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A POSTERIORI RELIABILITY FOR FITZHUGH-NAGUMO OPTIMAL CONTROL

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Abstract. We study distributed optimal control of a FitzHugh–Nagumo parabolic system with pointwise control constraints. We establish existence of optimal controls, derive continuous and fully discrete optimality systems, and obtain computable a posteriori reliability estimates for the states, adjoints, and control. The control estimate requires an explicit sufficient condition for strong convexity, which may restrict its practical applicability; the state and adjoint residual estimates do not require this condition. An exponential-memory interpretation and a manufactured benchmark for checking the consistency of the reconstructions and estimator are also presented.

Keywords. FitzHugh–Nagumo system, optimal control, a posteriori estimate, exponential memory.
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1. Introduction

The FitzHugh–Nagumo model describes excitation and recovery through a pair of coupled variables. Its origins lie in simplified descriptions of nerve-membrane dynamics [6, 12]. A numerical study of a time-fractional variant of the FitzHugh–Nagumo model was presented by Jabbar and Huseen [8]. In a spatial model, diffusion acts on the excitation variable, whereas the recovery variable may evolve without spatial diffusion. Distributed control modifies the excitation through an applied source, allowing the evolution of the system to be guided toward prescribed targets.

In this paper, we consider the controlled parabolic–ODE system

$$\begin{cases} \partial_t y - D\Delta y + f(y) + z = F + u & \text{in } Q, \\ \partial_t z + bz - \varepsilon y = 0 & \text{in } Q, \\ y = 0 & \text{on } \partial\Omega \times (0,T), \\ y(\cdot,0) = y_0, \qquad z(\cdot,0) = z_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

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where $Q = \Omega \times (0,T)$, $T > 0$, and Ω is either a bounded interval in $\mathbb{R}$ or a bounded Lipschitz polygon in $\mathbb{R}^2$. The reaction term is

$$f(s) = s(s-a)(s-1), \qquad 0 < a < 1.$$

Here y and z denote the excitation and recovery variables, respectively; $D > 0$ is the diffusion coefficient, $b > 0$ is the recovery decay rate, and $\varepsilon > 0$ is the coupling coefficient. The function F is a prescribed source, and u is the distributed control. Since the recovery equation contains no spatial derivatives, no boundary condition is prescribed for z.

Given desired states y_d and z_d, we study the optimal control problem

$$\min_{u \in \mathcal{U}_{\mathrm{ad}}} j(u), \qquad j(u) = \frac{1}{2}\|y(u) - y_d\|^2_{L^2(Q)} + \frac{1}{2}\|z(u) - z_d\|^2_{L^2(Q)} + \frac{\alpha}{2}\|u\|^2_{L^2(Q)}, \tag{1.2}$$

where $\alpha > 0$ and

$$\mathcal{U}_{\mathrm{ad}} = \big\{u \in L^2(Q) : u_a \le u \le u_b \text{ almost everywhere in } Q\big\},$$

with finite constants $u_a < u_b$. The states $(y(u), z(u))$ satisfy (1.1). Our objective is to derive computable a posteriori upper bounds for the errors in a fully discrete approximation of the optimal control, the two states, and the two adjoints.

Optimal control of the FitzHugh–Nagumo system has been studied in several settings. Casas, Ryll and Tröltzsch [3] investigated sparse control of the Schlögl and FitzHugh–Nagumo systems, and their subsequent analysis [4] developed second-order conditions and stability results. For semilinear parabolic control problems, Kammann, Tröltzsch and Volkwein [10] related a posteriori control estimates to second-order information. Other parabolic control studies address coupled linear equations [1], box-constrained diffusion-coefficient control with finite-element and implicit-Euler discretization [5], and a nonlinear inverse problem formulated through optimal control and finite elements [9]. Reaction–diffusion-constrained optimization has also been studied in a drug-delivery setting [2]. These works concern different state systems or error questions from the residual reliability estimates developed here. These studies provide a foundation for the analysis of nonlinear control problems and highlight the role of optimality conditions in estimating the control error.

