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NONLOCAL MIXED-TYPE PROBLEMS WITH BESSEL OPERATORS

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Abstract. We investigate three two-point nonlocal boundary value problems for a three-dimensional mixed elliptic-hyperbolic equation with Bessel operators in a parallelepiped. The coefficients are singular on two planes, including the type-change surface. We construct eigenfunctions, associated functions, and biorthogonal systems for the non-self-adjoint spectral problems. A common transmission operator yields double-series representations in terms of Bessel and modified Bessel functions. We establish uniqueness, sufficient existence conditions, and compatibility conditions at resonance. Uniform and polynomial Diophantine separation conditions control small denominators and ensure convergence. The third problem is a reflected counterpart of an earlier problem.

Keywords. Mixed-type equation; Bessel operator; nonlocal boundary condition; root functions; small denominators
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1. Introduction

Boundary value problems for mixed-type partial differential equations occupy a distinct position between elliptic and hyperbolic theories. In a single domain, they combine spatial smoothing with wave-like propagation, while the type-change surface couples two qualitatively different mechanisms. Classical investigations of mixed-type and degenerate equations by Tricomi, Frankl, Lavrent'ev, and Bitsadze showed that this coupling affects not only the admissible boundary data but also the functional setting in which existence and uniqueness can be established; see, e.g., the treatments and surveys in [3, 4, 25]. These issues become substantially more delicate when the differential expression contains singular or degenerate coefficients.

An important singular expression is the Bessel operator

$$B_q^t = \frac{\partial^2}{\partial t^2} + \frac{2q+1}{t}\frac{\partial}{\partial t}, \qquad q > -1.$$

It appears naturally after radial or cylindrical reductions of multidimensional equations. Its presence requires weighted spaces and a careful treatment of the solution and weighted flux at the singular hyperplane. The underlying special-function and operator-theoretic tools are developed in [17, 18, 24, 27].

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Nonlocal boundary conditions provide a second source of difficulty. Such conditions relate traces at distinct boundary points, opposite faces, or integral characteristics of the unknown function. They arise when pointwise observations are inaccessible but total mass, energy, or flux is measurable. Classical examples and later generalizations can be found in [2, 7, 8, 20, 21, 22, 23]. In many two-point problems, the spatial operator is non-self-adjoint. Its eigenfunctions alone may then be incomplete, and associated functions belonging to Jordan chains must be included. The basis theory needed for this construction goes back to Il'in and Naimark [5, 6, 19]. A related study in this journal treats nonlocal elliptic problems with transmission conditions and establishes Fredholmness [10].

Three-dimensional nonlocal problems with singular coefficients have been considered in rectangular and semi-infinite domains in [12, 13, 14, 26]. A mixed-type analogue in a semi-infinite parallelepiped was studied in [15]. Another nonlocal problem for a loaded mixed-type equation was studied in [9]. In this journal, Karimov and Toshpulatov [11] use separation of variables to study a wave-diffusion-wave problem and prove uniform convergence of the resulting series. Their fractional model differs from the singular Bessel equation considered here.

The purpose of this paper is to provide a unified treatment of three lateral conditions for the same three-dimensional mixed-type equation. The first two conditions have the same spectrum but different Jordan chains and different adjoint systems. The third condition is the spatial reflection of the differential boundary problem associated with the integral condition studied in [16]; it is included to make the symmetry between the three biorthogonal constructions transparent and is not presented as an independent new result. The genuinely new contributions are the simultaneous construction for the first two problems, a common transmission solution operator, a unified resonance criterion, and a quantitative small-denominator analysis applicable to all three systems.

More precisely, the main results are as follows: (i) normalized eigenfunctions and associated functions of the non-self-adjoint operators and their adjoints are constructed; (ii) the three-dimensional problems are reduced to a family of one-dimensional Bessel transmission problems; (iii) explicit double-series representations are derived; (iv) uniqueness and sufficient conditions for existence are proved; and (v) uniform and polynomial Diophantine separation conditions are formulated for the characteristic determinant. The last point separates the analytic part of the proof from the arithmetic properties of the domain and operator parameters.

2. Equation and formulation of the problems

Let

$$\Omega = \{(x, y, z) : 0 < x < a,\ -b_0 < y < b,\ 0 < z < c\}, \qquad a, b_0, b, c > 0.$$

Set $\Omega_+ = \Omega \cap \{y > 0\}$ and $\Omega_- = \Omega \cap \{y < 0\}$. We consider

$$\left(B^x_{-1/2} + \operatorname{sgn} y\, B^y_{\beta-1/2} + B^z_{\gamma-1/2}\right) u(x, y, z) = 0, \qquad (x, y, z) \in \Omega_+ \cup \Omega_-, \quad (2.1)$$

where

$$0 < \beta < \frac{1}{2}, \qquad 0 < \gamma < \frac{1}{2}.$$

In expanded form, equation (2.1) is

$$u_{xx} + \operatorname{sgn} y \left(u_{yy} + \frac{2\beta}{y} u_y \right) + u_{zz} + \frac{2\gamma}{z} u_z = 0.$$

It is elliptic in Ω_+ and hyperbolic in Ω_-. The planes $y = 0$ and $z = 0$ are singular, and $y = 0$ is additionally the type-change surface.

