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CELLULARITY OF TENSOR PRODUCTS OF m-PARTITION ALGEBRAS

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Abstract. In this article, we study the cellular structure of the tensor product of m-partition algebras using the method of iterated inflation developed by Green and Paget. We realize this algebra as an iterated inflation of tensor products of symmetric group algebras along diagram spaces indexed by the numbers of propagating lines in each component, and use this structure to establish its cellularity. We construct an explicit cellular basis and describe the corresponding cell modules and bilinear forms. Finally, over a field of characteristic p, we obtain a parametrization of the non-isomorphic simple modules in terms of p-regular partitions.

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1. Introduction

Cellular algebras were first introduced by Graham and Lehrer in [3]Their definition is based on the existence of a cellular basis indexed by a finite partially ordered set, together with an anti-involution satisfying a triangular condition relative to this order. This structure leads naturally to the construction of cell modules equipped with canonical bilinear forms, whose radicals play an important role in the classification of simple modules. The theory of cellular algebras has been widely applied in the representation theory of Hecke algebras, diagram algebras, and related combinatorial structures. Recent work on diagram algebras has also investigated cellular representations and cell modules, including Gram determinants and bilinear forms for Fuss–Catalan algebras [6, 7] .

A basis-free and structural characterization of cellular algebras was later developed by König and Xi [9]. They proved that this intrinsic formulation is equivalent to the original Graham-Lehrer definition. Moreover, their work connects cellularity with iterated inflation, providing a constructive interpretation in which cellular algebras arise through successive extensions from smaller cellular components. Each successive inflation produces a new cellular layer, and the

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process is compatible with anti-involution. This constructive viewpoint is particularly effective for diagram algebras, where combinatorial stratifications, such as the number of propagating lines, naturally induce the inflation layers.

Further contributions were made by Green and Paget [4], who extended the iterated inflation framework to a boarder class of diagram algebras. In particular, they presented various technical details and conditions which must be satisfied in order for such a construction to yield a cellular algebra.

The partition algebra $P_k(\delta)$ was originally introduced by Martin in the context of statistical mechanics, specifically for the Potts model. Its algebraic significance was further illuminated by Jones, who established a version of Schur-Weyl duality between $P_k(\delta)$ and the symmetric group. The cellular structure of the partition algebra was studied by Xi in [12], leading to explicit constructions of cell modules, descriptions of bilinear forms, and classifications of simple modules in terms of partitions and propagating numbers.

More recently, these algebras have been expanded to more complex structures, most notably the tensor product of m-partition algebras $P_k(\delta_1) \otimes P_k(\delta_2) \otimes \cdots \otimes P_k(\delta_m)$. This algebra was introduced and studied by Kennedy and Jaish in [8]. They realized this algebra as a centralizer algebra of the direct product $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}$ of symmetric groups acting on W^k, where $W = \mathbb{C}^{n_1 n_2 \cdots n_m}$. They also studied its inequivalent irreducible representations and their dimensions.

Although it is known that the tensor product of cellular algebras is again cellular, this general result does not provide an explicit cellular structure in the setting of partition algebras. In particular, the construction of the cellular basis, the corresponding cell modules and bilinear forms, and the parametrization of the simple modules require a detailed analysis of the diagrammatic structure.

Motivated by this, in this paper we provide an explicit cellular structure for the tensor product of m-partition algebras. Our approach is based on the theory of iterated inflation developed by Green and Paget [4], which shows that any algebra constructed as an iterated inflation of cellular algebras is itself cellular. Since each partition algebra $P_k(\delta_i)$ is known to be cellular, our approach is to realise their tensor product as an iterated inflation of tensor product of symmetric group algebras.

We construct a filtration of $A = P_k(\delta_1) \otimes \cdots \otimes P_k(\delta_m)$ by two-sided ideals according to the number of propagating lines in each component. The corresponding successive quotients are described in terms of tensor products of symmetric group algebras $F[S_{t_1} \times \cdots \times S_{t_m}]$ and suitable tensor products of spaces spanned by half-partition diagrams. We define the bilinear forms governing multiplication in each layer and construct an anti-involution induced by diagram reflection together with inversion in the symmetric groups. By verifying the hypotheses of the iterated inflation theorem, we establish the cellularity of A.
We then describe an explicit cellular basis for A. The indexing set is given by pairs $(\mathbf{t}, [\lambda])$, where $\mathbf{t} = (t_1, \cdots, t_m)$, $1 \leq t_i \leq k$, and $[\lambda] = (\lambda^{(1)}, \cdots, \lambda^{(m)})$ with $\lambda^{(i)}$ is a partition of t_i. The corresponding cell modules are described explicitly in terms of the space of half-partition diagrams and the cell module of $F[S_{t_1} \cdots \times S_{t_m}]$.
We also determine the bilinear form on each cell module and show that it is non-zero precisely when the corresponding bilinear form on the associated cell module

of $F[S_{t_1} \times \cdots \times S_{t_m}]$ is non-zero.
Finally, using the result of Dipper and James [1] on the simple modules of symmetric group algebras, we obtain a parametrization of the simple A-modules over a field of characteristic p. Assuming that $\delta_1, \cdots, \delta_m \neq 0$, the non-isomorphic simple A-modules are indexed by pairs $(\mathbf{t}, [\lambda])$ such that each $\lambda^{(i)}$ is a p-regular partition of t_i.
The paper is organized as follows. In section 2, we recall the necessary preliminaries on cellular algebras, partition algebras and iterated inflation. In section 3, we study the tensor product of m-partition algebras considered by Kennedy and Jaish [8], and establish the filtration used throughout the paper. Section 4 contains the main results. We describe the iterated inflation structure, construct the cellular basis and cell modules, determine their bilinear forms, and obtain the parametrization of the simple modules.

2. PRELIMINARIES

2.1. **Cellular algebras.** We briefly recall the notion of a cellular algebra introduced by Graham and Lehrer [3].
Let A be an R-algebra. A cellular structure on A consists of a cell datum (Λ, M, C, i), where Λ is a finite partially ordered set, $M(\lambda)$ is a finite indexing set for each $\lambda \in \Lambda$, and

$$\{C^{\lambda}_{S,T} : \lambda \in \Lambda,\ S, T \in M(\lambda)\}$$

is a set of elements of A. The cell datum satisfies the following conditions:

(C1) The set $\{C^{\lambda}_{S,T} : \lambda \in \Lambda,\ S, T \in M(\lambda)\}$ is an R-basis of A.
(C2) The map $i : A \to A$ is an R-linear anti-involution satisfying

$$i(C^{\lambda}_{S,T}) = C^{\lambda}_{T,S}.$$

(C3) For each $a \in A$ and $C^{\lambda}_{S,T}$, there exist coefficients $r_a(S', S) \in R$, independent of T, such that

$$aC^{\lambda}_{S,T} \equiv \sum_{S' \in M(\lambda)} r_a(S', S) C^{\lambda}_{S',T} \pmod{A^{<\lambda}},$$

where $A^{<\lambda}$ is the R-span of the basis elements indexed by $\mu < \lambda$.

For each $\lambda \in \Lambda$, the cellular basis determines a cell module $\Delta(\lambda)$ together with an R-bilinear form

$$\varphi_{\lambda} : \Delta(\lambda) \times \Delta(\lambda) \longrightarrow R.$$

When R is a field, the non-zero forms determine the non-isomorphic simple modules through the corresponding quotients of the cell modules [3].

2.2. **The symmetric group.** Let n be a non-negative integer. A partition of n, denoted by $\lambda \vdash n$, is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers satisfying

$$\lambda_1 \geq \lambda_2 \geq \cdots, \qquad |\lambda| = \sum_i \lambda_i = n.$$

The Young diagram of $\lambda = (\lambda_1, \ldots, \lambda_l)$ is

$$[\lambda] = \{(i, j) : 1 \leq i \leq l,\ 1 \leq j \leq \lambda_i\}.$$

A λ-tableau is a filling of the boxes of $[\lambda]$ with $1, \dots, n$ without repetition. It is standard if its entries increase from left to right in each row and from top to bottom in each column.

