Gulf Journal of Mathematics
Vol 24, Issue 2 (2026) 1-11

ISSN: 2309-4966
https://doi.org/10.56947/h5bwaw71

THE GEODETIC LOCAL RESOLVING NUMBER OF GRAPHS

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Abstract. For an ordered vertex set W of a connected graph G, the code of a vertex is its distance vector with respect to W. A set W is called a geodetic local resolving set if it distinguishes every pair of adjacent vertices and is geodetic. The minimum cardinality of such a set is called the geodetic local metric dimension, denoted by $glmd(G)$. In this paper, we establish sharp bounds for $glmd(G)$, characterize the graphs attaining these bounds, investigate its relationship with several classical graph invariants.

Keywords. local Basis, geodetic set, geodetic local dimension
2020 Mathematics Subject Classification. Primary 05C12

1. Introduction

Let $G = (V, E)$ be a finite, simple, connected graph of order n and size m. For $u, v \in V(G)$, the distance $d(u, v)$ is the length of a shortest u-v path, called a geodesic. For $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$, the code of $v \in V(G)$ with respect to W is defined by $code(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. The set W is a resolving set if distinct vertices have distinct codes, and the minimum cardinality of such a set is the metric dimension, denoted by $dim(G)$ [14, 9]. We refer to [7, 5] for standard graph-theoretic terminology. For $u, v \in V(G)$, let $I[u, v]$ denote the set of vertices lying on some shortest u-v path. For $S \subseteq V(G)$, the geodetic closure is $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set S is geodetic if $I[S] = V(G)$, and the minimum cardinality of a geodetic set is the geodetic number $g(G)$ [8]. The study of resolving and geodetic sets has been motivated, in particular, by problems in chemical graph theory and molecular identification [9, 6]. The concept of local metric dimension was introduced and studied in [12]. Several variants have been introduced by combining resolving sets with additional graph-theoretic properties [13, 10, 11]. In particular, Ananthakumar [1, 3, 2, 4] studied distance-based and geodetic resolving concepts. A resolving set W is called a geodetic resolving set if $I[W] = V(G)$. The minimum cardinality of such a set is the geodetic resolving number, denoted by $gr(G)$, and a minimum geodetic resolving set is called a geodetic basis.

Date: Received: Sep 4, 2026; Revised: Oct 4, 2026; Accepted: Oct 9, 2026.
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We discuss the motivation and potential applications of geodetic local resolving sets in technological and chemical science domains.

1.1. **Motivation and Application in Robot Navigation.** Robot navigation in a structured environment can naturally be modeled by a graph $G = (V, E)$, where the vertices represent possible robot locations and the edges represent feasible movements between locations. Let $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ denote a set of selected navigation landmarks. For each $v \in V(G)$, its distance representation with respect to W is defined by $r(v \mid W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$, where $d(u, v)$ denotes the shortest-path distance between u and v. The concept of a geodetic local resolving set provides a natural framework for landmark-based robot navigation. The local resolving property ensures that $r(u \mid W) \neq r(v \mid W)$ for every pair of adjacent vertices $u, v \in V(G)$, thereby enabling the robot to distinguish between neighboring positions. On the other hand, the geodetic property guarantees that every vertex $v \in V(G) \setminus W$ lies on a shortest path between some pair of landmarks $w_i, w_j \in W$. Hence, the two properties respectively provide local positional discrimination and shortest-path coverage of the environment.

Thus, a geodetic local resolving set may be interpreted as a collection of navigation landmarks that simultaneously supports local localization and geodesic coverage, expressed as Geodetic Local Resolving Set = Local Positional Discrimination+ Shortest-Path Coverage.

From an application-oriented perspective, the vertices of G may represent storage locations, junctions, rooms, buildings, or other robot positions, while the edges represent navigable corridors or feasible movements. The selected landmarks may correspond to fixed beacons or identifiable reference points. Consequently, the geodetic local resolving number can be interpreted as the minimum number of landmarks required to achieve both local positional discrimination and shortest-path coverage. This interpretation motivates potential applications in autonomous navigation systems, particularly in structured and GPS-denied environments such as warehouses, hospitals, factories, campuses, and tunnels. It also provides a graph-theoretic framework for investigating optimal landmark placement for autonomous robot localization and navigation.

