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INFINITELY MANY SOLUTIONS FOR VARIABLE–EXPONENT ELLIPTIC SYSTEMS

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Abstract. We prove the existence of a sequence of nontrivial weak solutions converging to zero for an elliptic system involving the $p(x)$-Laplacian-like operator, coupled with nonlinear boundary conditions. Our technique is based on variational arguments and a symmetric version of the mountain pass theorem.

Keywords. $p(x)$-Laplacian-Like operator, Nonlinear elliptic systems, Symmetric mountain pass lemma, Infinitely many solutions, Variational methods.
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1. Introduction and main result

Variational methods and critical point theory are powerful tools for studying the existence and multiplicity of solutions to nonlinear partial differential equations; see, for instance, [1, 4, 19, 20, 25, 26, 27] and the references therein.

In this work, we use these methods to establish an existence result for infinitely many nontrivial weak solutions to the following nonlinear elliptic system:

$$\begin{cases} \Delta^l_{p(x)} u = |u|^{p(x)-2}u, \quad \Delta^l_{q(x)} v = |v|^{q(x)-2}v & \text{in } \Omega, \\ \mathcal{B}_{p(x)} u = F_u(x,u,v), \quad \mathcal{B}_{q(x)} v = F_v(x,u,v) & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$ and the divergence terms are governed by the so-called $\theta(x)$-Laplacian-like operator, acting from $W^{1,\theta(x)}(\Omega)$ into its dual $W^{-1,\theta'(x)}(\Omega)$, defined by

$$\Delta^l_{\theta(x)} w = \operatorname{div}\left(\left(1 + \frac{|\nabla w|^{\theta(x)}}{\sqrt{1+|\nabla w|^{2\theta(x)}}}\right)|\nabla w|^{\theta(x)-2}\nabla w\right). \tag{1.2}$$

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To formulate the boundary conditions, we introduce the operator $\mathcal{B}_{\theta(x)}$ defined by

$$\mathcal{B}_{\theta(x)}w = \left(\left(1 + \frac{|\nabla w|^{\theta(x)}}{\sqrt{1+|\nabla w|^{2\theta(x)}}}\right)|\nabla w|^{\theta(x)-2}\nabla w\right).\eta \tag{1.3}$$

where η denotes the unit outward normal vector to $\partial\Omega$.

Since the end of the last century, numerous studies have highlighted the importance of problems with nonstandard growth conditions in modeling various phenomena arising in elastic mechanics [36], image restoration [2, 10], porous media [5, 23], electromagnetism [6, 35] or electrorheological [3, 11, 21, 31]. The natural framework for solving such problems is that of generalized Lebesgue and Sobolev spaces, whose origins date back to the pioneering work of Orlicz [32]. Many studies have investigated elliptic systems under both standard and nonstandard growth conditions. We refer the reader to [8, 9, 12, 14] and the references therein. Our study is partly motivated by the work [7], where the authors established several multiplicity results under various assumptions for the following p-Laplacian system:

$$-\Delta_p u = |u|^{p-2}u, \quad -\Delta_q v = |v|^{q-2}v \text{ in } \Omega,$$

with the nonlinear coupling at the boundary given by

$$|\nabla u|^{p-2}\frac{\partial u}{\partial \eta} = F_u(x,u,v), \quad |\nabla v|^{q-2}\frac{\partial v}{\partial \eta} = F_v(x,u,v) \quad \text{on } \partial\Omega.$$

More recently, [34] extended the results of [7] by establishing the existence of nontrivial solutions, via a variant of the mountain pass theorem, for the following problem:

$$\Delta_{p(x)} u = |u|^{p(x)-2}u, \quad \Delta_{q(x)} v = |v|^{q(x)-2}v \text{ in } \Omega,$$

with the nonlinear coupling at the boundary given by

$$|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \eta} = F_u(x,u,v), \quad |\nabla v|^{q(x)-2}\frac{\partial v}{\partial \eta} = F_v(x,u,v), \text{ on } \partial\Omega.$$

The main novelty of this work lies in extending the above results to a problem involving the more general $\theta(x)$-Laplacian-like operator introduced in (1.2), which encompasses both the θ-Laplacian and the $\theta(x)$-Laplacian as particular cases.