For a partition λ, let $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_l}$ be the Young subgroup of S_n, which is the row stabilizer of the initial tableau t^λ. For a row-standard tableau t, write

$$t = t^\lambda d(t), \qquad d(t) \in S_n.$$

We use the dominance order

$$\lambda \trianglerighteq \mu \quad \Longleftrightarrow \quad \sum_{i=1}^{r} \lambda_i \geq \sum_{i=1}^{r} \mu_i \quad \text{for all } r.$$

Let F be a field. Denote by Λ_n the set of partitions of n with the dominance order, and by $\mathcal{T}(\lambda)$ the set of standard λ-tableaux. The involution on FS_n is given by

$$\pi^* = \pi^{-1}, \qquad \pi \in S_n.$$

For $\lambda \in \Lambda_n$, let $x_\lambda = \sum_{w \in S_\lambda} w$. For $s, t \in \mathcal{T}(\lambda)$, define $m^\lambda_{s,t} = d(s) x_\lambda d(t)^*$.
The group algebra FS_n is a cellular algebra with cellular basis

$$\left\{ m^\lambda_{s,t} : \lambda \in \Lambda_n,\ s, t \in T(\lambda) \right\},$$

known as the Murphy basis [11, Theorem 3.20].

We extend this notation to direct products of symmetric groups. Let $n_1, \dots, n_m$ be non-negative integers and set $n = n_1 + \cdots + n_m$.
An m-partition of size n is an ordered m-tuple

$$[\lambda] = (\lambda^{(1)}, \dots, \lambda^{(m)}), \qquad \lambda^{(i)} \vdash n_i.$$

An m-tableau of shape $[\lambda]$ is an m-tuple

$$[t] = (t^{(1)}, \dots, t^{(m)}),$$

where $t^{(i)}$ has shape $\lambda^{(i)}$. It is standard if each $t^{(i)}$ is standard. We write

$$\mathcal{T}^m([\lambda]) = \mathcal{T}(\lambda^{(1)}) \times \cdots \times \mathcal{T}(\lambda^{(m)}).$$

There is a natural algebra isomorphism

$$FS_{n_1} \otimes \cdots \otimes FS_{n_m} \cong F[S_{n_1} \times \cdots \times S_{n_m}],$$

given by

$$\pi_1 \otimes \cdots \otimes \pi_m \longmapsto (\pi_1, \dots, \pi_m).$$

Proposition 2.1. [5, Proposition 1], [2, section 3.4].
The group algebra $F[S_{n_1} \times \cdots \times S_{n_m}]$ is cellular with respect to the involution $(\pi_1, \dots, \pi_m)^* = (\pi_1^{-1}, \dots, \pi_m^{-1})$, and cell-indexing poset $\Lambda_{n_1,\dots,n_m} = \Lambda_{n_1} \times \cdots \times \Lambda_{n_m}$, ordered component-wise by dominance. Its cellular basis is

$$m^{[\lambda]}_{[s],[t]} = \bigotimes_{i=1}^{m} m^{\lambda^{(i)}}_{s^{(i)},t^{(i)}}, \qquad [s], [t] \in \mathcal{T}^m([\lambda]).$$

Moreover, the corresponding cell module is

$$S^{[\lambda]} = S^{\lambda^{(1)}} \otimes \cdots \otimes S^{\lambda^{(m)}}.$$

2.3. **Iterated inflation of cellular algebras.** Iterated inflation was introduced by König and Xi [9]. We use the formulation given by Green and Paget [4].

Let A be an F-algebra with an anti-involution $*$. Suppose that, as an F-vector space,

$$A \cong \bigoplus_{i \in I} V_i \otimes B_i \otimes V_i,$$

where I is a finite partially ordered set, each V_i is an F-vector space, and each B_i is a cellular F-algebra. We refer to $V_i \otimes B_i \otimes V_i$ as the i-th layer of A. Assume that the anti-involution is compatible with the decomposition, so that

$$(u \otimes b \otimes v)^* = v \otimes b^* \otimes u.$$

For each $i \in I$, choose bases $\mathcal{V}_i$ of V_i and $\mathcal{B}_i$ of B_i. Let $\mathcal{A}$ be a basis of A that contains all elements $u \otimes b \otimes v$ arising from these bases. For each $i \in I$, suppose that there are maps

$$\phi_i : \mathcal{A} \times \mathcal{V}_i \longrightarrow V_i, \qquad \theta_i : \mathcal{A} \times \mathcal{V}_i \longrightarrow B_i,$$

such that, for $a \in \mathcal{A}$ and $u, v \in \mathcal{V}_i$, $b \in \mathcal{B}_i$,

$$a(u \otimes b \otimes v) \equiv \phi_i(a, u) \otimes \theta_i(a, u)b \otimes v \pmod{J(< i)},$$

where

$$J(< i) = \bigoplus_{j < i} V_j \otimes B_j \otimes V_j.$$

Under these conditions, the iterated inflation theorem implies that A is cellular. Its cell indices are pairs (i, λ), where λ is a cell index of B_i, and the corresponding cellular basis elements have the form

$$x \otimes C^{\lambda}_{X,Y} \otimes y.$$

See [4, Theorem 1]. Moreover, for each $i \in I$, the multiplication within the i-th layer is governed by a bilinear form of B_i values.

$$\psi_i : V_i \times V_i \longrightarrow B_i$$

such that

$$(x \otimes c \otimes y)(u \otimes b \otimes v) \equiv x \otimes c\,\psi_i(y, u)b \otimes v \pmod{J(< i)},$$

with the compatibility

$$\psi_i(y, u) = \psi_i(u, y)^*.$$

The corresponding cell module for (i, λ) is $V_i \otimes \Delta(\lambda)$, where $\Delta(\lambda)$ is the cell module of B_i. Its bilinear form is induced by ψ_i and the bilinear form on $\Delta(\lambda)$ [4].

2.4. **The Partition algebra $P_k(\delta)$.** Let F be a field and let $\delta \in F$. A k-partition diagram consists of two rows of k vertices, with the $2k$ vertices partitioned into blocks represented by connected components. We identify diagrams that determine the same set partition.

The product of two k-partition diagrams is defined by concatenation. Place the first diagram above the second and identify the bottom row of the first with the top row of the second. After removing the middle row, each connected component contained entirely in the middle row contributes a factor of δ. Thus,

$$d_1 d_2 = \delta^{\alpha} d_3,$$

where α is the number of connected components removed from the middle row and d_3 is the resulting k-partition diagram.

Extending this multiplication linearly gives the partition algebra $P_k(\delta)$, whose basis consists of all k-partition diagrams. Its dimension is the Bell number

$$\dim P_k(\delta) = B(2k) = \sum_{\ell=1}^{2k} S(2k, \ell),$$

where $S(2k, \ell)$ denotes the Stirling number of the second kind. The identity element is the diagram consisting of k propagating lines joining corresponding vertices in the two rows.

3. The tensor product of m-Partition algebras

Let F be a field and let $\delta_1, \cdots, \delta_m \in F$. We consider the tensor product $A = P_k(\delta_1) \otimes \cdots \otimes P_k(\delta_m)$. Its natural basis is

$$\mathfrak{M}_k = \{(d_1 \otimes \cdots \otimes d_m) \, : \, d_1, d_2, \cdots, d_m \text{ are the k-partition diagrams}\}.$$

Hence $dim_F A = [B(2k)]^m$.
For $d = (d_1 \otimes \cdots \otimes d_m)$, $d' = (d_1' \otimes d_2' \otimes \cdots d_m') \in \mathfrak{M}_k$, the product is defined component-wise by

$$dd' = (d_1 d_1') \otimes \cdots \otimes (d_m d_m').$$

If $d_i d_i' = \delta_i^{\alpha_i} d_i''$, then

$$dd' = \delta_1^{\alpha_1} \cdots \delta_m^{\alpha_m} (d_1'' \otimes \cdots \otimes d_m'').$$

For a partition diagram d, let $i(d)$ be the diagram obtained by reflecting the diagram d upside down. Then $d \mapsto i(d)$ extends linearly to the whole partition algebra is an anti-involution of $P_k(\delta)$; (See [12, Lemma 2.5]). We therefore define an ant-involution on A by reflecting each component upside down, and we have the following.

Lemma 3.1. *Define $i : A \to A$ on basis elements by*

$$i(d_1 \otimes \cdots \otimes d_m) = i(d_1) \otimes \cdots \otimes i(d_m),$$

where $i(d_j)$ is obtained by reflecting d_j upside down. Then i is an anti-involution of A; that is

$$i(ab) = i(b)i(a)$$

for all $a, b \in A$, and $i^2 = \mathrm{id}$.