Definition 2.1. A function u is called a regular solution if it satisfies

$$u \in C(\overline{\Omega}) \cap C^{2,2,2}(\Omega_+ \cup \Omega_-), \qquad u_x,\ z^{2\gamma} u_z \in C(\overline{\Omega}).$$

The one-sided limits of $|y|^{2\beta} u_y$ as $y \to \pm 0$ must also exist uniformly with respect to (x, z).

On the horizontal faces we impose

$$\begin{aligned} u(x, -b_0, z) = f_1(x, z), \qquad & u(x, b, z) = f_2(x, z), \\ u(x, y, 0) = 0, \qquad & u(x, y, c) = 0. \end{aligned} \tag{2.2}$$

Across $y = 0$ we require

$$\begin{aligned} u(x, -0, z) &= u(x, +0, z), \\ \lim_{y \to -0} (-y)^{2\beta} u_y(x, y, z) &= \lim_{y \to +0} y^{2\beta} u_y(x, y, z). \end{aligned} \tag{2.3}$$

Here f_1 and f_2 are prescribed on $\overline{\Pi} = [0, a] \times [0, c]$.

Problem 2.2 (B_1)**.** Find a regular solution of (2.1), (2.2)–(2.3) satisfying

$$u(0, y, z) = u(a, y, z), \qquad u_x(0, y, z) = 0. \tag{2.4}$$

Problem 2.3 (B_2)**.** Find a regular solution of (2.1), (2.2)–(2.3) satisfying

$$u(0, y, z) = u(a, y, z), \qquad u_x(a, y, z) = 0. \tag{2.5}$$

Problem 2.4 (B_3)**.** Find a regular solution of (2.1), (2.2)–(2.3) satisfying

$$u(a, y, z) = 0, \qquad u_x(0, y, z) = u_x(a, y, z). \tag{2.6}$$

All lateral equalities are understood for $-b_0 \le y \le b$ and $0 \le z \le c$. Compatibility at the edges requires $f_j(x, 0) = f_j(x, c) = 0$ and the corresponding x-conditions inherited from (2.4), (2.5), or (2.6).

Remark 2.5. Under the reflection $\widehat{u}(x, y, z) = u(a - x, y, z)$, condition (2.6) becomes $\widehat{u}(0, y, z) = 0$ and $\widehat{u}_x(0, y, z) = \widehat{u}_x(a, y, z)$. This is the differential form associated with the integral condition studied in [16]. Accordingly, the role of Problem B_3 here is comparative: its reflected root system completes the unified spectral picture, whereas the new nonlocal configurations are Problems B_1 and B_2.

3. Non-self-adjoint spectral problems in the x-variable

Separation of variables leads to

$$X''(x)+\mu X(x)=0, \qquad 0<x<a.$$

The corresponding boundary conditions are

$$\begin{aligned} B_1^x:&\quad X(0)=X(a), \quad X'(0)=0,\\ B_2^x:&\quad X(0)=X(a), \quad X'(a)=0,\\ B_3^x:&\quad X(a)=0, \quad X'(0)=X'(a). \end{aligned} \tag{3.1}$$

Put

$$\kappa_n=\frac{2\pi n}{a}, \qquad \mu_n=\kappa_n^2, \qquad n=0,1,2,\dots.$$

For $n\geq 1$, the eigenvalue μ_n has algebraic multiplicity two and geometric multiplicity one.

For B_1^x we take

$$X_0^{(1)}(x)=1, \qquad X_n^{(1)}(x)=\cos\kappa_n x, \qquad \widetilde{X}_n^{(1)}(x)=x\sin\kappa_n x. \tag{3.2}$$

Then

$$\left(\frac{d^2}{dx^2}+\mu_n\right)\widetilde{X}_n^{(1)}=2\kappa_n X_n^{(1)}.$$

The adjoint conditions are $T(a)=0$, $T'(0)=T'(a)$, and a normalized dual chain is

$$T_0^{(1)}=\frac{2(a-x)}{a^2}, \qquad T_n^{(1)}=\frac{4(a-x)}{a^2}\cos\kappa_n x, \qquad \widetilde{T}_n^{(1)}=\frac{4}{a^2}\sin\kappa_n x.$$

For B_2^x we choose

$$X_0^{(2)}(x)=1, \qquad X_n^{(2)}(x)=\cos\kappa_n x, \qquad \widetilde{X}_n^{(2)}(x)=(a-x)\sin\kappa_n x, \tag{3.3}$$

so that

$$\left(\frac{d^2}{dx^2}+\mu_n\right)\widetilde{X}_n^{(2)}=-2\kappa_n X_n^{(2)}.$$

The adjoint conditions are $T(0)=0$, $T'(0)=T'(a)$, and

$$T_0^{(2)}=\frac{2x}{a^2}, \qquad T_n^{(2)}=\frac{4x}{a^2}\cos\kappa_n x, \qquad \widetilde{T}_n^{(2)}=\frac{4}{a^2}\sin\kappa_n x.$$

For the reflected condition B_3^x in (3.1),

$$X_0^{(3)}(x)=a-x, \qquad X_n^{(3)}(x)=\sin\kappa_n x, \qquad \widetilde{X}_n^{(3)}(x)=(a-x)\cos\kappa_n x. \tag{3.4}$$

