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CAPUTO–VOLTERRA DYNAMICS WITH STATE-DEPENDENT DELAY, MEMORY, AND STABILITY

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Abstract. We study a scalar Caputo fractional differential equation coupling state-dependent delayed feedback with Volterra-type memory. Because the state-dependent evaluation operator is not generally Lipschitz on the natural mild-solution space, existence and uniqueness are established through separate arguments: Schaefer's fixed-point theorem yields existence on arbitrary finite intervals, while uniqueness follows from an explicit local Lipschitz condition on the delayed evaluation map. For globally integrable memory kernels, a quadratic Caputo estimate combined with a first-maximum argument yields an explicit sufficient dissipativity condition for uniform stability. Numerical experiments illustrate the roles of fractional order, feedback and memory strength, delay amplitude, and kernel integrability.

Keywords. Caputo fractional equations, state-dependent delay, Volterra memory, fixed-point theory, uniform stability
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1. Introduction

Fractional differential equations model hereditary effects, while delay equations describe dependence on past states. Their interaction becomes more delicate when the delay depends on the current state, because the delayed evaluation changes both the sampled value and its sampling point. We study this interaction for a scalar dissipative Caputo equation with state-dependent feedback and Volterra memory, with particular emphasis on the distinct assumptions required for existence, uniqueness, and long-time stability.

Caputo equations and their integral formulations are classical [21, 18, 7], and Lyapunov methods for fractional systems are well developed [3, 2]. Fixed-point and stability methods for related Caputo-type integro-differential problems have also been developed in [4]. State-dependent delay equations have an established

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classical theory [13]; in the fractional setting, practical stability, abstract well-posedness, and phase-space regularity have been studied in [1, 15, 16]. A recent *Gulf Journal of Mathematics* contribution treats existence and stability for Caputo-tempered fractional delay integro-differential equations [12], while Ulam-type stability for nonlinear implicit Caputo problems is considered in [17]. Related fractional Volterra–Fredholm problems with state-dependent delay have been treated by fixed-point and Ulam-stability techniques [10]; further background on fractional delay and integral-memory models is available in [6, 11, 22, 20]. For nonlinear fractional integro-differential systems, recent well-posedness and stability results further emphasize nonlocal memory effects [5]; complementary numerical work in this journal addresses Volterra equations with proportional delay [14].

The present contribution concerns the combined Caputo–Volterra model

$$ {}^{C}D_t^{\alpha} x(t) = -a x(t) + b f\big(x(t - \tau(x(t)))\big) + \int_0^t K(t-s)\, g(x(s))\, ds, \quad t > 0, \tag{1.1} $$

with history

$$ x(t) = \phi(t), \quad t \in [-\tau_M, 0]. \tag{1.2} $$

Here $0 < \alpha < 1$, $a > 0$, b is the feedback gain, τ is a bounded state-dependent delay, and K is the memory kernel; the precise hypotheses are given in Section 2. The central analytical point is that the map $x \mapsto x(\cdot - \tau(x(\cdot)))$ is not automatically Lipschitz on the natural space of Caputo mild solutions. Indeed, such solutions may be only Hölder continuous near the initial time, so a direct contraction argument cannot be justified from the basic assumptions alone. We therefore obtain existence by compactness through Schaefer's fixed-point theorem and state uniqueness separately under an explicit local Lipschitz condition on the delayed evaluation map.

Main contributions. The contributions are fourfold. First, existence on arbitrary finite intervals is established without imposing artificial time-Lipschitz regularity, while uniqueness is isolated under a transparent phase-space condition reflecting the genuine difficulty of state-dependent evaluation [16, 15]. Second, a quadratic Caputo estimate combined with a first-maximum argument yields the explicit sufficient condition $a > bL_f + L_g M_K$ for uniform stability without differentiating a functional with moving delay limits. Third, the scalar equation unifies state-dependent fractional delay, constant-delay Caputo–Volterra, and distributed-memory Caputo models, thereby making the roles of the two nonlocal mechanisms explicit. Finally, the numerical experiments distinguish the effects of fractional order, feedback and memory strengths, delay amplitude, and kernel integrability, while separating regimes covered by the stability theorem from exploratory cases outside its hypotheses.

Section 2 formulates the assumptions, Section 3 develops existence and conditional uniqueness, Section 4 proves the stability result, Section 5 presents the numerical illustrations, and Section 6 concludes the paper with a brief discussion of model interpretations and future directions.

2. Model Formulation and Assumptions

We consider the scalar Caputo fractional differential equation

$$\begin{cases} {}^{C}D_t^{\alpha} x(t) = -ax(t) + bf(x(t-\tau(x(t)))) \\ \qquad\qquad + \displaystyle\int_0^t K(t-s)g(x(s))\,ds, & t > 0, \\ x(t) = \phi(t), & t \in [-\tau_M, 0], \end{cases} \tag{2.1}$$

where $0 < \alpha < 1$, $a > 0$, $b \geq 0$, and $\tau_M > 0$. The first term on the right-hand side describes instantaneous dissipation, the second represents a delayed feedback whose delay depends on the present state, and the last term accounts for distributed memory through a Volterra kernel K.