1.2. **Application in Computer Networks.** A geodetic local resolving set has potential applications in network monitoring, fault localization, and routing. Let $G = (V, E)$ represent a communication network, where vertices denote routers and edges denote communication links. A geodetic local resolving set can be employed as a small set of monitoring nodes. The local resolving property enables adjacent routers to be distinguished through their distance codes, facilitating the localization of link or node failures. Moreover, the geodetic property ensures that the selected reference nodes provide shortest-path coverage, which is useful for efficient routing and spatial addressing in large-scale networks.

Example. Consider the cycle network C_4 with vertex set $V = \{A, B, C, D\}$. The set $S = A, C$ is a geodetic local resolving set. Indeed, the codes of the vertices with respect to S are $c(A) = (0, 2), \quad c(B) = (1, 1), \quad c(C) = (2, 0), \quad c(D) = (1, 1)$. Although B and D have identical codes, they are nonadjacent; hence S satisfies the local resolving property. Further, both B and D lie on shortest A–C

paths, namely $A-B-C$ and $A-D-C$, respectively. Thus, S is geodetic as well. In this model, the vertices in S may serve as monitoring probes. A change in the distance code caused by a link failure can be used to identify the affected region. Hence, geodetic local resolving sets provide a mathematically efficient framework for network fault localization and shortest-path-based routing.

1.3. **Application in Chemical Informatics.** In chemical graph theory, a molecular structure is represented by a graph $G = (V, E)$, where vertices correspond to atoms and edges represent chemical bonds. A geodetic local resolving set provides a compact topological coordinate system in which adjacent atoms can be distinguished using their distance codes. This property can support molecular identification, structural comparison, and molecular descriptor construction.

Example. Consider the carbon skeleton of benzene, represented by the cycle graph C_6 with vertices $V = \{C_1, C_2, C_3, C_4, C_5, C_6\}$. Let $S = \{C_1, C_4\}$. The corresponding codes are

$c(C_1) = (0, 3), \quad c(C_2) = (1, 2), \quad c(C_3) = (2, 1),$

$c(C_4) = (3, 0), \quad c(C_5) = (2, 1), \quad c(C_6) = (1, 2).$

The only repeated codes occur for nonadjacent pairs (C_2, C_6) and (C_3, C_5); hence S is a local resolving set. Moreover, every vertex lies on a shortest path between C_1 and C_4, through either C_2, C_3 or C_6, C_5. Therefore, S is a geodetic local resolving set. Thus, a small number of reference atoms can encode the local topology of a molecular structure, providing a compact representation useful in molecular databases, structural identification, and isomer differentiation.

2. Bounds For Geodetic Local Dimension

In this section, we establish sharp lower and upper bounds for the geodetic local resolving number in terms of the order and diameter of a graph. We also characterize the graphs attaining these extremal bounds, thereby providing insight into the relationship between the geodetic local resolving number and the structural properties of graphs.

Definition 2.1. Let $G = (V, E)$ be a connected graph. An ordered subset $W = \{w_1, w_2, \ldots, w_k\} \subseteq V$ is called a geodetic local resolving set of G if W is both a geodetic set and a local resolving set of G. Equivalently, every vertex $v \in V \setminus W$ lies on a geodesic joining two vertices of W, and for every pair of adjacent vertices $x, y \in V \setminus W$, we have $code(x) \neq code(y)$, where $code(v) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. A geodetic local resolving set of minimum cardinality is called a geodetic local basis of G. The cardinality of a geodetic local basis is called the geodetic local dimension of G, and is denoted by $glmd(G)$.

Example 2.2. Consider the graph G depicted in Figure 1. It can be observed that $\{x, z\}$ is contained in every minimum geodetic set of G, whereas every local resolving set of G contains at least one vertex from $\{a, b, c, d\}$. Therefore, $W = \{x, z, b\}$ is a minimum geodetic local resolving set of G, and hence $glmd(G) = 3$. Moreover, it is readily verified that $\{x, z\}$ is a minimum geodetic set, $\{a, c, y\}$ is a minimum

resolving set, $\{a, c, x, z\}$ is a minimum geodetic resolving set, and $\{a, c\}$ is a minimum local resolving set of G.