Indeed, these functions satisfy $X(a)=0$ and $X'(0)=X'(a)$, and

$$\left(\frac{d^2}{dx^2}+\mu_n\right)\widetilde{X}_n^{(3)}=2\kappa_n X_n^{(3)}.$$

The adjoint boundary conditions, obtained from Green's formula, are $T(0)=T(a)$ and $T'(0)=0$. The corresponding normalized system is

$$T_0^{(3)}=\frac{2}{a^2}, \qquad T_n^{(3)}=\frac{4x}{a^2}\sin\kappa_n x, \qquad \widetilde{T}_n^{(3)}=\frac{4}{a^2}\cos\kappa_n x. \tag{3.5}$$

Define

$$p_n^{(1)} = p_n^{(3)} = 2\kappa_n, \qquad p_n^{(2)} = -2\kappa_n. \tag{3.6}$$

Thus, for $r \in \{1,2,3\}$,

$$\left(\frac{d^2}{dx^2} + \mu_n\right) \widetilde{X}_n^{(r)} = p_n^{(r)} X_n^{(r)}. \tag{3.7}$$

Lemma 3.1. *For each $r \in \{1,2,3\}$, the systems*

$$\mathcal{X}_r = \{X_0^{(r)}, X_n^{(r)}, \widetilde{X}_n^{(r)} : n \geq 1\}, \qquad \mathcal{T}_r = \{T_0^{(r)}, T_n^{(r)}, \widetilde{T}_n^{(r)} : n \geq 1\}$$

are biorthonormal in $L_2(0,a)$. In particular,

$$\int_0^a X_0^{(r)} T_0^{(r)}\, dx = 1.$$

$$\int_0^a X_n^{(r)} T_l^{(r)}\, dx = \delta_{nl}, \qquad \int_0^a \widetilde{X}_n^{(r)} \widetilde{T}_l^{(r)}\, dx = \delta_{nl}.$$

All cross-pairings vanish.

Proof. The identities follow by direct integration and trigonometric orthogonality. The normalizations use

$$\int_0^a x \sin^2 \kappa_n x\, dx = \int_0^a (a-x) \sin^2 \kappa_n x\, dx = \frac{a^2}{4},$$

and the analogous identities with $\cos^2 \kappa_n x$. Completeness and the basis property with parentheses follow from the general theory of regular but not strongly regular second-order two-point problems [5, 6, 19]. □

4. Weighted spectral problem in the z-variable

The separated problem in z is the self-adjoint weighted problem

$$B^z_{\gamma-1/2} Z(z) + \eta Z(z) = 0, \qquad Z(0) = Z(c) = 0.$$

Let $\nu_\gamma = 1/2 - \gamma$, and denote by σ_m the mth positive zero of J_{ν_γ}. Then

$$\eta_m = \left(\frac{\sigma_m}{c}\right)^2, \qquad Z_m(z) = z^{1/2-\gamma} J_{1/2-\gamma}\left(\frac{\sigma_m z}{c}\right), \qquad m \geq 1. \tag{4.1}$$

The system $\{Z_m\}$ is orthogonal and complete in $L_2((0,c); z^{2\gamma} dz)$. Moreover,

$$\|Z_m\|^2_{2,\gamma} = \int_0^c z^{2\gamma} Z_m^2(z)\, dz = \frac{c^2}{2} J^2_{3/2-\gamma}(\sigma_m).$$

$$d_m = \|Z_m\|^{-2}_{2,\gamma} = \frac{2}{c^2 J^2_{3/2-\gamma}(\sigma_m)}.$$

We shall use the combined eigenvalues

$$\lambda_{nm} = \mu_n + \eta_m = \left(\frac{2\pi n}{a}\right)^2 + \left(\frac{\sigma_m}{c}\right)^2, \qquad n \geq 0,\ m \geq 1.$$

5. The transmission problem in the y-variable

Let $\lambda > 0$, $\rho = \sqrt{\lambda}$, and $\nu = 1/2 - \beta \in (0, 1/2)$. Introduce

$$\mathcal{L}_\lambda q = \operatorname{sgn} y \left(q'' + \frac{2\beta}{y} q' \right) - \lambda q.$$

We first solve

$$\mathcal{L}_\lambda q = h(y), \qquad y \in (-b_0, 0) \cup (0, b), \tag{5.1}$$

subject to

$$\begin{aligned} q(-b_0) &= r_-, \qquad q(b) = r_+, \\ q(-0) &= q(+0), \qquad \lim_{y \to -0} (-y)^{2\beta} q'(y) = \lim_{y \to +0} y^{2\beta} q'(y). \end{aligned} \tag{5.2}$$

5.1. **A matched fundamental system.** Define

$$\overline{Y}_\nu(t) = \frac{\pi}{2 \sin \pi\nu} \left[J_\nu(t) + J_{-\nu}(t) \right] = \frac{\pi}{2 \cos \beta\pi} \left[J_\nu(t) + J_{-\nu}(t) \right]. \tag{5.3}$$

The functions

$$F_\lambda(y) = \begin{cases} y^\nu I_\nu(\rho y), & 0 \le y \le b, \\ -(-y)^\nu J_\nu(\rho(-y)), & -b_0 \le y \le 0, \end{cases} \tag{5.4}$$

$$G_\lambda(y) = \begin{cases} y^\nu K_\nu(\rho y), & 0 \le y \le b, \\ (-y)^\nu \overline{Y}_\nu(\rho(-y)), & -b_0 \le y \le 0, \end{cases}$$