The assumptions needed for existence on a bounded interval are deliberately kept separate from those used later for uniqueness and long-time stability.

(A1) **Nonlinearities.** The functions $f, g : \mathbb{R} \to \mathbb{R}$ are globally Lipschitz continuous: there exist constants $L_f, L_g > 0$ such that

$$|f(u) - f(v)| \leq L_f|u - v|, \qquad |g(u) - g(v)| \leq L_g|u - v|, \qquad u, v \in \mathbb{R}.$$

We also assume

$$f(0) = g(0) = 0.$$

The normalization ensures that $x^* = 0$ is an equilibrium whenever the remaining terms are evaluated at zero.

(A2) **State-dependent delay.** The delay function

$$\tau : \mathbb{R} \to [0, \tau_M]$$

is continuous. Thus every delayed argument satisfies

$$t - \tau(x(t)) \in [-\tau_M, t].$$

When a quantitative uniqueness estimate is required, additional regularity of the state-dependent evaluation map will be stated explicitly.

(A3) **Memory kernel.** The kernel is non-negative and locally integrable,

$$K \in L^1_{\mathrm{loc}}([0, \infty)), \qquad K(t) \geq 0 \quad \text{a.e.}$$

For $T > 0$, set

$$M_K(T) := \int_0^T K(s)\,ds. \tag{2.2}$$

Local integrability is sufficient for the existence theory on finite time intervals.

(A4) **Initial history.** The history function satisfies

$$\phi \in C([-\tau_M, 0], \mathbb{R}).$$

(A5) **Global memory bound for the stability analysis.** When a stability statement is required on the whole half-line, we additionally assume

$$K \in L^1([0, \infty)), \qquad M_K := \int_0^{\infty} K(s)\,ds < \infty. \tag{2.3}$$

This assumption is not needed merely to construct a solution on a bounded interval; it is introduced only when a uniform-in-time memory bound is used.

The distinction between (A3) and (A5) is useful in applications. For example, an exponential kernel $K(t) = K_0 e^{-\mu t}$ satisfies both assumptions, whereas a locally integrable kernel with a non-integrable tail may satisfy (A3) but not (A5). The latter case is therefore covered by the existence result below but falls outside the global stability criterion developed later.

Integral formulation. For $0 < \alpha < 1$, problem (2.1) is equivalent, in the mild sense, to

$$x(t) = \phi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Bigg[-ax(s) + bf(x(s - \tau(x(s)))) + \int_0^s K(s-\sigma) g(x(\sigma))\, d\sigma \Bigg] ds, \qquad 0 \le t \le T, \tag{2.4}$$

together with $x(t) = \phi(t)$ for $t \in [-\tau_M, 0]$.

Definition 2.1. Let $T > 0$. A function $x \in C([-\tau_M, T], \mathbb{R})$ is called a mild solution of (2.1) on $[-\tau_M, T]$ if

$$x(t) = \phi(t), \qquad -\tau_M \le t \le 0,$$

and (2.4) holds for every $t \in [0, T]$.

A basic regularity point is worth recording before the fixed-point argument. A mild solution of a Caputo equation need not be Lipschitz at the initial time. Indeed,

$${}^C D_t^\alpha x(t) = 1, \qquad x(0) = 0,$$

has the solution

$$x(t) = \frac{t^\alpha}{\Gamma(1+\alpha)},$$

which is not Lipschitz at $t = 0$ for any $0 < \alpha < 1$. Consequently, existence below is obtained through compactness rather than by assuming an a priori Lipschitz regularity of the unknown solution.

3. Existence of Mild Solutions and Conditional Uniqueness

The state-dependent delay creates a genuine distinction between existence and uniqueness. Continuity of the delayed evaluation is sufficient for a compactness argument leading to existence. A contraction estimate, on the other hand, requires additional control of

$$x \longmapsto x(\,\cdot - \tau(x(\cdot)))\,.$$

We therefore treat the two issues separately.

Existence on an arbitrary finite interval. Fix $T > 0$ and work in the closed subspace

$$X_T^0 := \{u \in C([0,T],\mathbb{R}) : u(0) = 0\}, \qquad \|u\|_T := \sup_{0\le t\le T} |u(t)|.$$

Since X_T^0 is closed in $C([0,T],\mathbb{R})$, it is a Banach space. For $u \in X_T^0$, define

$$\widehat{u}(t) = \begin{cases} \phi(t), & -\tau_M \le t \le 0, \\ \phi(0) + u(t), & 0 < t \le T. \end{cases} \tag{3.1}$$

Define $\mathcal{A} : X_T^0 \to X_T^0$ by

$$(\mathcal{A}u)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F_{\widehat{u}}(s)\, ds, \tag{3.2}$$

where

$$F_{\widehat{u}}(s) = -a\widehat{u}(s) + bf(\widehat{u}(s - \tau(\widehat{u}(s)))) + \int_0^s K(s-\sigma) g(\widehat{u}(\sigma))\, d\sigma. \tag{3.3}$$

Because $(\mathcal{A}u)(0) = 0$, the operator indeed maps X_T^0 into itself. Moreover, the choice $u(0) = 0$ makes the extension (3.1) continuous at $t = 0$. A fixed point $u = \mathcal{A}u$ therefore yields a mild solution through $x(t) = \widehat{u}(t)$.