Definition 1.1. We say that $(u,v) \in W^{1,p(x)}(\Omega) \times W^{1,q(x)}(\Omega)$ is a weak solution to (1.1) if

$$\begin{aligned}
&\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)\nabla z dx + \int_\Omega |u|^{p(x)-2}uz dx \\
&+ \int_\Omega \left(|\nabla v|^{q(x)-2}\nabla v + \frac{|\nabla v|^{2q(x)-2}\nabla v}{\sqrt{1+|\nabla v|^{2q(x)}}}\right)\nabla w dx + \int_\Omega |v|^{q(x)-2}vw dx \\
&- \int_{\partial\Omega} \frac{\partial F}{\partial u}(x,u,v)z dx - \int_{\partial\Omega} \frac{\partial F}{\partial v}(x,u,v)w dx = 0
\end{aligned} \tag{1.4}$$

for all $(z,w) \in W^{1,p(x)}(\Omega) \times W^{1,q(x)}(\Omega)$.

Before stating our main result concerning the existence of infinitely many nontrivial solutions to Problem (1.1), we impose the following conditions on F:

$$F \in C^1\left(\partial\Omega \times \mathbb{R}^2, \mathbb{R}\right) \text{ and } F(x,0,0) = 0; \tag{1.5}$$

$$F(x,-s) = F(x,s), x \in \partial\Omega, \text{ and } s \in \mathbb{R}^2. \tag{1.6}$$

In view of the Sobolev trace embedding, we assume that

$$|F(x,u,v)| \leq c_1 + c_2|u|^{p_1(x)} + c_3|v|^{q_1(x)}; \tag{1.7}$$

$$|\frac{\partial F}{\partial u}(x,u,v)| \leq c_4 + c_5|u|^{p_1(x)-1} + c_6|v|^{q_1(x)-1}; \tag{1.8}$$

$$|\frac{\partial F}{\partial v}(x,u,v)| \leq c_7 + c_8|u|^{p_1(x)-1} + c_9|v|^{q_1(x)-1} \tag{1.9}$$

for all $(x,u,v) \in (\partial\Omega, \mathbb{R}^2)$, where $p_1, q_1 \in C_+(\bar{\Omega})$ satisfy

$$p_1(x) < p_\partial^*, \quad q_1(x) < q_\partial^* \quad \text{for all } x \in \bar{\Omega}$$

and

$$p_1^+ < p^-, \quad q_1^+ < q^-.$$

Theorem 1.2. *Under assumptions (1.5)-(1.9), Problem (1.1) admits infinitely many nontrivial weak solutions.*

The remainder of this paper is organized as follows: In Section 2, we state elementary properties of the generalized Lebesgue–Sobolev spaces and embedding results. We also collect the main tools and auxiliary results needed in the proof. In Section 3, we establish our main result, namely Theorem 1.2.

2. Variational framework

In this section, we recall some fundamental results concerning Lebesgue–Sobolev spaces with variable exponents. For a detailed exposition of this theory, including definitions, properties, and advanced results, we refer the reader to Fan and Zhang [18] and Kováčik and Rákosník [31].