Lemma 5.1. *The functions F_λ and G_λ satisfy $\mathcal{L}_\lambda q = 0$ on both subintervals and both transmission conditions in (5.2). They are linearly independent.*

Proof. The differential equations follow from the Bessel and modified Bessel equations. To pass through $y = 0$, use the small-argument expansions of $J_{\pm\nu}$, I_ν, and K_ν. The signs in (5.4) make the one-sided limits of $|y|^{2\beta} F_\lambda'(y)$ equal. Formula (5.3) is chosen so that the constant terms and weighted derivative terms of G_λ also agree. The weighted Wronskian is nonzero; hence the two matched solutions are linearly independent. □

Consequently,

$$q(y) = A F_\lambda(y) + B G_\lambda(y) \tag{5.5}$$

is the general matched homogeneous solution. Set

$$\begin{aligned} \Delta_\lambda(s,t) &= I_\nu(\rho t) \overline{Y}_\nu(\rho s) + K_\nu(\rho t) J_\nu(\rho s), \\ \Delta(\lambda) &= \Delta_\lambda(b_0, b), \\ A_\lambda(y,b) &= I_\nu(\rho b) K_\nu(\rho y) - K_\nu(\rho b) I_\nu(\rho y), \\ D_\lambda(b_0, s) &= J_\nu(\rho b_0) \overline{Y}_\nu(\rho s) - \overline{Y}_\nu(\rho b_0) J_\nu(\rho s). \end{aligned} \tag{5.6}$$

If $\Delta(\lambda) \neq 0$, define

$$\mathcal{H}_\lambda[r_-, r_+](y) = r_- H_\lambda^-(y) + r_+ H_\lambda^+(y), \tag{5.7}$$

where

$$H_\lambda^-(y) = \begin{cases} \left(\dfrac{y}{b_0}\right)^\nu \dfrac{A_\lambda(y,b)}{\Delta(\lambda)}, & 0 \le y \le b, \\ \left(\dfrac{-y}{b_0}\right)^\nu \dfrac{\Delta_\lambda(-y,b)}{\Delta(\lambda)}, & -b_0 \le y \le 0, \end{cases} \tag{5.8}$$

$$H_\lambda^+(y) = \begin{cases} \left(\dfrac{y}{b}\right)^\nu \dfrac{\Delta_\lambda(b_0,y)}{\Delta(\lambda)}, & 0 \le y \le b, \\ \left(\dfrac{-y}{b}\right)^\nu \dfrac{D_\lambda(b_0,-y)}{\Delta(\lambda)}, & -b_0 \le y \le 0. \end{cases} \tag{5.9}$$

Direct substitution gives

$$H_\lambda^-(-b_0) = H_\lambda^+(b) = 1, \qquad H_\lambda^-(b) = H_\lambda^+(-b_0) = 0.$$

5.2. **The inhomogeneous solution operator.** Introduce the state vector

$$V(y) = \begin{pmatrix} q(y) \\ p(y) \end{pmatrix}, \qquad p(y) = |y|^{2\beta} q'(y).$$

Then (5.1) is equivalent to

$$V'(y) = \mathbb{A}_\lambda(y) V(y) + b_h(y).$$

Here

$$\mathbb{A}_\lambda(y) = \begin{pmatrix} 0 & |y|^{-2\beta} \\ \operatorname{sgn} y\, \lambda |y|^{2\beta} & 0 \end{pmatrix}, \qquad b_h(y) = \begin{pmatrix} 0 \\ \operatorname{sgn} y\, |y|^{2\beta} h(y) \end{pmatrix}.$$

Both components of V are continuous through $y = 0$. A fundamental matrix for the homogeneous system is

$$\Phi_\lambda(y) = \begin{pmatrix} F_\lambda(y) & G_\lambda(y) \\ |y|^{2\beta} F_\lambda'(y) & |y|^{2\beta} G_\lambda'(y) \end{pmatrix}.$$

Let $e_1 = (1,0)$ and

$$M_\lambda = \begin{pmatrix} e_1 \Phi_\lambda(-b_0) \\ e_1 \Phi_\lambda(b) \end{pmatrix}.$$

The determinant of M_λ equals $-b_0^\nu b^\nu \Delta(\lambda)$; hence it is invertible if $\Delta(\lambda) \neq 0$. Define $\mathcal{R}_\lambda h$ as the first component of

$$V_h(y) = \Phi_\lambda(y) \left[C_h + \int_{-b_0}^{y} \Phi_\lambda^{-1}(t) b_h(t)\, dt \right], \tag{5.10}$$

where

$$C_h = -M_\lambda^{-1} \begin{pmatrix} 0 \\ e_1 \Phi_\lambda(b) \displaystyle\int_{-b_0}^{b} \Phi_\lambda^{-1}(t) b_h(t)\, dt \end{pmatrix}. \tag{5.11}$$

These integrals are defined as the sums of improper integrals over $[-b_0, 0)$ and $(0, b]$. They converge because $0 < \beta < 1/2$. By construction,

$$\mathcal{L}_\lambda(\mathcal{R}_\lambda h) = h, \qquad (\mathcal{R}_\lambda h)(-b_0) = (\mathcal{R}_\lambda h)(b) = 0.$$