Theorem 3.1. *Assume* (A1)–(A4). *Then, for every $T > 0$, problem* (2.1) *admits at least one mild solution $x \in C([-\tau_M, T], \mathbb{R})$.*

Proof. We apply Schaefer's fixed-point theorem [9]. Let $u_n \to u$ in X_T^0. Then $\widehat{u}_n \to \widehat{u}$ uniformly on $[-\tau_M, T]$. Since the family is uniformly bounded and τ is continuous, τ is uniformly continuous on the compact range of these functions; hence

$$q_n(s) := s - \tau(\widehat{u}_n(s)) \longrightarrow q(s) := s - \tau(\widehat{u}(s))$$

uniformly on $[0,T]$. Moreover,

$$|\widehat{u}_n(q_n(s)) - \widehat{u}(q(s))| \le \|\widehat{u}_n - \widehat{u}\|_\infty + |\widehat{u}(q_n(s)) - \widehat{u}(q(s))| \longrightarrow 0$$

uniformly. Assumptions (A1) and (A3) therefore imply $F_{\widehat{u}_n} \to F_{\widehat{u}}$ uniformly on $[0,T]$, and consequently $\mathcal{A}u_n \to \mathcal{A}u$ in X_T^0. Thus $\mathcal{A}$ is continuous.

If $B \subset X_T^0$ is bounded, choose $R > 0$ such that $\|\widehat{u}\|_{C([-\tau_M,T])} \le R$ for $u \in B$. From $f(0) = g(0) = 0$,

$$|F_{\widehat{u}}(s)| \le \big(a + bL_f + L_g M_K(T)\big) R =: C_T R, \qquad 0 \le s \le T. \tag{3.4}$$

Hence

$$|(\mathcal{A}u)(t)| \le \frac{C_T R}{\Gamma(1+\alpha)} t^\alpha.$$

The standard mapping properties of the fractional integral [7] give, uniformly for $u \in B$,

$$|(\mathcal{A}u)(t_2) - (\mathcal{A}u)(t_1)| \le C(\alpha, T) C_T R |t_2 - t_1|^\alpha.$$

Thus $\mathcal{A}(B)$ is bounded and equicontinuous; by the Arzelà–Ascoli theorem, $\mathcal{A}$ is completely continuous.

It remains to bound the Schaefer set

$$\mathcal{S} := \{u \in X_T^0 : \ u = \lambda \mathcal{A}u \text{ for some } \lambda \in [0,1]\}.$$

For $u \in \mathcal{S}$, set $x = \widehat{u}$ and

$$Y(t) := \max\left\{\|\phi\|_\infty, \sup_{0\le r\le t}|x(r)|\right\}.$$

Since $s - \tau(x(s)) \in [-\tau_M, s]$, both the delayed state and the memory states are bounded by $Y(s)$. Using $0 \le \lambda \le 1$ and (A1),

$$Y(t) \le \|\phi\|_\infty + \frac{C_T}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Y(s)\,ds.$$

The fractional Grönwall inequality yields

$$Y(t) \le \|\phi\|_\infty E_\alpha(C_T t^\alpha), \qquad 0 \le t \le T. \tag{3.5}$$

This estimate is independent of λ, so $\mathcal{S}$ is bounded. Schaefer's theorem therefore gives a fixed point of $\mathcal{A}$, whose extension (3.1) is a mild solution of (2.1). □

Remark 3.2. Theorem 3.1 requires no lower threshold on α beyond $0 < \alpha < 1$. It also requires only local integrability of the memory kernel. The stronger global condition (A5) is reserved for the long-time stability analysis.

A global existence consequence.

Corollary 3.3. *Under assumptions* (A1)–(A4)*, problem* (2.1) *admits at least one mild solution defined on* $[-\tau_M, \infty)$.

Proof. For each integer $n \ge 1$, Theorem 3.1 provides a mild solution x_n on $[-\tau_M, n]$. Fix $m \in \mathbb{N}$. For every $n \ge m$, estimate (3.5), with $T = m$, bounds x_n uniformly on $[-\tau_M, m]$ by a constant depending only on the data and on m. The integral representation also yields a Hölder estimate on $[0, m]$ whose constant depends only on m, α, the data, and the preceding uniform bound, and therefore not on n. Hence the restrictions $x_n|_{[-\tau_M,m]}$ form a relatively compact family in $C([-\tau_M, m])$.