Proof. Clearly, i is a linear map. Since $i^2(d_j) = i(i(d_j)) = d_j$ for each diagram d_j in $P_k(\delta_j)$, $j = 1, \cdots, m$, we have

$$i^2(d_1 \otimes \cdots d_m) = i^2(d_1) \otimes \cdots \otimes i^2(d_m) = d_1 \otimes \cdots \otimes d_m$$

So $i^2 = \mathrm{id}$ for all $d_1 \otimes \cdots \otimes d_m \in \mathfrak{M}_k$.
Now for $(d_1 \otimes d_2 \otimes \cdots \otimes d_m)$, $(d_1^{'} \otimes d_2^{'} \otimes \cdots d_m^{'}) \in \mathfrak{M}_k$, we have

$$\begin{aligned} i((d_1 \otimes \cdots \otimes d_m).(d_1^{'} \otimes \cdots \otimes d_m^{'})) &= i(\delta_1^{\alpha_1} \cdots \delta_m^{\alpha_m}(d_1 d_1^{'} \otimes \cdots \otimes d_m d_m^{'})) \\ &= \delta_1^{\alpha_1} \cdots \delta_m^{\alpha_m}(i(d_1 d_1^{'}) \otimes \cdots \otimes i(d_m d_m^{'}) \\ &= \delta_1^{\alpha_1} \cdots \delta_m^{\alpha_m}(i(d_1^{'}).i(d_1) \otimes \cdots \otimes i(d_m^{'}) i(d_m)) \\ &= \delta_1^{\alpha_1} \cdots \delta_m^{\alpha_m} i(d_i^{'} \otimes \cdots \otimes d_m^{'}).i(d_1 \otimes \cdots \otimes d_m). \end{aligned}$$

Thus i is an anti-involution of A. □

For a partition diagram d, let $\#^p(d)$ denote the number of propagating lines of d. For two partition diagrams d_1, d_2, we have

$$\#^p(d_1 d_2) \leq \min\{\#^p(d_1), \#^p(d_2)\}.$$

The following shows that the multiplication of tensor product of partition diagrams cannot increase the number of propagating lines.

Lemma 3.2. *Let $d = d_1 \otimes \cdots \otimes d_m$, $d' = d_1^{'} \otimes \cdots \otimes d_m^{'}$ be basis elements of A. Then*

$$\#^p(dd') \leq \min\{\#^p(d), \#^p(d')\}.$$

Proof. Since the multiplication in A is defined component-wise, we have

$$dd' = (d_1 d_1') \otimes \cdots \otimes (d_m d_m'),$$

Therefore,

$$\#^p(dd') = \sum_{i=1}^{m} \#^p(d_i d_i').$$

For each i, we have

$$\#^p(d_i d_i') \leq \min\{\#^p(d_i), \#^p(d_i')\}.$$

Hence,

$$\begin{aligned} \#^p(dd') &\leq \sum_{i=1}^{m} \min\{\#^p(d_i), \#^p(d_i')\} \\ &\leq \min\{\sum_{i=1}^{m} \#^p(d_i), \sum_{i=1}^{m} \#^p(d_i')\} \\ &= \min\{\#^p(d), \#^p(d')\}. \end{aligned}$$

□

For $t \in \{0, 1, \cdots, mk\}$, let $W^{(\leq t)}$ be the F-subspace of $A = P_k(\delta_1) \otimes \cdots \otimes P_k(\delta_m)$ spanned by all tensor products $d_1 \otimes \cdots \otimes d_m$ of partition diagrams whose total number of propagating lines is at most t. More precisely, if

$$t_i = \#^p(d_i), \quad 0 \leq t_i \leq k, \quad \text{for}, 1 \leq i \leq m, \text{ and}$$

$$\mathbf{t} = (t_1, \cdots, t_m), \qquad |\mathbf{t}| = \sum_{i=1}^{m} t_i, \quad \text{then}$$

$$W^{(\leq t)} = \text{span}_F\{d_1 \otimes \cdots \otimes d_m \ : \ |\mathbf{t}| \leq t\}.$$

By Lemma 3.2, $W^{(\leq t)}$ is a two-sided ideal of A. Hence, we obtain the filtration

$$\langle 0 \rangle \subseteq W^{(\leq 0)} \subseteq W^{(\leq 1)} \subseteq \cdots \subseteq W^{(\leq mk)} = A.$$

For $t \geq 1$, define $B_t = W^{(\leq t)}/W^{(\leq t-1)}$, and set $B_0 = W^{(\leq 0)}$. Then B_t is spanned by the cosets of tensor products $d_1 \otimes \cdots \otimes d_m$, whose total number of propagating lines is exactly t. Equivalently,

$$B_t = \text{span}_F\{d_1 \otimes \cdots \otimes d_m + W^{(\leq t-1)} \ : \ |\mathbf{t}| = t\}.$$

This filtration will be used in the next section to construct the iterated inflation structure of A.

4. Cellular structure of $A = P_k(\delta_1) \otimes \cdots \otimes P_k(\delta_m)$

4.1. The iterated inflation structure of $A = P_k(\delta_1) \otimes \cdots \otimes P_k(\delta_m)$.

Definition 4.1. A **partition half diagram with r propagating lines** is a diagram consisting of one row of k vertices labelled $1, \cdots, k$, together with exactly r propagating lines extending from vertices in the row, while the remaining $k - r$ vertices are joined by arcs or occur as isolated vertices.

Let $\mathcal{H}_r(P_k)$ denote the set of all partition half diagrams with r propagating lines, and let V_r be the F-vector space spanned by $\mathcal{H}_r(P_k)$.

For $\mathbf{t} = (t_1, \cdots, t_m)$, $0 \leq t_i \leq k$, define $V_{\mathbf{t}} = V_{t_1} \otimes \cdots \otimes V_{t_m}$. Let

$$\mathcal{H}_{\mathbf{t}} = \{x_1 \otimes \cdots \otimes x_m \ : \ x_i \in \mathcal{H}_{t_i}(P_k), 1 \leq i \leq m\}.$$

Thus $V_{\mathbf{t}}$ is spanned by $\mathcal{H}_{\mathbf{t}}$. We write $|\mathbf{t}| = \sum_{i=1}^{m} t_i$. In particular, if $|\mathbf{t}| = t$, then every element of $\mathcal{H}_{\mathbf{t}}$ has exactly t propagating lines in total.

For $\mathbf{t} = (t_1, \cdots, t_m)$, let $S_{\mathbf{t}} = S_{t_1} \times \cdots \times S_{t_m}$.
The following lemma describes B_t in terms of the vector spaces $V_{\mathbf{t}}$ and the groups $S_{\mathbf{t}}$.

Lemma 4.2. *For $t \in \{0, 1, \cdots, mk\}$, let*

$$\mathcal{T}_t = \{\boldsymbol{t} = (t_1, \cdots, t_m) \ : \ 0 \leq t_i \leq k, |\boldsymbol{t}| = t\}.$$

Then as an F-vector space

$$B_t \cong \bigoplus_{\boldsymbol{t} \in \mathcal{T}_t} (V_{\boldsymbol{t}} \otimes F[S_{\boldsymbol{t}}] \otimes V_{\boldsymbol{t}}).$$

Proof. Let $d = d_1 \otimes \cdots \otimes d_m$ be a basis element of B_t. For each $1 \leq i \leq m$, let $t_i = \#^p(d_i)$, where $\#^p(d_i)$ denotes the number of propagating lines of d_i. Since d has exactly t propagating lines in total, $|\mathbf{t}| = \sum_{i=1}^m t_i = t$, and hence $\mathbf{t} = (t_1, \cdots, t_m) \in \mathcal{T}_t$.
We decompose d componentwise by cutting each d_i along its t_i propagating lines. For each i, the resulting upper and lower parts determine partition half diagrams $x_i, y_i \in \mathcal{H}_{t_i}(P_k)$, respectively. The connections between the t_i propagating lines determine a unique permutation $\pi_i \in S_{t_i}$. Thus, each d_i is uniquely determined by the triple (x_i, π_i, y_i). Therefore, d determines the triple (x, π, y), where $x = x_1 \otimes \cdots \otimes x_m \in \mathcal{H}_{\mathbf{t}}$, $y = y_1 \otimes \cdots \otimes y_m \in \mathcal{H}_{\mathbf{t}}$ and $(\pi_1, \cdots, \pi_m) \in S_{\mathbf{t}}$. Hence every basis diagram of B_t determines an element of $\mathcal{H}_{\mathbf{t}} \times S_{\mathbf{t}} \times \mathcal{H}_{\mathbf{t}}$.
Conversely, let $(x, \pi, y) \in \mathcal{H}_{\mathbf{t}} \times S_{\mathbf{t}} \times \mathcal{H}_{\mathbf{t}}$, where $\mathbf{t} \in \mathcal{T}_t$. Write $x = x_1 \otimes \cdots \otimes x_m$, $y = y_1 \otimes \cdots \otimes y_m$, with $x_i, y_i \in \mathcal{H}_{t_i}(P_k)$ and $\pi = (\pi_1, \cdots, \pi_m)$, $\pi_i \in S_{t_i}$. For each i, connect the t_i propagating vertices of x_i to those of y_i according to the permutation π_i. . This uniquely determines a partition diagram d_i with exactly t_i propagating lines. Hence $d = d_1 \otimes \cdots \otimes d_m$ is a basis diagram with $\#^p(d) = \sum_{i=1}^m t_i = |\mathbf{t}| = t$. Thus d represents a basis element of B_t.
The two constructions above are inverse to each other. Therefore, the basis diagrams of B_t are in one-to-one correspondence with $\bigcup_{\mathbf{t} \in \mathcal{T}_t} (\mathcal{H}_{\mathbf{t}} \times S_{\mathbf{t}} \times \mathcal{H}_{\mathbf{t}})$. Since $V_{\mathbf{t}}$ is the F-vector space spanned by $\mathcal{H}_{\mathbf{t}}$, it follows that

$$B_t \cong \bigoplus_{\mathbf{t} \in \mathcal{T}_t} (V_{\mathbf{t}} \otimes F[S_{\mathbf{t}}] \otimes V_{\mathbf{t}}).$$