Let Ω be an open bounded domain in $\mathbb{R}^N$ $(N \geq 2)$, with smooth boundary $\partial\Omega$ and $\text{meas}(\Omega) > 0$. We define

$$C_+(\bar{\Omega}) = \{p : p \in C(\bar{\Omega}),\ p(x) > 1,\ \text{for all } x \in \bar{\Omega}\}.$$

For any function $p \in C_+(\bar{\Omega})$, we set

$$1 < p^- := \min_{x\in\bar{\Omega}} p(x) \leq p^+ := \max_{x\in\bar{\Omega}} p(x) < \infty$$

and we introduce the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) := \left\{u; u : \Omega \to \mathbb{R} \text{ is a measurable and } \int_\Omega |u|^{p(x)}dx < +\infty\right\},$$

endowed with the Luxemburg norm

$$|u|_{p(x)} := \inf\left\{\alpha > 0; \int_\Omega \left|\frac{u(x)}{\alpha}\right|^{p(x)} dx \leq 1\right\}$$

which is a separable and reflexive Banach space (see [13, 24, 28, 29, 31]), and its dual space is isomorphic to $L^{p'(.)}(\Omega)$, where $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$, for all $x\in\Omega$. For any $u\in L^{p(.)}(\Omega)$ and $v\in L^{p'(.)}(\Omega)$, the following Hölder type inequality holds true

$$\left|\int_\Omega uvdx\right|\leq\left(\frac{1}{p^-}+\frac{1}{(p')^-}\right)|u|_{p(x)}|v|_{p'(x)}\leq 2|u|_{p(x)}|v|_{p'(x)}. \tag{2.1}$$

Let us define the Sobolev space

$$W^{1,p(x)}(\Omega):=\left\{u\in L^{p(x)}(\Omega);|\nabla u|\in L^{p(x)}(\Omega)\right\}$$

equipped with the norm

$$\|u\|=\inf\left\{\alpha>0;\int_\Omega\left|\frac{\nabla u(x)}{\alpha}\right|^{p(x)}+\left|\frac{u(x)}{\alpha}\right|^{p(x)}dx\leq 1\right\},\quad\forall u\in W^{1,p(x)}(\Omega).$$

Throughout, let $p^*(x)$ be the critical Sobolev exponent of $p(x)$ defined by

$$p^*=\begin{cases}\frac{Np(x)}{N-p(x)}, & \text{if } p(x)<N\\ \infty, & \text{if } p(x)\geq N\end{cases},$$

$$p^*_\partial=\begin{cases}\frac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x)<N\\ \infty, & \text{if } p(x)\geq N\end{cases}.$$

The following propositions collect some fundamental properties of the space $W^{1,p(x)}(\Omega)$, as well as the embedding results that will be used throughout the paper.

Proposition 2.1 ([16, 18]). *$W^{1,p(x)}(\Omega)$ is separable reflexive Banach space.*

Proposition 2.2 ([18, 17]). *If $q\in C_+(\bar\Omega)$ and $q(x)<p^*(x)$ for any $x\in\bar\Omega$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous.*

Proposition 2.3 ([22]). *If $q\in C_+(\bar\Omega)$ and $q(x)<p^*_\partial(x)$ for any $x\in\bar\Omega$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\partial\Omega)$ is compact and continuous.*

The mapping

$$\rho(u):=\int_\Omega\left[|\nabla u|^{p(x)}+|u|^{p(x)}\right]dx,\forall u\in W^{1,p(x)}(\Omega), \tag{2.2}$$

plays a fundamental role in the analysis of variable exponent spaces, particularly in describing convergence properties and the behaviour of the norm.

The following propositions provide important relationships between the modular function $\rho(.)$ defined in (2.2) and the norm in $W^{1,p(x)}(\Omega)$,which will be used in the sequel.

Proposition 2.4 ([15]). *For $u,u_k\in W^{1,p(x)}(\Omega);k=1,2,\ldots$, we have:*

(i) $\|u\|\geq 1$ *implies* $\|u\|^{p^-}\leq\rho(u)\leq\|u\|^{p^+}$,
(ii) $\|u\|\leq 1$ *implies* $\|u\|^{p^-}\geq\rho(u)\geq\|u\|^{p^+}$,
(iii) $\|u_k\|\to 0$ *if and only if* $\rho(u_k)\to 0$,
(iv) $\|u_k\|\to\infty$ *if and only if* $\rho(u_k)\to\infty$.