Applying the Arzelà–Ascoli theorem successively for $m = 1, 2, \dots$ and using a diagonal subsequence gives a function $x \in C([-\tau_M, \infty))$ such that the selected subsequence converges uniformly to x on every compact interval. Passing to the limit in (2.4), using the continuity argument from Theorem 3.1, shows that x satisfies the mild equation on every finite interval. Thus x is a global mild solution. □

The state-dependent evaluation and uniqueness. The preceding result establishes existence without requiring a Lipschitz regularity of the solution in time. Uniqueness is more delicate. Define

$$(\mathcal{D}x)(t) := x(t - \tau(x(t)))\,. \tag{3.6}$$

For two functions x and y,

$$\begin{aligned}|(\mathcal{D}x)(t) - (\mathcal{D}y)(t)| \le\ & |x(t-\tau(x(t))) - x(t-\tau(y(t)))| \\ & + |x(t-\tau(y(t))) - y(t-\tau(y(t)))|.\end{aligned}$$

The second term is bounded by $\|x-y\|_\infty$, whereas the first depends on the time regularity of x. If x is only Hölder continuous with exponent $\beta \in (0,1)$ and τ is Lipschitz with constant L_τ, then

$$|x(t-\tau(x(t))) - x(t-\tau(y(t)))| \le [x]_\beta L_\tau^\beta |x(t)-y(t)|^\beta.$$

Thus the state-dependent evaluation is not automatically Lipschitz in the supremum norm. We therefore isolate the additional property needed for the standard contraction argument.

(U) Local Lipschitz property of the state-dependent evaluation. On the phase-space set under consideration, there exists $L_{\mathcal{D}} > 0$ such that

$$\|\mathcal{D}x - \mathcal{D}y\|_{C([0,T])} \le L_{\mathcal{D}} \|x-y\|_{C([-\tau_M,T])}. \tag{3.7}$$

Theorem 3.4. *Assume* (A1)–(A4) *and* (U). *If*

$$\frac{\big(a + bL_f L_{\mathcal{D}} + L_g M_K(T)\big)T^\alpha}{\Gamma(1+\alpha)} < 1, \tag{3.8}$$

then problem (2.1) *has at most one mild solution on* $[-\tau_M, T]$ *in the phase-space set on which* (U) *holds.*

Proof. Let x and y be two mild solutions with the same history. Subtracting their integral equations and using (A1) and (U) gives

$$|x(t)-y(t)| \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(a + bL_f L_{\mathcal{D}} + L_g M_K(T)\Big) \|x-y\|_\infty \, ds.$$

Taking the supremum over $t \in [0,T]$ yields

$$\|x-y\|_\infty \le \frac{\big(a + bL_f L_{\mathcal{D}} + L_g M_K(T)\big)T^\alpha}{\Gamma(1+\alpha)} \|x-y\|_\infty.$$

Condition (3.8) forces $\|x-y\|_\infty = 0$. □

Remark 3.5. A concrete phase space on which *(U)* holds is

$$\mathcal{X}_{R,L_*} := \big\{ x \in C([-\tau_M,T]) : \ \|x\|_\infty \le R, \ |x(t)-x(s)| \le L_*|t-s| \text{ for all } s,t \in [-\tau_M,T] \big\}.$$

Assume, in addition, that τ is Lipschitz on $[-R,R]$ with constant L_τ. For $x, y \in \mathcal{X}_{R,L_*}$ and $t \in [0,T]$, the boundedness $0 \le \tau \le \tau_M$ ensures that both delayed arguments belong to $[-\tau_M, T]$, and hence

$$\begin{aligned} |(\mathcal{D}x)(t) - (\mathcal{D}y)(t)| &\le |x(t-\tau(x(t))) - x(t-\tau(y(t)))| \\ &\quad + |x(t-\tau(y(t))) - y(t-\tau(y(t)))| \\ &\le L_* L_\tau |x(t)-y(t)| + \|x-y\|_\infty \\ &\le (1 + L_* L_\tau) \|x-y\|_\infty. \end{aligned}$$

Therefore *(U)* is satisfied on $\mathcal{X}_{R,L_*}$ with the explicit constant $L_{\mathcal{D}} = 1 + L_* L_\tau$. This example also clarifies why *(U)* is imposed separately: uniform time-Lipschitz regularity is an additional phase-space property and is not an automatic consequence of a Caputo equation with $0 < \alpha < 1$.

Discussion. The results above separate two questions that are often merged in short fixed-point arguments. Compactness of the fractional integral is enough to obtain existence under natural continuity assumptions. Uniqueness is more sensitive because a change in the state changes both the value being sampled and the point at which it is sampled. Assumption (U) makes this extra requirement explicit instead of hiding it inside an unsupported regularity claim.

4. Stability Analysis

We now turn to the stability of the zero equilibrium. The argument is kept separate from the uniqueness discussion in Section 3: the estimate below is an estimate along a solution, and does not require the state-dependent evaluation operator to be a contraction.