□

Remark 4.3. By lemma 4.2, $A = P_k(\delta_1) \otimes \cdots \otimes P_k(\delta_m)$ has the following decomposition as an F-vector space

$$A \cong \bigoplus_{t=0}^{mk} \bigoplus_{\mathbf{t} \in \mathcal{T}_t} (V_{\mathbf{t}} \otimes F[S_{\mathbf{t}}] \otimes V_{\mathbf{t}}).$$

This decomposition will allow us to exhibit the desired iterated inflation structure.

To define the multiplication on the components appearing in the above decomposition, we first introduce the following bilinear form.

Definition 4.4. For $\mathbf{t} = (t_1, \cdots, t_m) \in \mathcal{T}_t$, consider the map

$$\varphi_{\mathbf{t}} : V_{\mathbf{t}} \times V_{\mathbf{t}} \to F[S_{\mathbf{t}}],$$

which is defined componentwise from the maps

$$\varphi_{t_i} : V_{t_i} \times V_{t_i} \to F[S_{t_i}], \qquad 1 \leq i \leq m.$$

For each i, the map $\varphi_{t_i} : V_{t_i} \times V_{t_i} \to F[S_{t_i}]$ is defined as follows; see also [12].
Let $x = x_1 \otimes \cdots \otimes x_m$, $y = y_1 \otimes \cdots \otimes y_m$, where $x_i, y_i \in \mathcal{H}_{t_i}(P_k)$, $1 \leq i \leq m$.
For each i, let x_i be a bottom half diagram and y_i a top half diagram. Form $x_i y_i$ by identifying their labelled vertices, and denote the resulting graph by Γ_i. The connected components of Γ_i consist of paths and closed loops. If a path

contains exactly one propagating line, or if its two endpoints are propagating lines belonging to the same half diagram, set

$$\varphi_{t_i}(x_i, y_i) = 0.$$

Otherwise, the remaining paths connect the propagating lines of x_i with those of y_i, and hence determine a unique permutation $\pi_i \in S_{t_i}$. Label the propagating lines of x_i and y_i, from left to right, by $1, \cdots, t_i$. Then π_i is determined by requiring that the propagating line labelled j in x_i is connected to the propagating line labelled $(j)\pi_i$ in y_i. We then set

$$\varphi_{t_i}(x_i, y_i) = \delta_i^{n_i}\pi_i,$$

where n_i denotes the number of closed loops in Γ_i. By [12, Lemma 4.3], each φ_{t_i} is bilinear. Therefore, the componentwise definition above gives

$$\varphi_{\mathbf{t}} : V_{\mathbf{t}} \times V_{\mathbf{t}} \to F[S_{\mathbf{t}}]$$

by

$$\varphi_{\mathbf{t}}(x, y) = \begin{cases} 0, & \text{if } \varphi_{t_i}(x_i, y_i) = 0 \text{ for some } i; \\ \delta_1^{n_1} \cdots \delta_m^{n^m}(\pi_1, \cdots, \pi_m), & \text{if } \varphi_{t_i}(x_i, y_i) = \delta_i^{n_i}\pi_i \text{ for all } i. \end{cases}$$

Hence, φ_t is an F-bilinear form.

Lemma 4.5. *For a fixed t, the quotient $B_t = W^{(\leq t)}/W^{(\leq t-1)}$ is isomorphic, as an F-algebra, to*

$$\bigoplus_{\boldsymbol{t} \in \mathcal{T}_t} (V_{\boldsymbol{t}} \otimes F[S_{\boldsymbol{t}}] \otimes V_{\boldsymbol{t}}).$$

For each $\boldsymbol{t} \in \mathcal{T}_t$, the multiplication on the corresponding component $V_{\boldsymbol{t}} \otimes F[S_{\boldsymbol{t}}] \otimes V_{\boldsymbol{t}}$ is given by

$$(u \otimes \pi \otimes x)(y \otimes \pi' \otimes v) = u \otimes \pi\varphi_{\boldsymbol{t}}(x, y)\pi' \otimes v,$$

for $u, x, y, v \in V_{\boldsymbol{t}}$, $\pi, \pi' \in F[S_{\boldsymbol{t}}]$.

Proof. Let

$$\theta : B_t \to \bigoplus_{\mathbf{t} \in \mathcal{T}_t} (V_{\mathbf{t}} \otimes F[S_{\mathbf{t}}] \otimes V_{\mathbf{t}})$$

be the map defined on basis diagrams as follows. For a basis diagram $d \in B_t$, let $\mathbf{t} \in \mathcal{T}_t$ be determined by the numbers of propagating lines in its components. Then

$$\theta(d) = u \otimes \pi \otimes x \in V_{\mathbf{t}} \otimes F[S_{\mathbf{t}}] \otimes V_{\mathbf{t}},$$

where u and x are the top and bottom half diagrams of d, respectively, and $\pi \in S_{\mathbf{t}}$ records the connections between their propagating lines. Extend θ linearly to B_t. By Lemma 4.2, θ is a vector-space isomorphism. It remains to show that

$$\theta(dd') = \theta(d)\theta(d').$$

Let

$$\theta(d) = u \otimes \pi \otimes x, \qquad \theta(d') = y \otimes \pi' \otimes v.$$

We distinguish two cases according to the number of propagating lines in the product dd'.

Case 1: If $\#^p(dd') < t$. Then $dd' \in W^{(\leq t-1)}$, and hence $dd' = 0$ in the quotient

$B_t = W^{(\leq t)}/W^{(\leq t-1)}$. Moreover, in the middle row of dd', the condition $\#^p(dd') < t$ implies that, for some i, $\varphi_{t_i}(x_i, y_i) = 0$. Thus $\varphi_{\mathbf{t}}(x, y) = 0$ and consequently

$$\theta(dd') = 0 = \theta(d)\theta(d').$$

Case 2: if $\#^p(dd') = t$. Since $t = \sum_{i=1}^m t_i$ and the product in each factor satisfies $\#^p(d_i d_i') \leq t_i$, we have $\#^p(d_i d_i') = t_i$ for all $i = 1, \cdots, m$. Thus, by the multiplication rule in $P_k(\delta_i)$, we have

$$d_i d_i' = \delta_i^{n_i} d_i'',$$

where n_i is the number of connected components removed from the middle row of $d_i d_i'$. The resulting diagram d_i'' has t_i propagating lines and consists of the top half of d_i, the bottom half of d_i', together with a permutation $\sigma_i \in S_{t_i}$. On the other hand, by the definition of φ_{t_i},

$$\varphi_{t_i}(x_i, y_i) = \delta_i^{n_i} \pi_i'', \qquad \pi_i'' \in S_{t_i},$$

where π_i'' is determined by the paths joining the propagating lines of x_i to those of y_i. The same paths occur in the middle row of $d_i d_i'$, so the scalar factors agree. Moreover, for $1 \leq j \leq t_i$, $(j)(\pi_i \pi_i'' \pi_i') = (j)\sigma_i$. Hence $\pi_i \pi_i'' \pi_i' = \sigma_i$. Therefore,

$$\pi(\pi_1'', \cdots, \pi_m'')\pi' = (\sigma_1, \cdots, \sigma_m),$$

and consequently,

$$\theta(dd') = u \otimes \pi\varphi_{\mathbf{t}}(x, y)\pi' \otimes v = \theta(d)\theta(d').$$