Proposition 2.5 ([18]). *If $u,u_n\in L^{p(x)}(\Omega),n=1,2,\ldots$, then the following statements are mutually equivalent:*

(1) $\lim_{n\to\infty} |u_n - u|_{p(x)} = 0$,
(2) $\lim_{n\to\infty} \rho(u_n - u) = 0$,
(3) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to\infty} \rho(u_n) = \rho(u)$.

The key argument in the proof of Theorem 1.2 is the following version of the symmetric mountain pass theorem.

Theorem 2.6 ([30]). *Let E be an infinite dimensional Banach space and $I \in C^1(E, \mathbb{R})$ satisfy the following two assumptions:*

(A_1) *$I(\cdot)$ is even, bounded from below; $I(0) = 0$ and $I(u)$ satisfies the Palais-Smale condition (PS);*
(A_2) *For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u\in A_k} I(u) < 0$, where Γ_k denotes the family of closed symmetric subsets A of E such that $0 \notin A$ and $\gamma(A) \geq k$. Here*

$$\gamma(A) := \inf\left\{k \in \mathbb{N}; \exists\ h : A \to \mathbb{R}^k \backslash \{0\} \text{ such that } h \text{ is continuous and odd }\right\}$$

is the genus of A.

Then $I(\cdot)$ admits a sequence of critical points u_k such that $I(u_k) < 0; u_k \neq 0$ and $u_k \to 0$, as $k \to \infty$.

Another ingredient in the proof of our main result is the specific structure of the divergence operator defined in (1.2). To this end, consider the following functional:

$$\mathcal{K}(w) = \int_\Omega \frac{1}{\theta(x)}\left(|\nabla w|^{\theta(x)} + \sqrt{1 + |\nabla w|^{2\theta(x)}} - 1\right)dx, \quad w \in X := W^{1,\theta(x)}(\Omega).$$

It is readily seen that $\mathcal{K}$ is continuously Gâteaux differentiable on X, and its Gâteaux derivative at $w \in X$ is the functional $\mathcal{T} = \mathcal{K}'(w) \in X^*$, given by

$$\langle \mathcal{T}w, \varphi\rangle = \int_\Omega \left(|\nabla w|^{\theta(x)-2}\nabla w + \frac{|\nabla w|^{2\theta(x)-2}\nabla w}{\sqrt{1 + |\nabla w|^{2\theta(x)}}}\right)\nabla\varphi dx, \quad \forall w, \varphi \in X. \quad (2.3)$$

Definition 2.7. The mapping $\mathcal{T} : X \longrightarrow X^*$ is said to be of class (S_+), if for any sequence $(w_n) \subset X$ with $w_n \rightharpoonup w$ in X and

$$\limsup_{n\to\infty} \langle \mathcal{T}w_n, w_n - w\rangle \leq 0,$$

we have $w_n \to w$ strongly in X.

The following lemma summarizes the main properties of $\mathcal{T}$.

Lemma 2.8 (see [33], Proposition 3.1). *The operator $\mathcal{T} : X \longrightarrow X^*$ defined by*

$$\langle \mathcal{T}w, \varphi\rangle = \int_\Omega \left(|\nabla w|^{\theta(x)-2}\nabla w + \frac{|\nabla w|^{2\theta(x)-2}\nabla w}{\sqrt{1 + |\nabla w|^{2\theta(x)}}}\right)\nabla\varphi dx,$$

is continuous, bounded, strictly monotone, and of class (S_+).

