$.

This is the standard Schauder-type argument [8] since $d + c \cdot M(r)$ grows strictly slower than r, the two curves must cross, guaranteeing such an r_0.

For this r_0, if $\| y \| \leq r_0$, then:

$$|x(t)| \leq d + c \cdot M(r_0) \leq r_0.$$

Define the ball $B_{r_0} = \{y \in \mathcal{X} : \|y\| \leq r_0\}$. Then the operator P maps B_{r_0} into itself. As f is continuous and uniformly bounded for all $t \in J$, then the operator P is continuous and uniformly bounded on J. Therefore, we have $\|x(t_1) - x(t_2)\|$ tends to zero when $t_1 \to t_2$ for all $t_1, t_2 \in J$ with, $t_1 < t_2$ and for all $y \in \mathcal{X}$. Thus $P(B_{r_0})$ is equicontinuous and therefore the operator P is compact by the Arzelà-Ascoli theorem. Thus P is completely continuous. We get the operator P has a fixed point $y \in B_{r_0}$ according to the Schauder fixed point theorem since B_{r_0} is closed, bounded, and convex. That is

$$P(y) = x = y.$$

Therefore

$$x(t) = \varphi(0) + \int_0^t K(t,s)u(s)ds + \int_0^t K(t,s)f(s,x_s)\,ds$$

which means that $x(t)$ is the solution of system (4.3). Furthermore

$$x(b) = \varphi(0) + \int_0^b K(b,s)u(s)ds + \int_0^b K(b,s)f(s,y_s)\,ds$$

$$= \varphi(0) + \int_0^b K(b,s)f(s,y_s)\,ds + \int_0^b K(b,s)^2 W^{-1}$$

$$\times \left(x_1 - \varphi(0) - \int_0^b K(b,z)f(z,y_z)\,dz\right) ds$$

$$= \varphi(0) + \int_0^b K(b,s)f(s,y_s)\,ds + W^{-1}\left(x_1 - \varphi(0) - \int_0^b K(b,z)f(z,y_z)\,dz\right)$$

$$\times \int_0^b K(b,s)^2 ds$$

$$= \varphi(0) + \int_0^b K(b,s)f(s,y_s)\,ds + x_1 - \varphi(0) - \int_0^b K(b,z)f(z,y_z)\,dz$$

$$= x_1.$$

Therefore, the system (4.3) is controllable on J. □

5. Applications

Example 5.1. Consider the fractional system

$$\begin{cases} \dot{x}(t) + {}^{CF}D^{\frac{1}{2}}x(t) = \frac{1}{4}\sin\left(x(t-1)\right), & t \in [0,2] \\ x(t) = \frac{\pi}{2}, & t \in [-1,0]. \end{cases} \tag{5.1}$$

Then, $f\left(t, x_t\right) = \frac{1}{4}\sin\left(x(t-1)\right)$ satisfies the Lipschitz condition

$$|f(t,m) - f(t,n)| \leq \frac{1}{4}|m-n|, \qquad L = \frac{1}{4}$$

Since $b = 2$, we get

$$bL = 2 \times \frac{1}{4} = \frac{1}{2} < 1$$

By Theorem 3.1, the solution is given by:

$$x(t) = \varphi(0) + \int_0^t K(t,\xi)\, f\left(\xi, x_\xi\right)\, d\xi, t \in [0,2]$$

$$K(t,\xi) = \frac{\gamma}{\gamma+1} + \frac{1}{1+\gamma}e^{-c(t-\xi)} = \frac{1}{3} + \frac{2}{3}e^{-3(t-\xi)}$$

with initial condition $\varphi(0) = \frac{\pi}{2}$ and

$$f\left(\xi, x_\xi\right) = \frac{1}{4}\sin\left(x(\xi-1)\right).$$

The solution takes the following integral form:

$$x(t) = \frac{\pi}{2} + \frac{1}{4}\int_0^t \left(\frac{1}{3} + \frac{2}{3}e^{-3(t-\xi)} sin\left(x(\xi-1)\right)\, d\xi\right), t \in [0,2]$$

Solution on $t \in [0,1]$:

Here $\xi \in [0,t] \subseteq [0,1]$, so $\xi - 1 \in [-1,0]$, meaning $x(\xi-1) = \varphi(\xi-1) = \frac{\pi}{2}$. Hence $\sin\left(x(\xi-1)\right) = sin(\pi/2) = 1$ is constant, and the equation becomes a direct (non-implicit) integral:

$$x(t) = \frac{\pi}{2} + \frac{1}{4}\int_0^t \left(\frac{1}{3} + \frac{2}{3}e^{-3(t-\xi)}\right) d\xi$$

.