For the global result we strengthen the kernel assumption to

$$K(t) \geq 0, \qquad K \in L^1(0,\infty), \qquad M_K := \int_0^\infty K(s)\,ds < \infty. \tag{4.1}$$

The normalization $f(0) = g(0) = 0$ shows that $x^* = 0$ is an equilibrium.

Regularity used in the stability argument. The existence theory above is formulated for mild solutions. The stability argument uses pointwise Caputo inequalities and therefore requires a little more regularity. We shall call a global mild solution x *Caputo-regular* if, for every $T > 0$,

$$x \in C([-\tau_M, T]) \cap C^1((0,T]), \qquad x' \in L^1(0,T),$$

and (2.1) holds for every $t \in (0,T]$. For such a solution, x^2 has the same admissible regularity, the quadratic Caputo inequality of Aguila-Camacho, Duarte-Mermoud and Gallegos [3] applies, and the Caputo extremum principle may be used at a strict first maximum [19].

A quadratic fractional estimate. We use the quadratic function

$$V(x) = \frac{1}{2}x^2. \tag{4.2}$$

For a solution for which the Caputo derivative is defined, the standard quadratic inequality [3] gives

$${}^C D_t^\alpha V(x(t)) \leq x(t)\,{}^C D_t^\alpha x(t). \tag{4.3}$$

Substitution of (2.1), together with (A1), yields

$$\begin{aligned} {}^C D_t^\alpha V(x(t)) \leq\ & -a|x(t)|^2 + bL_f|x(t)|\,|x(t-\tau(x(t)))| \\ & + L_g|x(t)| \int_0^t K(t-s)|x(s)|\,ds. \end{aligned} \tag{4.4}$$

This estimate avoids differentiating a moving delay integral and, in particular, does not require an ordinary derivative $\dot{x}(t)$.

For $t \geq 0$, set

$$R(t) := \max_{-\tau_M \leq s \leq t} |x(s)|. \tag{4.5}$$

At any time t for which $|x(t)| = R(t)$, boundedness of the delay gives

$$|x(t - \tau(x(t)))| \le |x(t)|,$$

and the positivity and global integrability of K give

$$\int_0^t K(t-s)|x(s)|\,ds \le M_K|x(t)|.$$

Consequently,

$$^C D_t^\alpha V(x(t)) \le -\eta|x(t)|^2, \qquad \eta := a - bL_f - L_g M_K, \tag{4.6}$$

whenever $|x(t)| = R(t)$.

The estimate suggests the following transparent dissipativity condition:

$$\boxed{a > bL_f + L_g M_K.} \tag{4.7}$$

Notice that the bound does not involve the Lipschitz constant of the delay. This is not because the delay is dynamically irrelevant. Rather, at a running maximum the delayed state is bounded by the current amplitude regardless of the precise location selected by the bounded delay. The delay can therefore change the transient while disappearing from this particular sufficient stability condition.

Uniform stability. We use the Caputo extremum principle [19]: for a scalar function in the regularity class specified above, a strict new maximum at $t_* > 0$ satisfies $^C D_t^\alpha w(t_*) > 0$ (and a non-strict maximum gives the corresponding non-negative inequality). This is the fractional analogue of the first-maximum observation used for ordinary differential equations.

Theorem 4.1. *Assume (A1)–(A4) and (4.1). Let x be a Caputo-regular global solution of (2.1)–(1.2). If (4.7) holds, then*

$$|x(t)| \le \|\phi\|_{C([-\tau_M,0])}, \qquad t \ge 0. \tag{4.8}$$

In particular, the zero equilibrium is uniformly stable within the class of Caputo-regular solutions.

Proof. Let $r_0 := \|\phi\|_{C([-\tau_M,0])}$. Suppose, for the sake of contradiction, that $|x(t)| > r_0$ for some $t > 0$. Choose a first level $r > r_0$ reached by the solution and let $t_* > 0$ be the first time at which $|x(t_*)| = r$. Then

$$|x(s)| \le |x(t_*)| \qquad (-\tau_M \le s \le t_*),$$

so $V(x(t_*))$ is a strict new maximum of $V(x(\cdot))$ on $[0, t_*]$. The Caputo maximum principle therefore gives

$$^C D_t^\alpha V(x(t_*)) \ge 0.$$

On the other hand, (4.6) and (4.7) give

$$^C D_t^\alpha V(x(t_*)) \le -\eta|x(t_*)|^2 < 0,$$

a contradiction. Hence no level above r_0 can be reached, and (4.8) follows.

Given $\varepsilon > 0$, choosing $\delta = \varepsilon$ shows that $\|\phi\|_\infty < \delta$ implies $|x(t)| < \varepsilon$ for every $t \ge 0$. Thus the zero equilibrium is uniformly stable within the class of Caputo-regular solutions. □

Remark 4.2. Condition (4.7) is sufficient, not necessary. If it fails, the theorem becomes inconclusive; instability does not follow. Likewise, Theorem 4.1 proves uniform stability but does not, by itself, establish a Mittag–Leffler decay rate. Obtaining such a rate requires an additional fractional Razumikhin/Halanay comparison theorem with hypotheses matched to the present state-dependent delay and Volterra memory term. We do not impose that stronger conclusion here. This distinction is important because Razumikhin methods for Caputo delay equations require care; related state-dependent-delay and Volterra-memory results can be found in the fractional delay literature.