3. Proof of the main result

Let us define the functional $I : W^{1,p(x)}(\Omega) \times W^{1,q(x)}(\Omega) \to \mathbb{R}$ by

$$\begin{aligned} I(u,v) = &\int_\Omega \frac{1}{p(x)}\big(|\nabla u|^{p(x)} + |u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}} - 1\big)dx \\ &+ \int_\Omega \frac{1}{q(x)}\big(|\nabla v|^{q(x)} + |v|^{q(x)} + \sqrt{1+|\nabla v|^{2q(x)}} - 1\big)dx \\ &- \int_{\partial\Omega} F(x,u,v)d\sigma. \end{aligned} \tag{3.1}$$

Moreover, we denote

$$\begin{aligned} J(u,v) \;=\; &\int_\Omega \frac{1}{p(x)}\big(|\nabla u|^{p(x)} + |u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}} - 1\big)dx \\ &+ \int_\Omega \frac{1}{q(x)}\big(|\nabla v|^{q(x)} + |v|^{q(x)} + \sqrt{1+|\nabla v|^{2q(x)}} - 1\big)dx \end{aligned}$$

and

$$F(u,v) = -\int_{\partial\Omega} F(x,u,v)d\sigma.$$

Then, for every (u,v) and (φ,ψ) in $W^{1,p(x)}(\Omega) \times W^{1,q(x)}(\Omega)$, we have

$$J'(u,v)(\varphi,\psi) = D_1J(u,v)(\varphi) + D_2J(u,v)(\psi)$$

and

$$F'(u,v)(\varphi,\psi) = D_1F(u,v)(\varphi) + D_2F(u,v)(\psi),$$

where

$$D_1J(u,v)(\varphi) = \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u\nabla\varphi + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\nabla\varphi + |u|^{p(x)-2}u\varphi\right)dx,$$

$$D_2J(u,v)(\psi) = \int_\Omega \left(|\nabla v|^{q(x)-2}\nabla v\nabla\psi + \frac{|\nabla v|^{2q(x)-2}\nabla v}{\sqrt{1+|\nabla v|^{2q(x)}}}\nabla\psi + |v|^{q(x)-2}v\psi\right)dx,$$

$$D_1F(u,v)(\varphi) = -\int_{\partial\Omega} \frac{\partial F}{\partial u}(x,u,v)\varphi dx,$$

$$D_2F(u,v)(\psi) = -\int_{\partial\Omega} \frac{\partial F}{\partial v}(x,u,v)\psi dx.$$

It is well known that, the functional $I(u,v)$ is well defined and C^1 and we have

$$I'(u,v)(\varphi,\psi) = D_1I(u,v)(\varphi) + D_2I(u,v)(\psi).$$

In what follows, we denote by $W_{p(x),q(x)}$ the product space $W^{1,p(x)}(\Omega)\times W^{1,q(x)}(\Omega)$ equipped with the norm

$$\|(u,v)\| = \|u\| + \|v\|.$$

Obviously, if $F(x,0,0) = F_u(x,0,0) = F_v(x,0,0) = 0$ for all $x \in \partial\Omega$, then $u \equiv v \equiv 0$ is a trivial solution of the system.

To prove Theorem 1.2, we verify that the symmetric mountain pass theorem (Theorem 2.6) applies. We begin with the following lemmas.

Lemma 3.1. *Under assumptions (1.5)-(1.7), the functional I is even, bounded from below and coercive, with $I(0,0)=0$.*

Proof. It is clear that I is even, and $I(0,0)=0$.
Next, by dropping the positive terms that are not needed in (3.1), we obtain

$$\begin{aligned} I(u,v) \geq & \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)}+|u|^{p(x)}\right)dx+\int_\Omega \frac{1}{q(x)}\left(|\nabla v|^{q(x)}+|v|^{q(x)}\right)dx \\ & -\int_{\partial\Omega} F(x,u,v)d\sigma \\ \geq & \frac{1}{p^+}\rho(u)+\frac{1}{q^+}\rho(v)-\int_{\partial\Omega} c_1+c_2|u|^{p_1(x)}+c_3|v|^{q_1(x)}d\sigma. \end{aligned}$$

Using Proposition 2.3 and 2.4, we obtain

$$\begin{aligned} I(u,v) \geq & \frac{1}{p^+}\min\{\|u\|^{p+}_{p(x)};\|u\|^{p-}_{p(x)}\}+\frac{1}{q^+}\min\{\|v\|^{q^+}_{q(x)};\|v\|^{q^-}_{q(x)}\} \\ & -C\left(\max\{\|u\|^{p_1^+}_{p(x)};\|u\|^{p_1^-}_{p(x)}\}+\max\{\|v\|^{q_1^+}_{q(x)};\|v\|^{q_1^-}_{q(x)}\}\right). \end{aligned}$$