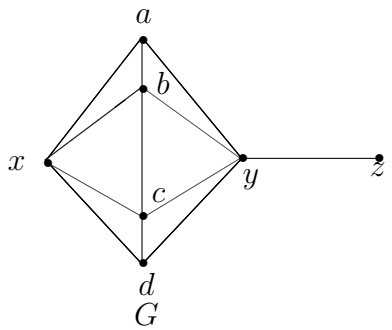


Figure 1.

The following propositions are immediate consequences of the definitions of the geodetic local resolving set and the geodetic local resolving number of a graph. These observations follow directly from the underlying definitions and provide several fundamental properties that are employed throughout the subsequent development of the theory. For the sake of completeness and future reference, we state these elementary results explicitly, as they serve as useful preliminary facts in establishing the main theorems of this paper.

Proposition 2.3.

(1.) *For any graph G, the union of a geodetic set and a local resolving set is a geodetic local resolving set of G. Moreover, every geodetic local basis of G is both a geodetic set and a local resolving set. Consequently,* $\max\{g(G), lmd(G)\} \leq glmd(G) \leq g(G) + lmd(G)$.

(2.) *Let T be a tree with l pendant vertices. Then $glmd(T) = l$. Indeed, the set of all pendant vertices of T is the unique minimum geodetic set of T. Furthermore, every pendant vertex belongs to a local resolving set of T. Hence, the set of all pendant vertices forms a minimum geodetic local resolving set of T. In particular, $glmd(P_n) = 2$ for the path P_n, and $glmd(K_{1,n}) = n$ for the star $K_{1,n}$.*

(3.) *For cycle C_n,* $glmd(C_n) = \begin{cases} 3 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even.} \end{cases}$

(4.) *For a complete graph K_n, $glmd(K_n) = n$.*

(5.) *Let $K_{m,n}$ be the complete bipartite graph with $m, n \geq 4$. Then $glmd(K_{m,n}) = 4$. Indeed, to cover every vertex of one partite set by a geodesic, any geodetic set must contain at least two vertices from the opposite partite set. Moreover, since every vertex of $K_{m,n}$ forms a local resolving basis, a minimum geodetic local resolving set must consist of exactly two vertices from each partite set. Consequently, $glmd(K_{m,n}) = 4$. Using the same argument, it follows that $glmd(K_{2,n}) = 2, \quad n \geq 2$, and $glmd(K_{3,n}) = 3, \quad n \geq 3$.*

In general, for any graph G of order n, $2 \leq glmd(G) \leq n$. The lower bound follows from the definition of a geodetic local resolving set, while the upper bound

is attained by considering the entire vertex set $V(G)$. In the following two theorems, we characterize the graphs G for which $glmd(G) = 2$ and $glmd(G) = n$, respectively.

Theorem 2.4. *Let G be a non-complete graph. Then $glmd(G) = 2$ if and only if G is obtained by arranging the graphs K_1, H_2, H_3, $\ldots$, H_d, K_1, in sequence, where each H_i is an empty graph (i.e., an independent set) and K_1 denotes a single vertex, with adjacency defined as follows:*

(i) *The induced subgraphs $\langle K_1 \cup H_2 \rangle$ and $\langle H_d \cup K_1 \rangle$ are stars.*
(ii) *For each $i = 3, 4, \ldots, d-1$, every vertex of H_i is adjacent to at least one vertex of both H_{i-1} and H_{i+1}.*
(iii) *No edges other than those specified in (i) and (ii) are present.*