Remark 4.3. The role of global integrability in (4.1) is also precise. It provides the uniform estimate

$$\int_0^t K(t-s)|x(s)|\,ds \le M_K R(t)$$

with a constant independent of t. The assumption is therefore sufficient for the present global criterion. It should not be interpreted as a necessary condition for stability of every system with a nonintegrable kernel.

5. Numerical Illustrations

The numerical experiments are used here as a check on the analytical framework and as a way to separate several effects that are combined in the model. In every experiment covered by Theorem 4.1, the sufficient condition (4.7) is evaluated from the prescribed parameters before the computation. A case outside the theorem is included only for comparison and is identified explicitly as such.

Unless stated otherwise, we take

$$f(x) = \tanh x, \qquad g(x) = x, \qquad \tau(x) = \tau_M \frac{|x|}{1+|x|},$$

so that $L_f = L_g = 1$ and $0 \le \tau(x) \le \tau_M$. For the exponential kernel $K(t) = K_0 e^{-\mu t}$,

$$M_K = \int_0^\infty K(s)\,ds = \frac{K_0}{\mu}.$$

The history is constant, $\phi(t) = 0.5$, on the corresponding delay interval.

The numerical approximation is based on the fractional Adams–Bashforth–Moulton predictor–corrector method [8, 7]. The Volterra convolution is evaluated by the composite trapezoidal rule on the same time grid. If the delayed argument lies between two grid points, the delayed state is reconstructed by linear interpolation; if it is negative, the prescribed history is used. At the corrector stage, the state-dependent delay at the new time level is evaluated from the predictor. To make the implementation reproducible, let $t_n = nh$ and denote the complete right-hand side of (2.1) by $F(t,x)$. The predictor used here is

$$x_{n+1}^P = \phi(0) + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{n} b_{j,n+1} F_j, \qquad b_{j,n+1} = (n+1-j)^\alpha - (n-j)^\alpha,$$

and the corrector is

$$x_{n+1} = \phi(0) + \frac{h^\alpha}{\Gamma(\alpha+2)} \left(F^P_{n+1} + \sum_{j=0}^{n} a_{j,n+1} F_j \right),$$

where

$$a_{0,n+1} = n^{\alpha+1} - (n-\alpha)(n+1)^\alpha,$$

and, for $1 \le j \le n$,

$$a_{j,n+1} = (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}.$$

Here F^P_{n+1} is evaluated at the predictor value. The Volterra term is approximated by the composite trapezoidal rule on the same mesh, and off-grid delayed states are reconstructed linearly. Unless otherwise noted, the figures below were computed with step size $h = 0.01$.

Example 1: effect of the fractional order inside the stable regime. Consider

$${}^C D^\alpha_t x(t) = -2x(t) + 0.2\tanh(x(t - \tau(x(t)))) + \int_0^t 0.3e^{-(t-s)} x(s)\, ds,$$

with $\tau_M = 1$. Here

$$M_K = 0.3, \qquad \eta = a - bL_f - L_g M_K = 2 - 0.2 - 0.3 = 1.5 > 0.$$

Thus the sufficient stability condition is satisfied for every $\alpha \in (0,1)$. We compare $\alpha \in \{0.6, 0.8, 0.95\}$ on $0 \le t \le 15$. All three trajectories remain below the initial amplitude and decay throughout the computed interval. At $t = 15$ the numerical values are approximately

$$x(15) = 0.03116, \quad 0.009377, \quad 0.001488$$

for $\alpha = 0.6, 0.8, 0.95$, respectively. The ordering illustrates the strong influence of the fractional order on the transient and long-time tail, while the analytical stability margin is unchanged.

Example 2: reducing the dissipativity margin. We fix $a = 2$, $\alpha = 0.9$, and $\tau_M = 1$, and compare

$$(b, K(t)) = (0.2, 0.3e^{-t})$$

with

$$(b, K(t)) = (0.4, 0.5e^{-0.5t}).$$

For the baseline case, $M_K = 0.3$ and $\eta = 1.5$. For the second case, $M_K = 1$ and

$$\eta = 2 - 0.4 - 1 = 0.6 > 0.$$

Both parameter sets are covered by Theorem 4.1, but the second has a smaller sufficient dissipativity margin. At $t = 15$ we find

$$x(15) \approx 0.003486 \qquad \text{and} \qquad x(15) \approx 0.02218,$$

respectively. The second trajectory therefore decays more slowly over the computed horizon. We stress that η is a sufficient margin in the estimate, not an asserted exact decay rate.