Thus θ preserves multiplication, and hence is an F-algebra isomorphism. Therefore,

$$B_t \cong \bigoplus_{\mathbf{t} \in \mathcal{T}_t} (V_{\mathbf{t}} \otimes F[S_{\mathbf{t}}] \otimes V_{\mathbf{t}}).$$

□

Remark 4.6. By the preceding Lemmas, we may identify the set $\mathfrak{M}_k$ with $\amalg_{t=0}^{mk} \amalg_{\mathbf{t} \in \mathcal{T}_t} \mathcal{H}_{\mathbf{t}} \times S_{\mathbf{t}} \times \mathcal{H}_{\mathbf{t}}$. Where

$$\mathcal{H}_{\mathbf{t}} = \mathcal{H}_{t_1}(P_k) \otimes \cdots \otimes \mathcal{H}_{t_m}(P_k), \qquad S_{\mathbf{t}} = S_{t_1} \times \cdots \times S_{t_m}.$$

Thus, whenever no ambiguity arises, an element $d \in B_t$ may be represented as

$$d = (x_1 \otimes \cdots \otimes x_m) \otimes (\pi_1, \cdots, \pi_m) \otimes (y_1 \otimes \cdots \otimes y_m),$$

where $x_i, y_i \in \mathcal{H}_{t_i}(P_k)$ and $\pi_i \in S_{t_i}$.
Under this identification, we can write

$$W^{(\leq t)} := \bigoplus_{l=0}^{t} \bigoplus_{\mathbf{s} \in \mathcal{T}_l} (V_{\mathbf{s}} \otimes F[S_{\mathbf{s}}] \otimes V_{\mathbf{s}})$$

where $\mathbf{s} = (s_1, \cdots, s_m) \in \mathcal{T}_l$, $V_{\mathbf{s}} = V_{s_1} \otimes \cdots \otimes V_{s_m}$, $S_{\mathbf{s}} = S_{s_1} \times \cdots \times S_{s_m}$.
Moreover, if $d, d' \in B_t$ with $d = u \otimes \pi \otimes x$, $d' = y \otimes \pi' \otimes v$ where $u, x, y, v \in \mathcal{H}_{\mathbf{t}}$, $\pi, \pi' \in S_{\mathbf{t}}$. Then

$$dd' = u \otimes \pi\varphi_{\mathbf{t}}(x, y)\pi' \otimes v \qquad (\text{mod } W^{(\leq t-1)}).$$

The bilinear form $\varphi_{\mathbf{t}}$ is compatible with the natural involution on $F[S_{\mathbf{t}}]$ in the following sense.

Lemma 4.7. *Let $i_{\boldsymbol{t}} : F[S_{\boldsymbol{t}}] \to F[S_{\boldsymbol{t}}]$ be the involution defined by*

$$i_{\boldsymbol{t}}(\pi_1, \cdots, \pi_m) = (\pi_1^{-1}, \cdots, \pi_m^{-1}).$$

Then, for all $x, y \in V_{\boldsymbol{t}}$ we have $i_{\boldsymbol{t}}(\varphi_{\boldsymbol{t}}(x, y)) = \varphi_{\boldsymbol{t}}(y, x)$.

Proof. Let $x = x_1 \otimes \cdots \otimes x_m$, $y = y_1 \otimes \cdots \otimes y_m$ be elements of $V_{\mathbf{t}}$. By Definition 4.4, we have

$$\varphi_{\mathbf{t}}(x, y) = \delta_1^{n_1} \cdots \delta_m^{n_m}(\pi_1, \cdots, \pi_m),$$

whenever the product $x_i y_i$ has t_i propagating lines for every i; otherwise, $\varphi_{\mathbf{t}}(x, y) = 0$.
Reflecting the diagram $x_i y_i$ upside down gives $y_i x_i$. Hence, the number of closed components and the scalar factor $\delta_i^{n_i}$ remain unchanged, while the permutation π_i is replaced by its inverse π_i^{-1}. Therefore,

$$\varphi_{\mathbf{t}}(y, x) = \delta_1^{n_1} \cdots \delta_m^{n_m}(\pi_1^{-1}, \cdots, \pi_m^{-1}) = i_{\mathbf{t}}(\varphi_{\mathbf{t}}(x, y)).$$

If $\varphi_{\mathbf{t}}(x, y) = 0$, then reflection also gives $\varphi_{\mathbf{t}}(y, x) = 0$, so the equality remains valid. Hence $i_{\mathbf{t}}(\varphi_{\mathbf{t}}(x, y)) = \varphi_{\mathbf{t}}(y, x)$. □

Lemma 4.8. *For $d = u \otimes \pi \otimes x \in V_{\boldsymbol{t}} \otimes F[S_{\boldsymbol{t}}] \otimes V_{\boldsymbol{t}}$, where $u, x \in V_{\boldsymbol{t}}$, $\pi = (\pi_1, \cdots, \pi_m) \in S_{\boldsymbol{t}}$. Define $i_{\boldsymbol{t}}$ on basis element by*

$$i_{\boldsymbol{t}}(d) = x \otimes \pi^{-1} \otimes u,$$

and extend $i_{\boldsymbol{t}}$ linearly. Then $i_{\boldsymbol{t}}$ is an anti-involution on $V_{\boldsymbol{t}} \otimes F[S_{\boldsymbol{t}}] \otimes V_{\boldsymbol{t}}$.

Proof. Let $d = u \otimes \pi \otimes x$ be a basis element. Reflection of the corresponding diagram upside down interchanges its top and bottom half diagrams and replaces each permutation π_i by π_i^{-1}. Hence,

$$i_{\mathbf{t}}(d) = x \otimes \pi^{-1} \otimes u.$$

Clearly, $i_{\mathbf{t}}^2 = u \otimes (\pi^{-1})^{-1} \otimes x = d$.
Now let, $d = u \otimes \pi \otimes x$, $d' = y \otimes \pi' \otimes v$. By Lemma 4.5,

$$dd' = u \otimes \pi\varphi_{\mathbf{t}}(x, y)\pi' \otimes v.$$

Therefore, using Lemma 4.7

$$\begin{aligned} i_{\mathbf{t}}(dd') &= v \otimes i_t(\pi\varphi_{\mathbf{t}}(x, y)\pi') \otimes u \\ &= v \otimes \pi'^{-1} i_{\mathbf{t}}(\varphi_{\mathbf{t}}(x, y))\pi^{-1} \otimes u \\ &= v \otimes (\pi')^{-1}\varphi_{\mathbf{t}}(y, x)^{-1} \otimes u \\ &= i_{\mathbf{t}}(d')i_{\mathbf{t}}(d). \end{aligned}$$

Thus, $i_{\mathbf{t}}(dd') = i_{\mathbf{t}}(d')i_{\mathbf{t}}(d)$, and hence $i_{\mathbf{t}}$ is an anti-involution. □

Lemma 4.9. *For each $0 \leq t \leq mk$ and $\boldsymbol{t} \in \mathcal{T}_t$, there are maps*

$$\phi_t : \mathfrak{M}_k \times \mathcal{H}_{\boldsymbol{t}} \to V_{\boldsymbol{t}} \qquad \text{and} \qquad \theta_t : \mathfrak{M}_k \times \mathcal{H}_{\boldsymbol{t}} \to F[S_{\boldsymbol{t}}],$$

such that for each $d = d_1 \otimes \cdots \otimes d_m \in \mathfrak{M}_k$, $u = u_1 \otimes \cdots \otimes u_m \in \mathcal{H}_{\boldsymbol{t}}$, $v = v_1 \otimes \cdots \otimes v_m \in \mathcal{H}_{\boldsymbol{t}}$ *and* $\pi = (\pi_1, \cdots, \pi_m) \in S_{\boldsymbol{t}}$, *we have*

$$d(u \otimes \pi \otimes v) = \phi_t(d,u) \otimes \theta_t(d,u)\pi \otimes v \pmod{W^{(\leq t-1)}}.$$

Proof. Let $d = d_1 \otimes \cdots \otimes d_m = x \otimes \sigma \otimes y$ be a basis element with $\#^p(d) = r = |\mathbf{r}|$, $x, y \in \mathcal{H}_{\mathbf{r}}$ and $\sigma \in S_{\mathbf{r}}$, and let $d' = u \otimes \pi \otimes v$, where $u, v \in \mathcal{H}_{\mathbf{t}}$, $\pi \in S_{\mathbf{t}}$, $|\mathbf{t}| = t$. By Lemma 3.2, $\#^p(dd') \leq \min\{r, t\}$.