Proof. Let $G = (V, E)$ be a graph with $glmd(G) = 2$. Then $g(G) = 2$. Let $S = \{a, b\}$ be a minimum geodetic local resolving set of (G). Since S is a geodetic basis of cardinality two, we have $d(a, b) = \text{diam}(G) = d$. Suppose that two vertices $x, y \in N_i(a)$, for some $i \in \{1, 2, \ldots, d-1\}$, are adjacent. Then $\text{code}(x) = \text{code}(y) = (i, d-i)$, which contradicts the fact that S is a local resolving set. Hence, the induced subgraph $\langle N_i(a) \rangle$ is an independent set for every $i = 1, 2, \ldots, d-1$. Since G is connected, every vertex in $N_i(a)$ is adjacent only to vertices in $N_{i-1}(a)$, $N_i(a)$, or $N_{i+1}(a)$. As $\langle N_i(a) \rangle$ is independent, it follows that the induced subgraph $\langle N_i(a) \cup N_{i+1}(a) \rangle$ is bipartite for each (i=1,2,. . . ,d-2). Moreover, the induced subgraphs $\langle a \cup N_1(a) \rangle$ and $\langle N_{d-1}(a) \cup b \rangle$ are stars. Since S is a geodetic set, every vertex of $V(G) \setminus S$ lies on an (a)-(b) geodesic. Consequently, every vertex of $N_i(a)$ is adjacent to at least one vertex in both $N_{i-1}(a)$ and $N_{i+1}(a)$ for each $i = 2, 3, \ldots, d-2$. Therefore, G has the structure described in the statement of the theorem.

Conversely, suppose that G has the structure described in the hypothesis. Let $S = \{v_1, v_{d+1}\}$, where $V(K_1) = \{v_1\}$ and $V(K_1) = \{v_{d+1}\}$. By construction, every vertex of $V(G) \setminus S$ lies on a geodesic joining v_1 and v_{d+1}. Hence, S is a geodetic set. Furthermore, any two vertices at the same distance from v_1 belong to the same independent set and are therefore non-adjacent. Thus, every pair of adjacent vertices is distinguished by their distances to the vertices of S, implying that S is a local resolving set. Consequently, S is a geodetic local resolving set of minimum cardinality, and therefore $glmd(G) = 2$. □

Theorem 2.5. *Let G be a graph of order n. Then $glmd(G) = n$ if and only if $G \cong K_n$.*

Proof. Assume that $glmd(G) = n$. Suppose that G contains a geodesic path $a - b - c$ of length at least 2. Then the set $W = V(G) \setminus \{b\}$ is both a geodetic set and a local resolving set of G. Consequently, $glmd(G) \leq n-1$, which contradicts the assumption that (glmd(G)=n). Hence, G contains no geodesic path of length at least (2), and therefore $G \cong K_n$. Conversely, let $G \cong K_n$. Since every geodetic set of K_n must contain all vertices, $V(K_n)$ is the unique geodetic set of K_n. Moreover, every local resolving set of K_n is equal to $V(K_n)$. Hence, $V(K_n)$ is the unique geodetic local metric set of K_n, and therefore $glmd(G) = n$. This completes the proof. □

Theorem 2.6. *Let G be a non-complete graph of order n. Then $glmd(G) = n-1$ if and only if $G = H + \bigcup_{i=1}^{k} H_i$, where H and H_i are complete graphs for each $i = 1, 2, \ldots, k-1$, and H_k is an edgeless graph.*

Proof. Assume that G is a non-complete graph with $glmd(G) = n - 1$. By Theorems (X) and (Y), we have $\text{diam}(G) = 2$. Let S be a cut set of G, and let $H_1, H_2, \ldots, H_k$ be the components of $G-S$, where H_k is the maximal independent component. Suppose that H_i is not complete for some $1 \leq i \leq k-1$. Then H_i contains a geodesic path $a - b - c$ of length 2. Let $u \in S$ be a vertex adjacent to a vertex of H_i. It follows that $V(G) \setminus \{b, u\}$ is a geodetic local resolving set of G, implying that $glmd(G) \leq n - 2$, a contradiction. Hence, H_i is complete for every $i = 1, 2, \ldots, k-1$. Moreover, $\bigcup_{i=1}^{k} V(H_i)$ is contained in every geodetic set of G. Suppose that there exist vertices $x \in V(H_i)$ and $u \in S$ such that x is not adjacent to u. Let P_1 be a geodesic joining a vertex of H_k to u, and let P_2 be a geodesic joining u to x. Then $V(G) \setminus \big(\text{Int}(P_1) \cup \text{Int}(P_2)\big)$ is a geodetic local resolving set of G, where $\text{Int}(P_1)$ and $\text{Int}(P_2)$ denote the sets of internal vertices of P_1 and P_2, respectively. Consequently, $glmd(G) \leq n - 2$, which is again a contradiction. Therefore, every vertex of each component H_i is adjacent to every vertex of S, for $i = 1, 2, \ldots, k$. We next show that the induced subgraph $\langle S \rangle$ is complete. Suppose, to the contrary, that $\langle S \rangle$ contains an independent set S' of order at least 2. Then $V(G) \setminus S'$ is a geodetic local resolving set of G, implying that $glmd(G) \leq n - 2$, contradicting the assumption that $glmd(G) = n - 1$. Hence, $\langle S \rangle$ is complete.