Example 3: influence of the state-dependent delay. Fix

$$a = 2, \qquad b = 0.8, \qquad \alpha = 0.8, \qquad K(t) = 0.3e^{-t}.$$

Then $M_K = 0.3$ and

$$\eta = 2 - 0.8 - 0.3 = 0.9 > 0.$$

We vary

$$\tau_M \in \{0.1, 1, 4, 10\}.$$

The sufficient condition is unchanged because the proof of Theorem 4.1 uses only that the delayed state belongs to the preceding trajectory. The numerics nevertheless show a visible delay effect. At $t = 15$ the computed values are approximately

$$0.01710, \quad 0.01729, \quad 0.01794, \quad 0.01959,$$

for $\tau_M = 0.1, 1, 4, 10$, respectively. Larger delay bounds produce a slower transient in this example. Thus the absence of τ_M from the coarse sufficient condition should not be interpreted as absence of a dynamical delay effect.

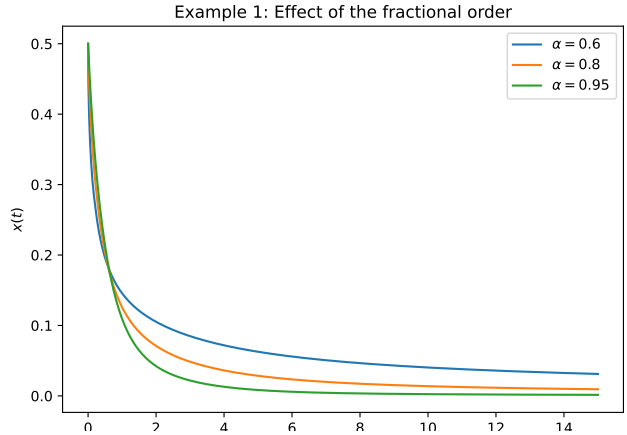
Example 4: comparison with a kernel outside the global stability hypothesis. We return to $a = 2$, $b = 0.2$, $\alpha = 0.8$, and $\tau_M = 1$. We compare

$$K_1(t) = 0.3e^{-t}, \qquad \int_0^\infty K_1(s)\, ds = 0.3,$$

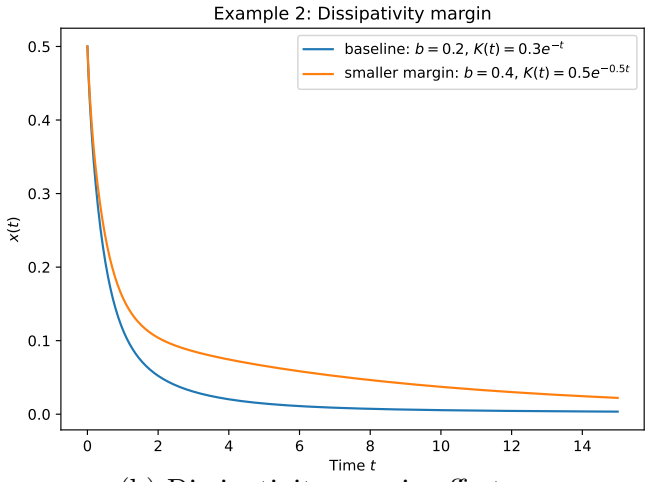
with

$$K_2(t) = \frac{0.6}{1+t}, \qquad \int_0^T K_2(s)\, ds = 0.6 \log(1+T) \longrightarrow \infty.$$

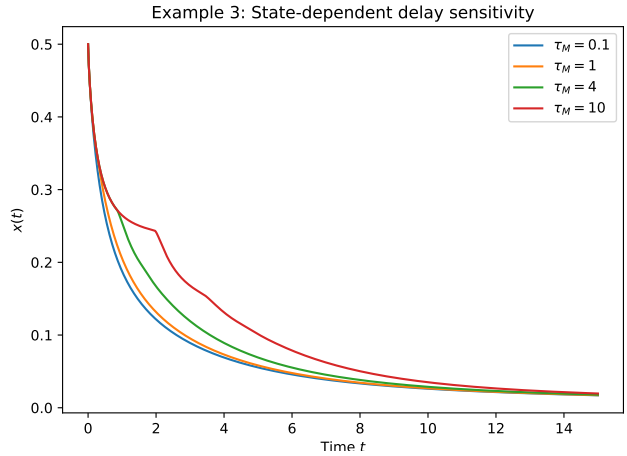
The first kernel satisfies the hypotheses of Theorem 4.1; the second does not. The exponential case continues to decay, with $x(40) \approx 0.003989$. For the non-integrable kernel, the solution decreases initially, reaches a numerical minimum $x(t) \approx 0.07712$ near $t \approx 7.87$, and subsequently rises to $x(40) \approx 0.1568$. This computation is deliberately outside the theorem. It illustrates that a slowly decaying memory tail can alter the long-time behavior, but it is not a proof that global integrability is necessary for stability.