Both matching conditions also hold. Therefore the unique nonresonant solution of (5.1)–(5.2) is

$$q(y) = \mathcal{H}_\lambda[r_-, r_+](y) + \mathcal{R}_\lambda h(y).$$

6. Spectral expansion and constructive solution

For $r \in \{1,2,3\}$, $j \in \{1,2\}$, $m \geq 1$, and $n \geq 1$, put

$$f_{j,0m}^{(r)} = d_m \int_0^c \int_0^a f_j(x,z) T_0^{(r)}(x) z^{2\gamma} Z_m(z)\, dx\, dz, \tag{6.1}$$

$$f_{j,nm}^{(r)} = d_m \int_0^c \int_0^a f_j(x,z) T_n^{(r)}(x) z^{2\gamma} Z_m(z)\, dx\, dz,$$

$$\widetilde{f}_{j,nm}^{(r)} = d_m \int_0^c \int_0^a f_j(x,z) \widetilde{T}_n^{(r)}(x) z^{2\gamma} Z_m(z)\, dx\, dz. \tag{6.2}$$

Under the compatibility and smoothness assumptions stated in Section 9,

$$f_j(x,z) = \sum_{m=1}^{\infty} f_{j,0m}^{(r)} X_0^{(r)}(x) Z_m(z) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left[f_{j,nm}^{(r)} X_n^{(r)}(x) + \widetilde{f}_{j,nm}^{(r)} \widetilde{X}_n^{(r)}(x) \right] Z_m(z).$$

The pairing inside the brackets follows the Jordan chain at the eigenvalue μ_n.

We seek the solution of Problem B_r in the form

$$u_r(x,y,z) = \sum_{m=1}^{\infty} q_{0m}^{(r)}(y) X_0^{(r)}(x) Z_m(z) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left[q_{nm}^{(r)}(y) X_n^{(r)}(x) + \widetilde{q}_{nm}^{(r)}(y) \widetilde{X}_n^{(r)}(x) \right] Z_m(z). \tag{6.3}$$

Using (3.7) and (4.1), substitution into (2.1) yields

$$\mathcal{L}_{\lambda_{0m}} q_{0m}^{(r)} = 0,$$

$$\mathcal{L}_{\lambda_{nm}} \widetilde{q}_{nm}^{(r)} = 0, \tag{6.4}$$

$$\mathcal{L}_{\lambda_{nm}} q_{nm}^{(r)} = -p_n^{(r)} \widetilde{q}_{nm}^{(r)}. \tag{6.5}$$

The endpoint data are the coefficients in (6.1)–(6.2). Write

$$\Delta_{nm} = \Delta(\lambda_{nm}).$$

If $\Delta_{nm} \neq 0$, then

$$q_{0m}^{(r)}(y) = \mathcal{H}_{\lambda_{0m}}[f_{1,0m}^{(r)}, f_{2,0m}^{(r)}](y), \tag{6.6}$$

$$\widetilde{q}_{nm}^{(r)}(y) = \mathcal{H}_{\lambda_{nm}}[\widetilde{f}_{1,nm}^{(r)}, \widetilde{f}_{2,nm}^{(r)}](y), \tag{6.7}$$

$$q_{nm}^{(r)}(y) = \mathcal{H}_{\lambda_{nm}}[f_{1,nm}^{(r)}, f_{2,nm}^{(r)}](y) - p_n^{(r)} \mathcal{R}_{\lambda_{nm}}[\widetilde{q}_{nm}^{(r)}](y). \tag{6.8}$$

Thus (6.3) and (6.6)–(6.8) give a common constructive representation for all three problems. Their differences are confined to the root systems (3.2), (3.3), (3.4), the dual coefficients, and the signs in (3.6).

7. Uniqueness

Theorem 7.1. *A regular solution of Problem B_r, $r \in \{1,2,3\}$, is unique if and only if*

$$\Delta_{nm} \neq 0 \qquad \text{for every } n \geq 0,\ m \geq 1. \tag{7.1}$$

Proof. Let $f_1 = f_2 = 0$. Project a regular solution onto the biorthogonal system in x and the weighted Bessel system in z. The coefficient of each associated function satisfies the homogeneous problem (6.4) with zero endpoint values. Under (7.1), formulas (5.7)–(5.9) imply $\widetilde{q}_{nm}^{(r)} \equiv 0$. Equation (6.5) then becomes homogeneous, and hence $q_{nm}^{(r)} \equiv 0$. The same argument gives $q_{0m}^{(r)} \equiv 0$. Completeness of $\mathcal{X}_r \otimes \{Z_m\}$ yields $u_r \equiv 0$.

Conversely, assume $\Delta_{n_0m_0} = 0$. The homogeneous transmission problem has a nonzero solution $Q_{n_0m_0} = AF_{\lambda_{n_0m_0}} + BG_{\lambda_{n_0m_0}}$ vanishing at $-b_0$ and b. Then

$$u(x,y,z) = X_{n_0}^{(r)}(x) Q_{n_0m_0}(y) Z_{m_0}(z).$$

This is a nontrivial solution of the corresponding homogeneous three-dimensional problem. Thus uniqueness fails. □

8. Characteristic determinant and small denominators

The determinant Δ_{nm} occurs in (5.8) and (5.9). Nonvanishing alone does not control the convergence of (6.3); a quantitative separation from zero is required. This is a small-denominator phenomenon in the sense of [1].