- If $\#^p(dd') < t$, then $dd' \in W^{(\leq t-1)}$ and hence $dd' \equiv 0 \pmod{W^{(\leq t-1)}}$.
- If $\#^p(dd') = t = \#^p(d')$. Then the bottom half of dd' is precisely v. Consider the concatenation of the part of d containing its propagating lines with u. By the multiplication rule, this gives

  $$d(u \otimes \pi \otimes v) \equiv \lambda z \otimes \gamma\pi \otimes v \pmod{W^{(\leq t-1)}},$$

  where $\lambda \in F$ is the scalar contributed by the connected components removed from the middle row,$z \in \mathcal{H}_{\mathbf{t}}$ and $\gamma \in S_{\mathbf{t}}$ is the permutation determined by the resulting propagating lines. Importantly, γ depends only on d and u, and is independent of π. We therefore define

  $$\phi_t(d,u) := \lambda z, \qquad \theta_t(d,u) := \gamma.$$

  It follows that

  $$dd' \equiv \phi_t(d,u) \otimes \theta_t(d,u)\pi \otimes v \pmod{W^{(\leq t-1)}}.$$

  If $r = t$, then the propagating lines of d are all preserved, so $z = x$ and the induced permutation satisfies $\gamma = \sigma\varphi_{\mathbf{t}}(y,u)$. Consequently,

  $$dd' \equiv \phi_t(d,u) \otimes \theta_t(d,u)\pi \otimes v \pmod{W^{(\leq t-1)}}.$$

  If $r < t$, then $\#^p(dd') < t$, and hence $dd' \equiv 0 \pmod{W^{(\leq t-1)}}$.

Thus the required factorization holds in all cases. □

The following theorem establishes the cellularity of $P_k(\delta_1) \otimes \cdots \otimes P_k(\delta_m)$.

Theorem 4.10. *Let* F *be a field,* $\delta_1, \cdots, \delta_m \in F$. $A = P_k(\delta_1) \otimes \cdots \otimes P_k(\delta_m)$ *is a cellular algebra over* F.

Proof. By Lemma 4.2, A has the following vector space decomposition,

$$A = \bigoplus_{t=0}^{mk} \bigoplus_{\mathbf{t} \in \mathcal{T}_t} (V_{\mathbf{t}} \otimes F[S_{\mathbf{t}}] \otimes V_{\mathbf{t}}).$$

Here, $S_{\mathbf{t}} = S_{t_1} \times \cdots \times S_{t_m}$. By Proposition 2.1, $F[S_{\mathbf{t}}]$ is a cellular algebra with involution $(\pi_1, \cdots, \pi_m)^* = (\pi_1^{-1}, \cdots, \pi_m^{-1})$. Moreover, by Lemmas 4.8 and 4.9, the above decomposition satisfies the hypotheses of the iterated inflation theorem of Green-Paget [4, Theorem 1]. Hence, A is a cellular algebra over F. □

As a consequence of the previous theorem and from [4, Proposition 3], we have the following.

Corollary 4.11. *The cell modules of* $A = P_k(\delta_1) \otimes \cdots \otimes P_k(\delta_m)$ *are indexed by* $\boldsymbol{t} = (t_1, \cdots, t_m) \in \mathcal{T}_t$ *and* $[\lambda] = (\lambda^{(1)}, \cdots, \lambda^{(m)}) \in \Lambda_{t_1} \times \cdots \times \Lambda_{t_m}$, *where* $\lambda^{(i)}$ *is a partition of* t_i. *They are given by*

$$\Delta^{\boldsymbol{t}}_{[\lambda]} = V_{\boldsymbol{t}} \otimes S^{[\lambda]},$$

where $V_{\boldsymbol{t}} = V_{t_1} \otimes \cdots \otimes V_{t_m}$ *and* $S^{[\lambda]} = S^{\lambda^{(1)}} \otimes \cdots \otimes S^{\lambda^{(m)}}$ *is the cell module of* $F[S_{\boldsymbol{t}}] = F[S_{t_1} \times \cdots \times S_{t_m}]$ *corresponding to* $[\lambda]$.

4.2. **The cell datum and cellular basis of** $A = P_k(\delta_1) \otimes \cdots \otimes P_k(\delta_m)$**.** In this subsection, we describe a cellular basis for $A = P_k(\delta_1) \otimes \cdots \otimes P_k(\delta_m)$ in the sense of Graham and Lehrer [3].

Definition 4.12. For $\mathbf{t} = (t_1, \cdots, t_m) \in \mathcal{T}_t$, $0 \leq t_i \leq k$, let

$$\Lambda_{\mathbf{t}} = \Lambda_{t_1} \times \cdots \times \Lambda_{t_m},$$

where Λ_{t_i} denotes the set of partitions of t_i. Thus,

$$[\lambda] = (\lambda^{(1)}, \cdots, \lambda^{(m)}) \in \Lambda_{\mathbf{t}}, \quad \text{where } \lambda^{(i)} \vdash t_i \text{ for } 1 \leq i \leq m.$$

Define,

$$\mathcal{P}_{mk} = \{(\mathbf{t}, [\lambda]) : \mathbf{t} \in \mathcal{T}_t, [\lambda] \in \Lambda_{\mathbf{t}}, 0 \leq t \leq mk\}.$$

For $(\mathbf{t}, [\lambda]), (\mathbf{l}, [\mu]) \in \mathcal{P}_{mk}$, where $\mathbf{t} = (t_1, \cdots, t_m)$, $\mathbf{l} = (l_1, \cdots, l_m)$, define an order relation in $\mathcal{P}_{mk}$ by

$$(\mathbf{t}, [\lambda]) \leq (\mathbf{l}, [\mu]) \text{ if and only if } (t_i, \lambda^{(i)}) \leq (l_i, \mu^{(i)}) \text{ for all } 1 \leq i \leq m,$$

where the order on each pair is given by,

$$(t_i, \lambda^{(i)}) \leq (l_i, \mu^{(i)}) \quad \text{if either} \quad t_i < l_i \text{ or } (t_i = l_i \text{ and } \lambda^{(i)} \trianglerighteq \mu^{(i)}).$$

Clearly this ordered relation is a partial order relation since the dominance order $\lambda^{(i)} \trianglerighteq \mu^{(i)}$ is a partial order.

Definition 4.13. For $(\mathbf{t}, [\lambda]) \in \mathcal{P}_{mk}$, define

$$M(\mathbf{t}, [\lambda]) = \{(x, [s]) \, : \, x \in \mathcal{H}_{\mathbf{t}}, [s] \in \mathcal{T}^m([\lambda])\},$$

where $[\lambda] = (\lambda^{(1)}, \cdots, \lambda^{(m)})$ and $\mathcal{T}^m([\lambda]) = \mathcal{T}(\lambda^{(1)}) \cdots \times \mathcal{T}(\lambda^{(m)})$.

We now use the cellular basis of $F[S_{n_1} \times \cdots \times S_{n_m}]$, given by Proposition 2.1 to construct a cellular basis of A as follows.

For $(\mathbf{t}, [\lambda]) \in \mathcal{P}_{mk}$ and $(x, [u]), (y, [v]) \in M(\mathbf{t}, [\lambda])$, define

$$m^{(\mathbf{t}, [\lambda])}_{(x,[u]),(y,[v])} = x \otimes m^{[\lambda]}_{[u][v]} \otimes y,$$

where $x, y \in \mathcal{H}_{\mathbf{t}}$, $[u], [v] \in \mathcal{T}^m([\lambda])$.
By Lemma 4.2, the tensor product of m-partition algebras A has the following basis

$$\{x \otimes \pi \otimes y \, : \, x, y \in \mathcal{H}_{\mathbf{t}}, \pi \in S_{\mathbf{t}}, \mathbf{t} \in \mathcal{T}_t, 0 \leq t \leq mk\}.$$

Here, $\mathbf{t} = (t_1, \cdots, t_m)$, $|\mathbf{t}| = \sum_{i=1}^{m} t_i = t$, and $S_{\mathbf{t}} = S_{t_1} \times \cdots \times S_{t_m}$.
For each $(\mathbf{t}, [\lambda]) \in \mathcal{P}_{mk}$, the elements

$$m^{(\mathbf{t},[\lambda])}_{(x,[u]),(y,[v])} = x \otimes m^{[\lambda]}_{[u][v]} \otimes y,$$

where $(x, [u]), (y, [v]) \in M(\mathbf{t}, [\lambda])$, form a basis of $V_{\mathbf{t}} \otimes F[S_{\mathbf{t}}] \otimes V_{\mathbf{t}}$. Since

$$\{m^{[\lambda]}_{[u],[v]} \ : \ [\lambda] \in \Lambda_{\mathbf{t}}, [u], [v] \in \mathcal{T}^m([\lambda])\}$$

is the cellular basis of $F[S_{\mathbf{t}}]$ given by Proposition 2.1 . Consequently,

$$\mathcal{M} = \{m^{(\mathbf{t},[\lambda])}_{(x,[u]),(y,[v])} \ : \ (\mathbf{t}, [\lambda]) \in \mathcal{P}_{mk}, (x, [u]), (y, [v]) \in M(\mathbf{t}, [\lambda])\}$$

forms a basis of A.
Thus, condition (C_1) stated in Section 2.1 is satisfied.