Conversely, assume that $G = H + \bigcup_{i=1}^{k} H_i$, where H and H_i are complete graphs for each $i = 1, 2, \ldots, k-1$, and H_k is an independent graph. It is immediate that $\bigcup_{i=1}^{k} V(H_i)$ is contained in every geodetic set of G. Let $S = V(G) \setminus \{x, y\}$, where x and y are adjacent vertices of H. Then $\text{code}(x) = \text{code}(y)$, and therefore S is not a local resolving set. Consequently, every geodetic local resolving set must contain at least one of x and y. It follows that $V(G) \setminus \{x\}$, where $x \in V(H)$, is a minimum geodetic local resolving set of G. Hence, $glmd(G) = n - 1$. □

Theorem 2.7. *There exists no graph G such that $glmd(G) = 3$ and $g(G) + lmd(G) = 3$.*

Proof. Suppose there exist a graph G such that $glmd(G) = 3$ and $g(G)+lmd(G) = 3$. Then $g(G) = 2$ and $lmd(G) = 1$ and hence G must be a bi-partite graph. Therefore the two vertices u and v (say) in the geodetic set must be diametrical vertices of G. If there exist two adjacent vertices a, b in G such that $d(u, a) = d(u, b)$, then the union of the geodetic paths joining the vertices $u - a$ and $u - b$ contains an odd cycle, which is a contradiction to G is bipartite. Therefore all the i^{th} neighborhoods of u are independent, implying that the set $\{u, v\}$ forms a geodetic local resolving set of G. Hence $glmd(G) = 2$ which is again a contradiction. So we can conclude that no such graph exists. □

Theorem 2.8. *Let $G = K_{n_1, n_2, \ldots, n_k}$ be any complete k-partite graph with $n_1 \leq n_2 \leq \ldots n_k$. Then*

$$glmd(G) = \begin{cases} k+1 & \text{if } n_{k-1} \geq 2 \\ (n-2) + n_k & \text{if } n_1 = n_2 = \cdots = n_{k-1} = 1 \text{and } n_k \geq 2. \end{cases}$$

Proof. Let $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ be the k-partite sets of G and let $V_i = \{v_{i_1}, v_{i_2}, \ldots, v_{i_{n_i}}\}$ for $i = 1, 2, \ldots, k$. Assume that $n_{k-1} \geq 2$. Now we prove that $W = \{v_{2_1}, v_{3_1}, \ldots, v_{(k-2)_1}\} \cup \{v_{(k-1)_1}, v_{(k-1)_2}, v_{k_1}, v_{k_2}\}$ is a minimum geodetic local resolving set of G (i.e) W is obtained from selecting exactly one vertex from the partite sets $V_1, V_2, \ldots, V_{k-2}$ and selecting two verties from the partite sets V_{k-1} and V_k). Let $v_{i_l}v_{j_m} \in E(G)$. Then $d(v_{i_l}, v_{i_1}) = 2$ and $d(v_{j_m}, v_{i_1}) = 1$ for $2 \leq i < j \leq k$. Let $v_{1_l}v_{j_m} \in E(G)$. Then $d(v_{1_l}, v_{j_1}) = 1$ and $d(v_{j_m}, v_{j_1}) = 2$. Hence any two adjacent to vertices in G has distinct codes with respect to the set W. Also every vertex in $V - V_k$ lies in the geodesic $v_{k_1} - x - v_{k_2}$ of length two joining the vertices v_{k_1} and v_{k_2}, and every vertex $x \in V_k$ lies in the geodesic of length two $v_{(k-1)_1} - x - v_{(k-2)_2}$. Also if any two partite sets say V_i and V_j have an empty intersection with any set $W_1 \subseteq V(G)$, then $code(v_{i_1}) = code(v_{j_1}) = (1, 1, \ldots, 1)$ with respect to the set W_1, so at most one partite set can have empty intersection with any minimum local resolving set. And from at least two partite sets, need to select minimum two vertices to cover every vertices in G by a geodesic path. Therefore $glmd(G) = (k-3) + 4 = k + 1$.
Now consider the case $n_1 = n_2 = \cdots = n_{k-1} = 1$ and $n_k \geq 2$. Then it can be verify that $W' = \{v_{2_1}, v_{3_1}, \ldots, v_{(k-1)_1}\} \cup \{v_{k_1}, v_{k_2}, \ldots, v_{k_{n_k}}\}$ is a minimum geodetic local resolving set of G as arguments above. Therefore $glmd(G) = (n-2) + n_k$. □