For $\rho \to \infty$, the standard asymptotic formulas give

$$\begin{aligned} I_\nu(\rho b) &= \frac{e^{\rho b}}{\sqrt{2\pi\rho b}} \left(1 + O(\rho^{-1})\right), \\ K_\nu(\rho b) &= \sqrt{\frac{\pi}{2\rho b}}\, e^{-\rho b} \left(1 + O(\rho^{-1})\right), \\ J_\nu(\rho b_0) &= \sqrt{\frac{2}{\pi\rho b_0}} \left[\cos\left(\rho b_0 - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(\rho^{-1})\right]. \end{aligned} \tag{8.1}$$

By (5.3),

$$\overline{Y}_\nu(\rho b_0) = \frac{\sqrt{\pi/2}}{\sin(\pi\nu/2)\sqrt{b_0}}\, \rho^{-1/2} \left[\sin\left(\rho b_0 + \frac{\pi}{4}\right) + O(\rho^{-1})\right], \tag{8.2}$$

where the displayed coefficient is nonzero because $0 < \nu < 1/2$. The term containing $K_\nu(\rho b)$ in (5.6) is exponentially smaller than the first term. Consequently,

$$\Delta(\lambda) = \frac{\lambda^{-1/2} e^{b\sqrt{\lambda}}}{2\sin(\pi\nu/2)\sqrt{bb_0}} \left[\sin\left(b_0\sqrt{\lambda} + \frac{\pi}{4}\right) + O(\lambda^{-1/2})\right]. \tag{8.3}$$

The factor $\lambda^{-1/2}$ is essential: it is the product of the two factors $\rho^{-1/2}$ in (8.1) and (8.2).

Lemma 8.1. *Assume that there exist $\delta > 0$ and $N \in \mathbb{N}$ such that*

$$\left|\sin\left(b_0\sqrt{\lambda_{nm}} + \frac{\pi}{4}\right)\right| \geq \delta, \qquad n + m \geq N. \tag{8.4}$$

Then, for some $C_0 > 0$,

$$|\Delta_{nm}| \geq C_0 \lambda_{nm}^{-1/2} e^{b\sqrt{\lambda_{nm}}}, \qquad n + m \geq N. \tag{8.5}$$

Proof. Insert $\lambda = \lambda_{nm}$ in (8.3). The remainder tends to zero uniformly as $n+m \to \infty$. Increasing N if necessary makes its absolute value smaller than $\delta/2$, which proves (8.5). □

Uniform separation is convenient but strong. The following version allows polynomial approach to resonance.

Lemma 8.2. *Suppose that for some $\delta > 0$, $0 \le \tau < 1/2$, and $N \in \mathbb{N}$,*

$$\left|\sin\left(b_0\sqrt{\lambda_{nm}} + \frac{\pi}{4}\right)\right| \ge \delta(1+\lambda_{nm})^{-\tau}, \qquad n+m \ge N. \tag{8.6}$$

Then

$$|\Delta_{nm}| \ge C_\tau \lambda_{nm}^{-1/2}(1+\lambda_{nm})^{-\tau} e^{b\sqrt{\lambda_{nm}}} \tag{8.7}$$

for all sufficiently large $n+m$.

Proof. After factoring out $\lambda_{nm}^{-1/2}e^{b\sqrt{\lambda_{nm}}}$ in (8.3), use (8.6). Since $\tau < 1/2$, the $O(\lambda_{nm}^{-1/2})$ remainder is of smaller order than $(1+\lambda_{nm})^{-\tau}$. This yields (8.7). □

Remark 8.3. Condition (8.6) is an arithmetic hypothesis on a, b_0, c, and γ. The analysis above does not assert that it holds for every parallelepiped. For fixed geometric parameters, it can be checked on a finite index set numerically; a rigorous tail estimate must use interval bounds or a separate number-theoretic argument. Stating this hypothesis explicitly avoids hiding the small-denominator issue inside an unspecified constant.

9. Existence and convergence

Let $A_r = -d^2/dx^2$ with the boundary conditions in (3.1), and let $C_\gamma = -B^z_{\gamma-1/2}$ with the Dirichlet conditions at $z = 0, c$. We first give convenient sufficient assumptions for the uniform-separation case. They are not intended to be minimal.

For $j = 1, 2$, suppose that:

(1) $f_j \in C^{4,6}(\overline{\Pi})$, and all expressions below involving $B^z_{\gamma-1/2}$ extend continuously to $z = 0$;
(2) $f_j(\cdot, z) \in D(A_r^2)$ for every $z \in [0, c]$;
(3) $C^s_\gamma f_j(x, 0) = C^s_\gamma f_j(x, c) = 0$ for $s = 0, 1, 2$;
(4) $z^{2\gamma}\partial_z(C^s_\gamma f_j)$ is continuous for $s = 0, 1, 2$.