Furthermore, by Lemma 4.8,

$$i_{\mathbf{t}}(m^{(\mathbf{t},[\lambda])}_{(x,[u]),(y,[v])}) = y \otimes m^{[\lambda]}_{[v][u]} \otimes x = m^{(\mathbf{t},[\lambda])}_{(y,[v]),(x,[u])}.$$

Hence, condition (C_2) stated in Section 2.1 is satisfied.

In order to show that $\mathcal{M}$ is a cellular basis of $P_k(\delta_1) \otimes \cdots \otimes P_k(\delta_m)$, it remains to verify condition (C_3), which is shown in the following.

Proposition 4.14. *Let $z = a \otimes \pi \otimes b$ be any basis element of A and $m^{(\boldsymbol{t},[\lambda])}_{(x,[u]),(y,[v])} \in \mathcal{M}$, then there exists $r_{(x',[s])} \in F$ such that*

$$z m^{(\boldsymbol{t},[\lambda])}_{(x,[u]),(y,[v])} \equiv \sum_{(x',[s]) \in M(\boldsymbol{t},[\lambda])} r_{(x',[s])} m^{(\boldsymbol{t},[\lambda])}_{(x',[s]),(y,[v])} \quad (mod\, \bar{A}^{(\boldsymbol{t},[\lambda])}),$$

where

$$\bar{A}^{(\boldsymbol{t},[\lambda])} := Span\{m^{(\boldsymbol{l},[\mu])}_{(x,[u]),(y,[v])} : (x, [u]), (y, [v]) \in M(\boldsymbol{l}, [\mu]), (\boldsymbol{l}, [\mu]) < (\boldsymbol{t}, [\lambda])\}$$

Proof. From Lemma 4.9 , we have

$$\begin{aligned} & z m^{(\mathbf{t},[\lambda])}_{(x,[u]),(y,[v])} \\ & = (a \otimes \pi \otimes b)(x \otimes m^{[\lambda]}_{[u][v]} \otimes y) \\ & \equiv \phi_t(z, x) \otimes \theta_t(z, x) m^{[\lambda]}_{[u][v]} \otimes y \pmod{\bar{A}^{(\mathbf{t},[\lambda])}} \\ & \equiv \alpha\, x' \otimes \sigma m^{[\lambda]}_{[u][v]} \otimes y \pmod{\bar{A}^{(\mathbf{t},[\lambda])}} \end{aligned}$$

where $x' \in \mathcal{H}_{\mathbf{t}}$ and $\sigma = (\sigma_1, \cdots, \sigma_m) \in S_{\mathbf{t}}$, $\alpha \in F$, these quantities are independent of $m^{[\lambda]}_{[u][v]}$.
Since $m^{[\lambda]}_{[u][v]}$ is a cellular basis element of $F[S_{\mathbf{t}}]$, we have

$$(\sigma_1, \cdots, \sigma_m) m^{[\lambda]}_{[u][v]} = \otimes_{i=1}^{m} \sigma_i m^{\lambda^{(i)}}_{u^{(i)} v^{(i)}}$$

where $m^{\lambda^{(i)}}_{u^{(i)}v^{(i)}}$ is a Murphy basis of FS_{t_i}, then (from [10, Theorem 3.20]), we have,

$$\begin{aligned}\sigma_i m^{\lambda^{(i)}}_{u^{(i)}v^{(i)}} &\equiv \sum_{s^{(i)}\in\mathcal{T}(\lambda^{(i)})} r_{s^{(i)}} m^{\lambda^{(i)}}_{s^{(i)}v^{(i)}} \ (mod\, \bar{A}^{(\lambda^{(i)})})\\ &= \sum_{s^{(i)}\in\mathcal{T}(\lambda^{(i)})} r_{s^{(i)}} m^{\lambda^{(i)}}_{s^{(i)}v^{(i)}} + \sum_{\mu^{(i)}\rhd\lambda^{(i)}} a_{p^{(i)}q^{(i)}} m^{\mu^{(i)}}_{p^{(i)}q^{(i)}}\end{aligned}$$

where $s^{(i)}$ depends on $\sigma_{(i)}$ and $u^{(i)}$, $r_{s^{(i)}} \in F$. Consequently,

$$\sigma m^{[\lambda]}_{[u][v]} = \sum_{[s]\in\mathcal{T}^m([\lambda])} r_{[s]} m^{[\lambda]}_{[s][v]} + \sum_{[\mu]\rhd[\lambda]} a_{[p][q]} m^{[\mu]}_{[p][q]}$$

where $[\mu] \rhd [\lambda]$ means $\mu^{(i)} \rhd \lambda^{(i)}$ for all $1 \le i \le m$.
Therefore,

$$\begin{aligned}&zm^{(\mathbf{t},[\lambda])}_{(x,[u]),(y,[v])}\\ &= \alpha x' \otimes \Big(\sum_{[s]\in\mathcal{T}^m([\lambda])} r_{[s]} m^{[\lambda]}_{[s][v]} + \sum_{[\mu]\rhd[\lambda]} a_{[p][q]} m^{[\mu]}_{[p][q]} \Big) \otimes y + \text{terms in } \bar{A}^{(\mathbf{t},[\lambda])}\\ &= \alpha x' \otimes \sum_{[s]\in\mathcal{T}^m([\lambda])} r_{[s]} m^{[\lambda]}_{[s][v]} \otimes y + \text{terms in } \bar{A}^{(\mathbf{t},[\lambda])}\\ &\equiv \sum_{[s]\in\mathcal{T}^m([\lambda])} \alpha r_{[s]} \big(x' \otimes m^{[\lambda]}_{[s][v]} \otimes y\big) \quad (mod\, \bar{A}^{(\mathbf{t},[\lambda])})\\ &\equiv \sum_{(x',[s])\in M(\mathbf{t},[\lambda])} r_{(x',[s])} m^{(\mathbf{t},[\lambda])}_{(x'[s])(y,[v])} \qquad (mod\, \bar{A}^{(\mathbf{t},[\lambda])})\end{aligned}$$

□

Therefore, the datum $(\mathcal{P}_{mk}, M, \mathcal{M}, i_{\mathbf{t}})$ is a cell datum for A.

Since, by Proposition 2.1, the cell module $S^{[\lambda]}$ of $F[S_{n_1} \times \cdots \times S_{n_m}]$ has basis

$$\{m^{[\lambda]}_{[v]}, [v] \in \mathcal{T}^m([\lambda])\},$$

we obtain that the set

$$\{x \otimes m^{[\lambda]}_{[v]} \ : \ x \in \mathcal{H}_{\mathbf{t}}\,, m^{[\lambda]}_{[v]} \text{is a basis element of } \ S^{[\lambda]}\}$$

is a basis of the cell module $\Delta^{\mathbf{t}}_{[\lambda]} = V_{\mathbf{t}} \otimes S^{[\lambda]}$ of A.

4.3. **The indexing set of the simple modules for $A = P_k(\delta_1) \otimes \cdots \otimes P_k(\delta_m)$.** In this subsection, we determine the indexing set of the simple modules. We first describe the bilinear form on the cell modules and then use a result of Dipper and James to determine when this form is non-zero.

Definition 4.15. For $(\mathbf{t},[\lambda]) \in \mathcal{P}_{mk}$, define
$\Phi_{(\mathbf{t},[\lambda])} : \Delta^{\mathbf{t}}_{[\lambda]} \times \Delta^{\mathbf{t}}_{[\lambda]} \to F$ by

$$m^{(\mathbf{t},[\lambda])}_{(x,[u]),(y,[v])} m^{(\mathbf{t},[\lambda])}_{(x',[u']),(y',[v'])} \equiv \Phi_{(\mathbf{t},[\lambda])}\big(y \otimes m^{[\lambda]}_{[v]}, x' \otimes m^{[\lambda]}_{[u']}\big)\ m^{(\mathbf{t},[\lambda])}_{(x,[u]),(y',[v'])}\ (mod\ \bar{A}^{(\mathbf{t},[\lambda])}).$$

From [4, Proposition 3] we have the following.