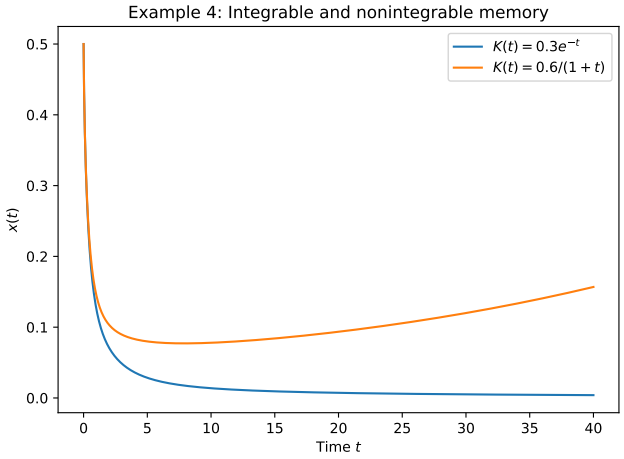
(a) Fractional-order effect.



(b) Dissipativity-margin effect.



(c) State-dependent delay sensitivity.



(d) Integrable versus nonintegrable memory.

FIGURE 1. Combined numerical illustrations. (a) Effect of the fractional order for a parameter set with $\eta = 1.5$. (b) Comparison of two parameter sets inside the sufficient stability region, with margins $\eta = 1.5$ and $\eta = 0.6$. (c) Effect of the state-dependent delay bound for parameter sets with the same margin $\eta = 0.9$. (d) Comparison of an integrable exponential kernel with the nonintegrable kernel $0.6/(1+t)$; the latter lies outside the hypotheses of Theorem 4.1 and is shown only as an exploratory case.

Step-size consistency. Because both the Caputo term and the distributed memory are nonlocal, and because the state-dependent delayed argument generally does not fall on the mesh, a step-size check is useful before interpreting the trajectories. For nested grids we define

$$E_h = \max_{t_j \le T} |x_h(t_j) - x_{h/2}(t_j)|.$$

Table 1 reports this quantity for one stable trajectory, the largest delay used in Example 3, and the exploratory nonintegrable-kernel case. In all three cases E_h decreases under refinement. The delay-sensitive case is the least forgiving, as expected, but still shows a clear reduction of the grid-to-grid difference.

TABLE 1. Nested-grid consistency check.

Case	h	E_h	$\log_2(E_h/E_{h/2})$
Example 1, $\alpha = 0.8$	0.04	2.735×10^{-4}	2.000
	0.02	6.839×10^{-5}	1.067
	0.01	3.264×10^{-5}	–
Example 3, $\tau_M = 10$	0.04	1.120×10^{-3}	1.489
	0.02	3.991×10^{-4}	1.546
	0.01	1.367×10^{-4}	–
Example 4, $K(t) = 0.6/(1+t)$	0.04	3.031×10^{-4}	2.000
	0.02	7.575×10^{-5}	1.226
	0.01	3.239×10^{-5}	–

The empirical values in the last column are included only as a grid-refinement diagnostic. Because the implementation couples a fractional predictor–corrector formula, trapezoidal convolution, and interpolation of a moving delayed argument, we do not use these three-grid values to claim a formal convergence order for the complete scheme.

6. Conclusion and Perspectives

We studied a scalar Caputo–Volterra equation in which distributed memory is coupled to a delay that depends on the current state. The analysis shows that existence, uniqueness, and long-time stability should not be bundled into a single contraction argument: the state-dependent evaluation affects these questions in different ways.

The scalar framework also admits natural phenomenological interpretations: the Volterra term may represent relaxation or distributed synaptic memory, while the bounded state-dependent delay may model an adaptive response latency. These interpretations are illustrative rather than calibrated applications; parameter identification and application-specific validation lie beyond the scope of the present work.

In summary, the analysis separates existence from uniqueness: under the basic continuity and Lipschitz assumptions, a bounded continuous state-dependent delay, and a locally integrable memory kernel, mild solutions follow from Schaefer's fixed-point theorem and compactness of the fractional Volterra operator, whereas uniqueness is imposed only under the additional condition (U). For a globally integrable nonnegative kernel, the quadratic function $V(x) = x^2/2$ and the Caputo maximum principle yield the sufficient condition (4.7) for uniform stability without a time-Lipschitz bound or differentiation of delay-dependent integral terms. The model therefore clarifies how state-dependent delay and Volterra memory

interact and which assumptions belong to finite-time existence versus global stability. The numerical experiments complement the analysis by distinguishing parameter regimes covered by the sufficient criterion from exploratory cases outside its hypotheses.

Future Research Directions. Natural extensions include deriving sharper structural conditions that imply (U), treating vector-valued, neutral, or abstract fractional systems, developing networked and control formulations with state-dependent delays, and identifying the memory kernel and delay law from observed trajectories. These directions preserve the central distinction among compactness-based existence, phase-space regularity, and dissipative stability while extending the present scalar framework.

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