Since $p_1^+<p^-$and $q_1^+<q^-$, it follows that I is bounded from below and coercive. □

Lemma 3.2. *Let (u_n,v_n) be a Palais-Smale sequence for the Euler-Lagrange functional I. If conditions (1.5)-(1.7) are satisfied, then (u_n,v_n) is bounded.*

Proof. Let (u_n,v_n) be a Palais-Smale sequence for the functional I. By definition, $I(u_n,v_n)$ is bounded and $\|I'(u_n,v_n)\|_*\to 0$ as $n\to\infty$. Since I is coercive and $I(u_n,v_n)$ is bounded, the sequence (u_n,v_n) must be bounded. □

Lemma 3.3. *Let (u_n,v_n) be a Palais-Smale sequence for the Euler-Lagrange functional I. If conditions (1.5)-(1.7) are satisfied, then (u_n,v_n) admits a strongly convergent subsequence.*

Proof. Let (u_n,v_n) be a Palais-Smale sequence for I. By Lemma 3.2, the sequence (u_n,v_n) is bounded in $W_{p(x),q(x)}$. Since $W_{p(x),q(x)}$ is a reflexive and separable Banach space, there exists a subsequence, still denoted by (u_n,v_n), such that

$$(u_n,v_n)\rightharpoonup(u,v) \quad \text{weakly in } W_{p(x),q(x)}. \tag{3.2}$$

We have

$$\begin{aligned} & \langle I'(u_n,v_n)-I'(u,v),(u_n-u,0)\rangle \\ & =\int_\Omega\left(|\nabla u_n|^{p(x)-2}\nabla u_n+\frac{|\nabla u_n|^{2p(x)-2}\nabla u_n}{\sqrt{1+|\nabla u_n|^{2p(x)}}}\right)\nabla(u_n-u)\,dx \\ & -\int_\Omega\left(|\nabla u|^{p(x)-2}\nabla u+\frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)\nabla(u_n-u)dx \\ & +\int_\Omega\left(|u_n|^{p(x)-2}u_n-|u|^{p(x)-2}u\right)(u_n-u)\,dx \\ & -\int_{\partial\Omega}\left(\frac{\partial F}{\partial u}(x,u_n,v_n)-\frac{\partial F}{\partial u}(x,u,v)\right)(u_n-u)\,d\sigma. \end{aligned}$$

By the definition of the operator $\mathcal{T}$ in (2.3), we have

$$\begin{aligned}\langle \mathcal{T}u_n - \mathcal{T}u, u_n - u\rangle &= \langle I'(u_n, v_n) - I'(u, v), (u_n - u, 0)\rangle \\ &\quad - \int_\Omega \left(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\right)(u_n - u)\,dx \\ &\quad + \int_{\partial\Omega} \left(\frac{\partial F}{\partial u}(x, u_n, v_n) - \frac{\partial F}{\partial u}(x, u, v)\right)(u_n - u)\,d\sigma.\end{aligned}$$

Since (u_n, v_n) is a Palais–Smale sequence, then, from (3.2), we have

$$\langle I'(u_n, v_n) - I'(u, v), (u_n - u, 0)\rangle \quad \to 0 \quad \text{as } n \to \infty. \tag{3.3}$$

Moreover, since $I'(u, v) \in (W_{p(x),q(x)})^*$ and $(u_n - u, 0) \rightharpoonup (0, 0)$, we have

$$\langle I'(u, v), (u_n - u, 0)\rangle \longrightarrow 0.$$

By the compact embeddings $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ and the corresponding compact trace embeddings, we obtain $u_n \to u \quad \text{in } L^{p(x)}(\Omega)$, and, under the assumptions on p_1 and q_1,

$$u_n \to u \quad \text{in } L^{p_1(x)}(\partial\Omega), \qquad u_n \to u \quad \text{in } L^{q_1(x)}(\partial\Omega).$$