Explicit closed-form solution on $[0,1]$:

$$x(t) = \frac{\pi}{2} + \frac{t}{12} + \frac{1}{18} - \frac{1}{18}e^{-3t}, \qquad t \in [0,1]$$

At $t = 1$:

$$x(1) = \frac{\pi}{2} + \frac{5}{36} - \frac{1}{18}e^{-3} \approx 1.706919$$

Solution on $t \in [1,2]$:
Since $x(s) = \frac{\pi}{2} + g(s)$ for $s \in [0,1]$, we have $\sin\left(x(\xi-1)\right) = \sin\ \left(\frac{\pi}{2} + g(\xi-1)\right)$

$$= cos\left(g(\xi-1)\right).$$

$$x(t) = \frac{\pi}{2} + \frac{1}{4}\left[\frac{1}{3} + \frac{2}{9}\left(e^{3-3t} - e^{-3t}\right)\right] + \frac{1}{4}\int_0^{t-1}\left(\frac{1}{3} + \frac{2}{3}e^{-3(t-1-s)}\right)\cos\left(g(s)\right)\, ds, t \in [1,2]$$

where:

$$g(s) = \frac{s}{12} + \frac{1}{18} - \frac{1}{18}e^{-3s}.$$

Hence, by Theorem 3.2, system (5.1) has a unique solution in $\mathcal{X}$.
Moreover, since $bL < 1$, Theorem 3.3 gives the stability constant

$$k = \frac{b}{1-bL} = \frac{2}{1-\frac{1}{2}} = 4$$

That means the fractional system (5.1) is Ulam-Hyers stable.

Example 5.2. Let us consider the fractional linear system

$$\begin{cases} \dot{x}(t) + {}^{CF}D^{\frac{1}{2}}x(t) = u(t) & t \in [0,\pi] \\ x(t) = 1, & t \in \left[-\frac{\pi}{2}, 0\right]. \end{cases} \tag{5.2}$$

Then, we have

$$K(t,\xi) = \frac{1}{3} + \frac{2}{3}e^{-3(t-\xi)}$$

and the scalar Gramian

$$W = \int_0^{\pi}\left(\frac{1}{3} + \frac{2}{3}e^{-3(\pi-\xi)}\right)^2 d\xi$$

.

Set $\eta = \ \pi - \xi$:

$$W = \int_0^{\pi}\ \left(\tfrac{1}{3} + \tfrac{2}{3}e^{-3(\eta)}\right)^2 d\eta$$

$$= \frac{\pi}{9} + \frac{4}{27}\left(1 - e^{-3\pi}\right) + \frac{2}{27}\left(1 - e^{-6\pi}\right) > 0$$

Therefore, the condition is met, and the system (5.2) is controllable.

Example 5.3. Consider the nonlinear system

$$\begin{cases} \dot{x}(t) + {}^{CF}D^{\frac{1}{2}}x(t) = u(t) + sin\left(x(t)\right), & t \in [0,\pi] \\ x(t) = 1, & t \in \left[-\frac{\pi}{2}, 0\right]. \end{cases} \tag{5.3}$$

Then, we use the kernel

$$K(t,\xi) = \frac{1}{3} + \frac{2}{3}e^{-3(t-\xi)}$$

with scalar Gramian in example 5.2. Since $f(t, x_t) = sin(x(t))$ is bounded, $|f(t, x_t)| \leq 1$, so

$$\lim_{\|x_t\|\to\infty} \frac{|f(t, x_t)|}{\| x_t \|} \leq \lim_{\|x_t\|\to\infty} \frac{1}{\| x_t \|} = 0.$$

Therefore, the condition of Theorem 4.3 is met, and since $W > 0$, the system (5.3) is controllable on $[0, \pi]$.

6. Conclusion

In this study, we investigated the stability and controllability of a nonlinear fractional differential system with a time delay, defined in terms of the Caputo-Fabrizio derivative. First, using the Laplace transform, we derived an explicit integral formula for the solution of the system in terms of a given kernel that depends on the derivative order γ. Second, using Banach fixed point theorem, we proved the existence and uniqueness of the solution under a Lipschitz condition on the function f, with a sufficient condition relating the length of the time interval to the Lipschitz constant. Third, we proved the Ulam-Hyers stability of the system: if the approximate solution is close to satisfying the equation, then there exists an exact solution of the system close to this approximate solution, with a distance directly proportional to the margin of error. Similarly, we proved the Ulam-Hyers-Rassias stability, a broader generalization based on a weight function. Fourth, regarding controllability:

For a linear system, we proved that the system is controllable if and only if the scalar controllability Gramian is strictly positive. For a nonlinear system, using Schauder fixed point theorem and a certain growth condition on the function f, we also proved that the system is controllable. Fifth, Example 5.1 confirmed the existence, uniqueness, and stability of a nonlinear system involving a sine function, while Examples 5.2 and 5.3 confirmed the controllability of a linear and a nonlinear system, respectively.

These results show that the Caputo-Fabrizio derivative, thanks to its non-singular exponential kernel, provides a flexible and explicitly computable mathematical framework for studying the stability and controllability of fractional systems with delays, compared to derivatives such as the Riemann-Liouville or classical Caputo derivatives.

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