Theorem 2.9. *Let $G = (V, E)$ be any connected graph of order n and diameter d. Then $glmd(G) \leq n - (d-1)$. Also, the bound is sharp for a graph G if and only if G is the graph obtained from the path $P' = P_{d-1} : (v_2, v_3, \ldots, v_d)$ and a collection of complete graphs $\{H_1, H_2, \ldots, H_k\}$ by identifying exactly one vertex of H_i^s to a vertex of path P_{d-1}, and no other edges.*

Proof. Let $P : u_1, u_2, \ldots, u_{d+1}$ be any diametrical path of G. Then $S = V - \{u_2, u_3, \ldots, u_d\}$ is a geodetic local resolving set of G for any vertex u_i, $2 \leq i \leq d$ lies in a geodesic path P joining the vertices v_1 and u_{d+1}, also for an edge u_iu_{i+1} $2 \leq i \leq d$ we have $d(u_1, u_i) = i - 1$ and $d(u_1, u_{i+1}) = i$ implies that $code(u_i) \neq code(u_{i+1})$. Here $|S| = n - (d-1)$ and hence $glmd(G) \leq n - (d-1)$.

The bound is sharp: Assume that $glmd(G) = n - (d-1)$. Now it is enough to prove that in G, every vertex x which is not lies in a diametrical path P of G is in every geodetic set or local resolving set of G. Let $x \in V - V(P)$, with $d(x, V(P)) = 2$. Then G has a geodesic path $x - y - v_i$ for some $v_i \in V(P)$ and $y \in V - V(P)$. In this case $S_1 = V - (\{v_1, v_2, \ldots, v_d\} \cup \{y\})$ is a geodetic local resolving set of G and hence $glmd(G) < n - (d-1)$, which is not possible. Let $x \in N(v_i) - V(P)$ and $y \in N(v_{i+1}) - V(P)$. Suppose $xy \in E$ or $yv_i \in E$. Then $S_1 = V - (\{v_2, v_3, \ldots, v_d\} \cup \{y\})$ is a geodetic local resolving set of G and $glmd(G) < n - (d-1)$, which is again not possible. Therefore the components of the induced subgraph $\langle V - V(P) \rangle$ are all complete.

Conversely, assume that G is a graph given in theorem. Then it is easy to verify that $V - V(P')$ is a minimum order geodetic local resolving set of G, where $V(P')$ is the set of all internal vertices of diametrical path P' of G. Hence $glmd(G) = n - (d-1)$. □

3. Realization Results

Studying the relationship between newly introduced graph parameters and existing graph invariants is an important aspect of graph theory. In this section, we present realization results for the geodetic local metric dimension, $glmd(G)$, with respect to the order of a graph, the geodetic resolving number, and the local resolving number. These results establish the existence of graphs with prescribed values of these parameters and provide insight into the interplay between $glmd(G)$ and these classical graph invariants.