Consequently,

$$\int_\Omega \left(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\right)(u_n - u), dx \longrightarrow 0. \tag{3.4}$$

Furthermore, the trace embedding, the Hölder type inequality and assumption (1.8) yield

$$\begin{aligned}&\left|\int_{\partial\Omega} \frac{\partial F}{\partial u}(x, u_n, v_n)(u_n - u)d\sigma\right| \\ &\quad \le c_4|u_n - u|_{L^1(\partial\Omega)} + c_5\left||u_n|^{p_1(x)-1}\right|_{L^{p_1'(x)}(\partial\Omega)}|u_n - u|_{L^{p_1(x)}(\partial\Omega)} \\ &\quad\quad + c_6\left||v_n|^{q_1(x)-1}\right|_{L^{q_1'(x)}(\partial\Omega)}|u_n - u|_{L^{q_1(x)}(\partial\Omega)}.\end{aligned}$$

Since (u_n, v_n) is bounded and

$$\left||u_n|^{p_1(x)-1}\right|_{L^{p_1'(x)}(\partial\Omega)}, \qquad \left||v_n|^{q_1(x)-1}\right|_{L^{q_1'(x)}(\partial\Omega)}$$

are bounded, while

$$|u_n - u|_{L^1(\partial\Omega)} \to 0, \quad |u_n - u|_{L^{p_1(x)}(\partial\Omega)} \to 0, \quad |u_n - u|_{L^{q_1(x)}(\partial\Omega)} \to 0,$$

we obtain

$$\int_{\partial\Omega} \frac{\partial F}{\partial u}(x, u_n, v_n)(u_n - u)d\sigma \longrightarrow 0. \tag{3.5}$$

Similarly,

$$\int_{\partial\Omega} \frac{\partial F}{\partial u}(x, u, v)(u_n - u)d\sigma \longrightarrow 0. \tag{3.6}$$

Combining (3.3)–(3.6), we obtain

$$\limsup_{n\to\infty} \langle \mathcal{T}u_n - \mathcal{T}u, u_n - u\rangle \le 0. \tag{3.7}$$

Since $\mathcal{T}$ is of class (S_+) by Lemma 2.8, it follows that $u_n \to u$ strongly in $W^{1,p(x)}(\Omega)$. By applying the same argument to the v-component, we obtain $v_n \to v$ strongly in $W^{1,q(x)}(\Omega)$. Therefore,

$$(u_n, v_n) \to (u, v) \qquad \text{strongly in } W_{p(x),q(x)}.$$

□

Lemma 3.4. *Assume (1.5)-(1.6) hold. Then, for every $k \in \mathbb{N}$, there exists $A_k \in \Gamma_k$ such that*

$$\sup_{(u,v)\in A_k} I(u,v) < 0.$$

Proof. Fix $k \in \mathbb{N}$. Let

$$w_i = (\varphi_i, \psi_i) \in C^\infty(\overline{\Omega}) \times C^\infty(\overline{\Omega}), \quad i = 1, \dots, k,$$

be such that

$$\overline{\{x \in \partial\Omega; w_i(x) \neq 0\}} \cap \overline{\{x \in \partial\Omega; w_j(x) \neq 0\}} = \emptyset, \text{ if } i \neq j$$

and

$$|\{x \in \partial\Omega; w_i(x) \neq 0\}| > 0 \quad \forall i \in \{1, 2, \dots k\}.$$