The condition $f_j(\cdot, z) \in D(A_r^2)$ is equivalent to the following compatibility relations. For B_1,

$$\begin{aligned} f_j(0,z) &= f_j(a,z), \qquad & f_{jx}(0,z) &= 0, \\ f_{jxx}(0,z) &= f_{jxx}(a,z), \qquad & f_{jxxx}(0,z) &= 0; \end{aligned} \tag{9.1}$$

for B_2,

$$\begin{aligned} f_j(0,z) &= f_j(a,z), \qquad & f_{jx}(a,z) &= 0, \\ f_{jxx}(0,z) &= f_{jxx}(a,z), \qquad & f_{jxxx}(a,z) &= 0; \end{aligned} \tag{9.2}$$

and for the reflected Problem B_3,

$$\begin{aligned} f_j(a,z) &= 0, \qquad & f_{jx}(0,z) &= f_{jx}(a,z), \\ f_{jxx}(a,z) &= 0, \qquad & f_{jxxx}(0,z) &= f_{jxxx}(a,z). \end{aligned} \tag{9.3}$$

Repeated integration by parts, using the adjoint boundary conditions, gives

$$|f_{j,nm}^{(r)}| + |\widetilde{f}_{j,nm}^{(r)}| \le C(1+n)^{-4}(1+m)^{-5}, \tag{9.4}$$

$$|f_{j,0m}^{(r)}| \le C(1+m)^{-5}. \tag{9.5}$$

The exact exponents can be increased by strengthening the smoothness assumptions.

The small- and large-argument estimates for the Bessel functions, together with Lemma 8.1 and (5.8)–(5.9), yield

$$\|H_{\lambda_{nm}}^{-}\|_{C[-b_0,b]} + \|H_{\lambda_{nm}}^{+}\|_{C[-b_0,b]} \le C(1+\lambda_{nm})^{\beta/2}, \tag{9.6}$$

$$\|\mathcal{R}_{\lambda_{nm}} h\|_{C[-b_0,b]} \le C\lambda_{nm}^{-1/2}(1+\lambda_{nm})^{\beta/2}\|h\|_{C[-b_0,b]}. \tag{9.7}$$

Here the factor $(1+\lambda_{nm})^{\beta/2}$ cannot in general be omitted. Indeed, with $\rho = \sqrt{\lambda}$, formula (5.8) and the small-argument expansion of K_ν give

$$H_\lambda^{-}(0+) = \frac{2^{\nu-1}\Gamma(\nu) b_0^{-\nu}\rho^{-\nu} I_\nu(\rho b)}{\Delta(\lambda)}.$$

Together with (8.3), this quantity has order $\rho^{1/2-\nu} = \rho^\beta = \lambda^{\beta/2}$ along uniformly separated spectral subsequences. To prove (9.6)–(9.7), split each interval into $\rho|y| \le 1$ and $\rho|y| \ge 1$, use the small-argument estimates in the first part and the standard oscillatory or exponential estimates in the second part, and then apply (8.5) to the denominator. The Green kernel defined by (5.10)–(5.11) is estimated in the same way.

Since $|p_n^{(r)}| = 2\kappa_n \le 2\sqrt{\lambda_{nm}}$, formulas (6.7) and (6.8), together with (9.4), (9.6), and (9.7), imply

$$\|q_{nm}^{(r)}\|_C + \|\widetilde{q}_{nm}^{(r)}\|_C \le C(1+\lambda_{nm})^{\beta}(1+n)^{-4}(1+m)^{-5}. \tag{9.8}$$

Theorem 9.1. *Fix $r \in \{1,2,3\}$. Assume the four regularity conditions above and the corresponding compatibility conditions (9.1), (9.2), or (9.3). Suppose that $\Delta_{nm} \neq 0$ for $n+m < N$ and that (8.4) holds for $n+m \ge N$. Then the series (6.3), with coefficients (6.6)–(6.8), converges absolutely and uniformly on $\overline{\Omega}$. The series for u_x and $z^{2\gamma}u_z$ converge uniformly on $\overline{\Omega}$, while those for u_{xx}, $B^y_{\beta-1/2}u$, and $B^z_{\gamma-1/2}u$ converge uniformly on every compact subset of $\Omega_+ \cup \Omega_-$. Its sum is the unique regular solution of Problem B_r.*

Proof. Uniformly in $x \in [0,a]$,

$$|X_n^{(r)}| + |\widetilde{X}_n^{(r)}| \le C, \tag{9.9}$$

$$|(X_n^{(r)})'| + |(\widetilde{X}_n^{(r)})'| \le C(1+n).$$

$$|(X_n^{(r)})''| + |(\widetilde{X}_n^{(r)})''| \le C(1+n)^2. \tag{9.10}$$

The standard asymptotics of Bessel zeros give $\sigma_m = \pi(m-\gamma/2) + O(m^{-1})$, together with polynomial bounds for Z_m, $z^{2\gamma}Z_m'$, and $B^z_{\gamma-1/2}Z_m$. Moreover,

$$(1+\lambda_{nm})^\beta \le C(1+n)^{2\beta}(1+m)^{2\beta}.$$

Because $2\beta < 1$, the decay in (9.4)–(9.5) still absorbs the factor in (9.8), as well as the factors created by the derivatives in (9.9)–(9.10) and by the z-operator. In particular, after two differentiations the worst n-majorant has order $(1+n)^{-2+2\beta}$

and the worst m-majorant has order $(1+m)^{-3+2\beta}$; both are summable. The Weierstrass test therefore gives the stated uniform convergence and permits termwise differentiation on compact subsets away from the singular planes.