Theorem 3.1. *For every pair of integers k and n satisfying $2 \leq k \leq n$, there exists a connected graph G of order n such that $glmd(G) = k$.*

Proof. Assume that $3 \leq k \leq n-1$. Let G be the graph obtained by joining the complete graph K_k, with vertex set $V_1 = \{v_1, v_2, \ldots, v_k\}$, and the path $P_{n-k} : v_{k+1}, v_{k+2}, \ldots, v_n$, by adding the single edge $v_k v_{k+1}$, and no other edges between K_k and P_{n-k}. Then $W = (V_1 \setminus \{v_k\}) \cup \{v_n\}$ is a minimum geodetic local resolving set of G. Consequently, $glmd(G) = |W| = k$. If $k = 2$, let $G = P_n$ with vertex sequence $(v_1, v_2, \ldots, v_n)$. Then the set $W = v_1, v_n$ is a minimum geodetic local resolving set of G. Hence, $glmd(G) = 2$. Finally, if $k = n$, let $G = K_n$. By Theorem 3.1, $glmd(G) = n$. Therefore, for every pair of integers k and n satisfying $2 \leq k \leq n$, there exists a connected graph of order n with geodetic local resolving number k. □

Theorem 3.2. *Given two natural numbers a, b with $2 \leq a \leq b \leq n$ there exists a connected graph G such that $glmd(G) = a$ and $gr(G) = b$.*

Proof. When $a = b = n$, take the graph $G = K_n$. Then $glmd(G) = gr(G) = n$.
Suppose $a = b = n-1$. Let $G = K_1 + (K_{n_1} \cup K_{n_2})$ be a graph on n vertices. Then by Theorem 2.6, it follows directly that $glmd(G) = n-1$. Furthermore, a straightforward calculation yields $gr(G) = n-1$.
Assume $a = b < n-1$. Then construct the graph G which is obtained from the graphs, path $P = P_{n-a+2}$ and complete graph K_a with vertex sets $V(P) = (v_1, v_2, \ldots, v_{n-a+2})$ and $V(K_a) = \{u_1, u_2, \ldots, u_a\}$ by identifying the edges $v_2 v_3$ of path P and $u_1 u_2$ of K_a and no other edges. Then it can be shown that $W = \{v_1, v_{n-a+2}\} \cup \{u_3, u_4, \ldots, u_a\}$ is a unique minimum order geodetic resolving set and geodetic local resolving set of G. Hence $glmd(G) = gr(G) = a$.
Suppose $a < b \leq n-2$. Then construct the graph G obtained from the graphs K_a, $P_{n-(b+5)}$ and $K_{2,\ b-a+1}$ with vertex sets $V(K_a) = \{v_1, v_2, \ldots, v_a\}$, $V(P_{n-(b+5)}) = \{u_1, u_2, \ldots, u_{n-(b+5)}\}$ and $V(K_{2,b-(a+1)}) = \{x_1, x_2\} \cup \{y_1, y_2, \ldots, y_{b-(a+1)}\}$ (bi-partition) by identifying the vertices v_a and u_1, and $u_{n-(b+5)}$ and x_1 and no other edges. In this case it can be easily check that $W_1 = \{v_1, v_2, \ldots, v_{a-1}\} \cup \{x_2\}$ is a minimum geodetic local resolving set of G and $W_2 = \{v_1, v_2, \ldots, v_{a-1}\} \cup \{x_2\} \cup \{y_1, y_2, \ldots, y_{b-a}\}$ is a minimum geodetic resolving set of G. Hence $glmd(G) = a$ and $gr(G) = b$. □

For every triple of integers a, b and c satisfying $2 \leq a \leq b \leq c \leq n$ and $c \geq 3$, there exists a connected graph G of order n such that $g(G) = a$, $\quad glmd(G) = b$, and $g(G) + lmd(G) = c$.

Theorem 3.3. *For every triple of integers a, b and c satisfying $2 \leq a \leq b \leq c \leq n$ and $c \geq 3$, there exists a connected graph G of order n such that $g(G) = a$, $glmd(G) = b$, and $g(G) + lmd(G) = c$.*

Proof. We distinguish the following cases.

Case 1. $a = b$.

Consider the star graph $G = K_{1,a}$. Since the set of all pendant vertices is the unique minimum geodetic set as well as the unique minimum geodetic local resolving set of G, it follows that $g(G) = a = glmd(G)$.

Case 2 $a < b$.