Set $F_k := \operatorname{span}\{w_1, \dots, w_k\}$, then $\dim F_k = k$.
Denote

$$S := \{w \in W_{p(x),q(x)}; \|w\| = 1\}$$

and, for $t \in (0, 1]$, define $A_k(t) = t\,(F_k \cap S)$.
Since F_k is k-dimensional, $F_k \cap S$ is symmetric and

$$\gamma(A_k(t)) = \gamma(F_k \cap S) = k.$$

Hence $A_k(t) \in \Gamma_k$.
We claim that, for $t > 0$ sufficiently small,

$$\sup_{(u,v)\in A_k(t)} I(u,v) < 0.$$

Indeed, for $(u, v) \in A_k(t)$, we have

$$\begin{aligned}
\sup_{(u,v)\in A_k(t)} I(u,v) &\le \sup_{w\in F_k\cap S} I(tw) = \sup_{(u_0,v_0)\in F_k\cap S} I(tu_0, tv_0) \\
&= \sup_{(u_0,v_0)\in F_k\cap S} \left\{ \int_\Omega \frac{1}{p(x)} \big(|\nabla t u_0|^{p(x)} + |t u_0|^{p(x)} + \sqrt{1 + |\nabla t u_0|^{2p(x)}} - 1\big) dx \right. \\
&\qquad + \int_\Omega \frac{1}{q(x)} \big(|\nabla t v_0|^{q(x)} + |t v_0|^{q(x)} + \sqrt{1 + |\nabla t v_0|^{2q(x)}} - 1\big) dx \\
&\qquad \left. - \int_{\partial\Omega} F\,(x, t u_0, t v_0)\, d\sigma \right\} \\
&\le \sup_{(u_0,v_0)\in F_k\cap S} \left\{ \int_\Omega \frac{t^{p(x)}}{p(x)} \big(2|\nabla u_0|^{p(x)} + |u_0|^{p(x)}\big) dx - c_2 \int_{\partial\Omega} |t u_0|^{p_1(x)} d\sigma \right. \\
&\quad \left. + \int_\Omega \frac{t^{q(x)}}{q(x)} \big(2|\nabla v_0|^{q(x)} + |v_0|^{q(x)}\big) dx - c_3 \int_{\partial\Omega} |t v_0|^{q_1(x)} d\sigma - c_1 |\partial\Omega| \right\}.
\end{aligned}$$

Consequently,

$$\sup_{(u,v)\in A_k(t)} I(u,v) \leq \sup_{(u_0,v_0)\in F_k\cap S} \left\{ 2\frac{t^{p^-}}{p^-}\rho\left(u_0\right) + 2\frac{t^{q^-}}{q^-}\rho\left(v_0\right) - c_1 t^{p_1^+} \int_{\partial\Omega} |u_0|^{p_1(x)} d\sigma \right.$$
$$\left. -ct^{q_1^+} \int_{\partial\Omega} |v_0|^{q_1(x)} d\sigma - c_1|\partial\Omega| \right\}.$$

Since F_k is finite-dimensional, all norms are equivalent on F_k. Moreover, by the choice of F_k and the compactness of $F_k \cap S$, there exists $C_k > 0$ such that

$$\int_{\partial\Omega} |u_0|^{p_1(x)}\, d\sigma + \int_{\partial\Omega} |v_0|^{q_1(x)}\, d\sigma \geq C_k$$

for every $(u_0, v_0) \in F_k \cap S$.
Since $p_1^+ < p^-$ and $q_1^+ < q^-$, the negative boundary terms dominate the positive terms as $t \to 0^+$. Hence, for $t > 0$ sufficiently small,

$$\sup_{(u,v)\in A_k(t)} I(u,v) < 0.$$

Taking such a t and setting $A_k := A_k(t)$ completes the proof. □

Proof of Theorem 1.2. By Lemmas 3.1–3.4 and Theorem 2.6, all the assumptions of the symmetric mountain pass theorem are satisfied. Hence, Problem (1.1) admits infinitely many nontrivial weak solutions $(u_k, v_k) \in W_{p(x),q(x)}$ such that

$$I(u_k, v_k) < 0 \qquad \text{and} \qquad (u_k, v_k) \to (0,0) \quad \text{in } W_{p(x),q(x)}$$

as $k \to \infty$. □

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