Every partial sum satisfies the equation, the appropriate lateral conditions, the Dirichlet conditions in z, and the transmission conditions in y. The endpoint properties of $\mathcal{H}_\lambda$ and $\mathcal{R}_\lambda$ recover (2.2). Passing uniformly to the limit proves that the sum belongs to the class of Definition 2.1 and solves Problem B_r. Uniqueness follows from Theorem 7.1. □

Proposition 9.2. *Assume (8.6). The conclusion of Theorem 9.1 remains valid if the data are smooth enough for their spectral coefficients to satisfy*

$$\sum_{m=1}^{\infty}(1+\lambda_{0m})^{\tau+2}|f_{j,0m}^{(r)}|+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(1+\lambda_{nm})^{\tau+2}\left(|f_{j,nm}^{(r)}|+|\widetilde{f}_{j,nm}^{(r)}|\right)<\infty, \quad j=1,2. \tag{9.11}$$

Proof. Estimate (8.7) replaces (9.6) and (9.7) by the same estimates multiplied by at most $(1+\lambda_{nm})^{\tau}$. Consequently, the associated coefficient can lose a factor of order $(1+\lambda_{nm})^{\beta/2+\tau}$, while the coupled coefficient in (6.8) can lose at most $(1+\lambda_{nm})^{\beta+2\tau}$. Since $\beta+\tau<1$, the weight $(1+\lambda_{nm})^{\tau+2}$ in (9.11) absorbs this loss and the factors created by two differentiations. The preceding proof then applies without change. □

10. The resonant case

Assume that $\Delta(\lambda)=0$. For $h=0$, the endpoint equations for the constants in (5.5) read

$$\begin{pmatrix} -J_\nu(\rho b_0) & \overline{Y}_\nu(\rho b_0) \\ I_\nu(\rho b) & K_\nu(\rho b)\end{pmatrix}\begin{pmatrix} A \\ B\end{pmatrix}=\begin{pmatrix} b_0^{-\nu}r_- \\ b^{-\nu}r_+\end{pmatrix}.$$

A necessary and sufficient compatibility condition is

$$b^\nu I_\nu(\rho b)r_- + b_0^\nu J_\nu(\rho b_0)r_+ = 0. \tag{10.1}$$

Equivalently, using the identities relating K_ν, $I_{\pm\nu}$, and $J_{\pm\nu}$,

$$b^\nu I_{-\nu}(\rho b)r_- - b_0^\nu J_{-\nu}(\rho b_0)r_+ = 0.$$

For the inhomogeneous problem, let $\ell_\lambda \neq 0$ be a left null vector of M_λ, so $\ell_\lambda^T M_\lambda = 0$. The Fredholm condition is

$$\ell_\lambda^T\left[\begin{pmatrix} r_- \\ r_+\end{pmatrix}-\begin{pmatrix} 0 \\ e_1\Phi_\lambda(b)\displaystyle\int_{-b_0}^{b}\Phi_\lambda^{-1}(t)b_h(t)\,dt\end{pmatrix}\right]=0. \tag{10.2}$$

Theorem 10.1. *Suppose that the set of resonant pairs (n,m) is finite. Problem B_r is solvable if and only if, for every resonant pair, condition (10.1) holds first for the endpoint data of $\widetilde{q}_{nm}^{(r)}$, and condition (10.2) then holds for $q_{nm}^{(r)}$ with $h=-p_n^{(r)}\widetilde{q}_{nm}^{(r)}$. If these conditions are satisfied, the solution set is a finite-dimensional affine space.*

Proof. The first condition is the solvability criterion for the homogeneous equation (6.4) with prescribed endpoint data. Once $\widetilde{q}_{nm}^{(r)}$ is fixed, equation (6.5) is inhomogeneous and its solvability is equivalent to (10.2). At every resonant pair, the coefficient functions are determined up to elements of the kernel. Hence one may add a finite linear combination of modes

$$X_n^{(r)}(x)Q_{nm}(y)Z_m(z),$$

where Q_{nm} is a nonzero homogeneous resonant solution. The number of independent kernel modes gives the dimension of the affine solution set. □

11. Conclusion

We have placed three two-point lateral problems for a three-dimensional mixed elliptic-hyperbolic equation with two singular coefficients in a common spectral framework. Although the three non-self-adjoint operators have the same eigenvalues, their eigenfunctions, associated functions, adjoint boundary conditions, and biorthogonal partners differ. The reflected condition $u(a, y, z) = 0$ in Problem B_3 leads specifically to the reflected chain (3.4) and the adjoint system (3.5).

The common transmission operator $\mathcal{H}_\lambda + \mathcal{R}_\lambda$ accounts simultaneously for the change of type and the singularity at $y = 0$. It yields explicit double-series representations and separates the spectral geometry in (x, z) from the one-dimensional transmission analysis in y. The characteristic determinant controls both uniqueness and resonance. Its asymptotic form exposes the small-denominator mechanism, while Lemmas 8.1 and 8.2 give two transparent ways of controlling it.

The paper also distinguishes the new content from the earlier integral-condition study: the third lateral condition is the reflection of the differential boundary problem associated there with the integral condition, whereas the first two problems, their paired Jordan chains, the unified solution operator, and the common resonant analysis constitute the principal contribution. Possible extensions include a loaded term, mixed boundary conditions in z, several type-change surfaces, and stable numerical truncations of the double spectral series.

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