Let G be the graph obtained as the sequential join of the four graphs $H_1 = K_1$, $H_2 = (n-(b+1))K_1 \cup K_{b-(a+1)}$, $H_3 = K_1$, $H_4 = (a-1)K_1$ whose vertex sets are given by $V(H_1) = \{u\}$, $V(H_2) = \{x_1, x_2, \ldots, x_{n-(b+1)}\} \cup \{y_1, y_2, \ldots, y_{b-(a+1)}\}$, $V(H_3) = \{v\}$, and $V(H_4) = \{z_1, z_2, \ldots, z_{a-1}\}$, respectively. It is straightforward to verify that $S = \{u, z_1, z_2, \ldots, z_{a-1}\}$ is a minimum geodetic set of G, whereas $W = S \cup \{y_1, y_2, \ldots, y_{b-a}\}$ is a minimum geodetic local resolving set of G. Consequently, $g(G) = a$ and $glmd(G) = b$.

Case 3. $b = c$.

Suppose that $b = c = k$, where $k \geq 4$. Consider the graph $G = 2K_1 + \left(\bigcup_{i=1}^{k-2} K_2\right)$, where the i-th copy of K_2 has vertex set $\{a_i, b_i\}$, and let the two isolated vertices of $2K_1$ be denoted by u and v. Then $W = \{u, v, a_1, a_2, \ldots, a_{k-2}\}$ is a minimum geodetic local resolving set of G, $S = \{u, v\}$ is a minimum geodetic set of G, and $L = \{a_1, a_2, \ldots, a_{k-2}\}$ is a minimum local resolving set of G. Therefore, $g(G) = 2$, $lmd(G) = k - 2$, $glmd(G) = k$, and hence $g(G) + lmd(G) = k = c$.

Case 4. $b < c$.

Let G be the graph obtained from the complete graph $K_s : (v_1, v_2, \ldots, v_s)$, the path $P_k : (u_1, u_2, \ldots, u_k)$, and the friendship graph $F_t = K_1 + tK_2$, whose vertex set is $\{x\} \cup \bigcup_{i=1}^{t} \{a_i, b_i\}$, where a_i and b_i are adjacent, by identifying the vertex v_s with the end-vertex u_1 of P_k, and identifying the other end-vertex u_k with the central vertex x of F_t. One can verify that $W = \{v_1, v_2, \ldots, v_{s-1}\} \cup \{a_1, a_2, \ldots, a_t\} \cup \{b_1, b_2, \ldots, b_t\}$ is a minimum geodetic local resolving set of G, while $L = \{v_1, v_2, \ldots, v_{s-1}\} \cup \{a_1, a_2, \ldots, a_t\}$ is a minimum local resolving set of G. Hence, $glmd(G) = s + 2t - 1$ and $lmd(G) = s + t - 1$. By choosing $b = s + t - 1$ and $c = s + 2t - 1$, we obtain $b \leq c$ and $g(G) + lmd(G) = c$. This completes the proof. □

Conclusion

In this paper, we introduced the notion of a geodetic local resolving set and the associated graph invariant, namely the geodetic local dimension, as a natural combination of the concepts of geodetic sets and local resolving sets. We established several fundamental properties of this newly defined parameter, including sharp lower and upper bounds, and characterized the extremal cases wherever possible. Furthermore, the geodetic local dimension was explicitly determined for several standard families of graphs, thereby illustrating its structural behavior and demonstrating its applicability to well-known graph classes. In addition, a comparative study was carried out between the geodetic local dimension and

other closely related graph invariants, such as the geodetic resolving number and the geodetic number. These comparisons yielded several realization results, highlighting the relationships, distinctions, and possible inequalities among these parameters.

The results presented in this work naturally lead to several directions for further investigation. Some interesting problems that remain open are listed below.

Problem 3.4. Characterize graphs G for which $glmd(G) = g(G)$.

Problem 3.5. Characterize graphs G for which $glmd(G) = lmd(G)$.

Problem 3.6. Characterize graphs G for which $glmd(G) = g(G) + lmd(G)$.

Acknowledgement. The Author is thankful to the management of SRMIST, Tiruchirappalli for its constant support and encouragement. The author also thanks the editor and the anonymous referees for their valuable suggestions.

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