Proposition 4.16.

$$\Phi_{(\boldsymbol{t},[\lambda])}\big(x \otimes m^{[\lambda]}_{[v]}, y \otimes m^{[\lambda]}_{[v']}\big) := \phi_{[\lambda]}(m^{[\lambda]}_{[v]}, \varphi_{\boldsymbol{t}}(x,y) m^{[\lambda]}_{[v']}).$$

where $x, y \in V_{\boldsymbol{t}}$ *and*

$$\phi_{[\lambda]} : S^{[\lambda]} \times S^{[\lambda]} \to F$$

is the symmetric F*-bilinear form on* $S^{[\lambda]}$ *of* $F[S_{\boldsymbol{t}}]$, $[\lambda] = (\lambda^{(1)}, \cdots, \lambda^{(m)})$, $\lambda^{(i)}$ *is a partition of* t_i.

Lemma 4.17. *Let* $\delta_1, \cdots, \delta_m$ *be non-zero elements in a field* F. *Then* $\Phi_{(\boldsymbol{t},[\lambda])} \neq 0$ *if and only if its corresponding bilinear form* $\phi_{[\lambda]}$ *for cellular algebra* $F[S_{\boldsymbol{t}}]$ *is non-zero.*

Proof. Write $\mathbf{t} = (t_1, \cdots, t_m)$, $\quad |\mathbf{t}| = \sum_{i=1}^m t_i = t$.
Suppose first that $t > 0$. Let $x = \bigotimes_{j=1}^m x_j \in V_t$, where $x_j \in V_{t_j}$. By proposition 4.16,

$$\Phi_{(\mathbf{t},[\lambda])}\big(x \otimes m^{[\lambda]}_{[v]}, i_{\mathbf{t}}(x) \otimes m^{[\lambda]}_{[v']}\big) = \phi_{[\lambda]}(m^{[\lambda]}_{[v]}, \varphi_{\mathbf{t}}(x, i_{\mathbf{t}}(x)) m^{[\lambda]}_{[v']}).$$

Since,

$$\varphi_{\mathbf{t}}(x, i_{\mathbf{t}}(x)) = \bigotimes_{j=1}^m \varphi_{t_j}(x_j, i_{\mathbf{t}}(x_j)),$$

we may choose, for each j, $x_j \in V_{t_j}$, such that, by Definition 4.4

$$\varphi_{t_j}(x_j, i_{\mathbf{t}}(x_j)) = \delta_j^{n_j} id_{S_{t_j}}.$$

where n_j is the number of closed components arising in the corresponding product. Hence

$$\varphi_{\mathbf{t}}(x, i_{\mathbf{t}}(x)) = \big(\prod_{j=1}^m \delta_j^{n_j}\big) id_{S_{\mathbf{t}}}.$$

Therefore,

$$\Phi_{(\mathbf{t},[\lambda])}\big(x \otimes m^{[\lambda]}_{[v]}, i_{\mathbf{t}}(x) \otimes m^{[\lambda]}_{[v']}\big) = \big(\prod_{j=1}^m \delta_j^{n_j}\big) \phi_{[\lambda]}(m^{[\lambda]}_{[v]}, m^{[\lambda]}_{[v']}). \qquad (1)$$

Since every $\delta_j \neq 0$, the scalar $\prod_{j=1}^m \delta_j^{n_j}$ is non-zero. It follows from (1) that if $\phi_{[\lambda]} \neq 0$, then $\Phi_{(\mathbf{t},[\lambda])} \neq 0$.
Conversely, suppose that $\phi_{[\lambda]} = 0$, then $\phi_{[\lambda]}(m^{[\lambda]}_{[v]}, \pi m^{[\lambda]}_{[v']}) = 0$ for all $[v], [v'] \in \mathcal{T}^m([\lambda])$, $\pi \in S_{\mathbf{t}}$. Hence, by Proposition 4.16, $\Phi_{(\mathbf{t},[\lambda])} = 0$.

It remains to consider the case $t = 0$. Then $t_j = 0$ for every j, and hence $S_0 = S_0 \cdots \times S_0 \cong F$.

So,for $x \in V_0$, we have $\varphi_0(x, i_{\mathbf{t}}(x)) = \prod_{j=1}^{m} \delta_j^{n_j}$.
Since all δ_j are non-zero, this scalar is non-zero. Thus, by Proposition 4.16, $\Phi_{(0,[\lambda])} \neq 0$ whenever $\phi_{[\lambda]} \neq 0$.
Therefore, $\Phi_{(\mathbf{t},[\lambda])} \neq 0 \Leftrightarrow \phi_{[\lambda]} \neq 0$. This completes the proof. □

Theorem 4.18. *Let $A = P_k(\delta_1) \otimes \cdots \otimes P_k(\delta_m)$ be the tensor product of m-partition algebras over a field F of characteristic p, where $p \geq 0$. If $\delta_1, \cdots, \delta_m \neq 0$, then the non-isomorphic simple A-modules are parameterized by*

$$\{(\boldsymbol{t}, [\lambda]) \in \mathcal{P}_{mk} : \lambda^{(i)} \text{ is a } p\text{-regular partition of } t_i,\ 1 \leq i \leq m\},$$

where $\boldsymbol{t} = (t_1, \cdots, t_m)$, $0 \leq t_i \leq k$, $[\lambda] = (\lambda^{(1)}, \cdots, \lambda_{(m)})$.

Recall that a partition is p-regular if it does not have p equal parts ($p \neq 0$); if $p = 0$, then all partitions are p-regular.

Proof. By Theorem 4.10 and [3, Theorem 3.4], the non-isomorphic simple A-modules are parameterized by those $(\mathbf{t}, [\lambda]) \in \mathcal{P}_{mk}$ for which $\Phi_{(\mathbf{t},[\lambda])} \neq 0$.
Suppose first that $|\mathbf{t}| = \sum_{i=1}^{m} t_i > 0$. By Lemma 4.17, $\Phi_{(\mathbf{t},[\lambda])} \neq 0$ if and only if the corresponding bilinear form $\phi_{[\lambda]}$ on the cell module $S^{[\lambda]}$ of $F[S_{\mathbf{t}}]$ is non-zero. Since $S_{\mathbf{t}} = S_{t_1} \times \cdots \times S_{t_m}$, and $S^{[\lambda]} = S^{\lambda^{(1)}} \otimes \cdots \otimes S^{\lambda^{(m)}}$, the bilinear form $\phi_{[\lambda]}$ is the tensor product of the corresponding bilinear forms $\phi_{(t_i,\lambda^{(i)})}$ on the cell modules of $F[S_{t_i}]$. Hence $\phi_{[\lambda]} \neq 0$ if and only if $\phi_{(t_i,\lambda^{(i)})} \neq 0$ for all $1 \leq i \leq m$.
By the result of Dipper and James [1, 7.6], $\phi_{(t_i,\lambda^{(i)})} \neq 0$ if and only if $\lambda^{(i)}$ is p-regular partition of t_i. Therefore $\Phi_{(\mathbf{t},[\lambda])} \neq 0$ if and only if each $\lambda^{(i)}$ is p-regular.
It remains to consider the case $|\mathbf{t}| = 0$. Then $t_1 = \cdots = t_m = 0$, and by Lemma 4.17, $\Phi_{(0,[\lambda])} \neq 0$ because $\delta_i \neq 0$ for all $1 \leq i \leq m$. Since the unique partition of 0 is p-regular, this case is also included in the stated parametrization.
Hence the non-isomorphic simple A-modules are parameterized by

$$\{(\mathbf{t}, [\lambda]) \in \mathcal{P}_{mk} : \lambda^{(i)} \text{ is a p-regular partition of } t_i,\ 1 \leq i \leq m\}.$$

This completes the proof. □

5. Conclusion

In this paper, we established an explicit cellular structure for the tensor product of m-partition algebras by realizing it as an iterated inflation of tensor products of symmetric group algebras. This approach makes explicit how the diagrammatic structure of partition algebras is linked to the representation theory of symmetric groups through the successive inflation layers.

The resulting cellular basis gives explicit descriptions of the cell modules and their bilinear forms and, for non-zero parameters, leads to a parametrization of the non-isomorphic simple modules by pairs $(\mathbf{t}, [\lambda])$, where each $\lambda^{(i)}$ is a p-regular partition of t_i. Thus, the iterated inflation framework provides an effective method for transferring structural and representation-theoretic information from symmetric group algebras to tensor products of partition algebras.

The explicit cellular structure developed here may serve as a basis for further study of the representation theory of these algebras, including decomposition numbers, radicals of cell modules, and behavior in positive characteristic.

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