The methodological starting point of the present work is Lu [11], who studied fully discrete finite-element approximations of nonlinear parabolic integro-differential control problems. That analysis combines intermediate continuous states associated with a discrete control and residual estimates for the state and adjoint equations. Applying this approach to (1.1) requires accounting for two features of the coupled system. First, the cubic reaction is not monotone, although it satisfies the global one-sided bound

$$f'(s) \ge -\kappa, \qquad \kappa = \frac{1 - a + a^2}{3}.$$

Second, the recovery equation provides no spatial gradient control. Its residual must therefore be estimated in its natural $L^2(\Omega)$ setting. The adjoint system has the same distinction between diffusive and nondiffusive variables, with different weights needed to cancel the coupling terms.

The system also has a precise connection with equations involving memory. Eliminating the recovery variable gives

$$z(t) = e^{-bt} z_0 + \varepsilon \int_0^t e^{-b(t-s)} y(s)\, \mathrm{d}s.$$

Consequently, the excitation equation contains an exponential-memory term acting on y. This representation differs from the gradient-memory operator considered in [11]. Related analysis of nonlinear reaction-diffusion equations with memory appears in Ho, Nguyen and Vo [7], while regularity and parameter-continuity questions for viscoelastic parabolic equations are treated by Thach, Le and Nguyen [14]. These works provide background for the memory interpretation considered here.

The principal contribution of this paper is a combined residual reliability estimate for the optimality system of (1.1)–(1.2). More specifically, the analysis provides the following results.

- We establish state and adjoint reliability estimates using the one-sided derivative bound for the cubic reaction and weighted energy identities. The estimates include separate recovery and recovery-adjoint residuals without spatial mesh-size weights, together with initial and terminal discrepancies.
- We derive the fully discrete adjoint and control conditions directly from the discrete objective and state constraints. Continuous reconstructions provide residuals that retain spatial and temporal approximation errors, data-approximation errors, and algebraic defects.
- We introduce a normal-cone residual for the box-constrained control and combine it with the adjoint residual estimate. An explicit sufficient condition on the regularization parameter guarantees strong convexity of the reduced objective and yields an upper bound for the optimal-control error. State and adjoint stability then give simultaneous bounds for all five variables.
- We obtain computable majorants for the parabolic dual residual norms through conforming fluxes and give an exact primitive formula on an interval. We also identify the recovery residual as the forcing term in the defect of the exponential-memory representation and construct a manufactured optimality system for a reproducible consistency benchmark.

The combined reliability theorem is stated in Section 7. The state and adjoint residual estimates hold independently of the strong-convexity condition, which is used to estimate the distance to the unique optimal control. This sufficient condition relies on bounds uniform over the admissible control set and can require a large regularization parameter or a sufficiently short time horizon. Its practical limitations are discussed after Theorem 3.4.

The bounds are established for fixed positive model parameters. Local efficiency, parameter-uniform robustness, and convergence or optimality of adaptive algorithms remain open to further analysis. The manufactured benchmark evaluates interpolants of a known optimality system. It checks consistency, but does

not assess an optimization solver or estimator-driven adaptive mesh refinement (AMR).

Section 2 establishes the state theory. Section 3 presents the continuous optimality system and a sufficient condition for strong convexity. Sections 4 and 5 describe the fully discrete scheme, continuous reconstructions, and residual computation. Sections 6 and 7 prove state, adjoint, and optimal-control reliability. Section 8 develops the memory interpretation and the manufactured benchmark. Section 9 presents the conclusions.

2. The state system and its stability

Let Ω be a bounded Lipschitz polygon in $\mathbb{R}^2$, or a bounded interval in $\mathbb{R}$. Write $Q = \Omega \times (0,T)$, $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$, equipped with $\|v\|_V = \|\nabla v\|_H$. Let C_P be any known Poincaré constant, so $\|v\|_H \le C_P \|v\|_V$. The pivot identification $H \subset V^*$ is understood. We use $(\cdot,\cdot)$ for the inner product in H, and $\|\cdot\|_Q$ for $L^2(Q)$.

Fix $D, \varepsilon, b, \alpha > 0$ and $0 < a < 1$. Define

$$f(s) = s(s-a)(s-1), \qquad \kappa = \frac{1-a+a^2}{3}, \qquad f'(s) = 3\left(s - \frac{1+a}{3}\right)^2 - \kappa. \tag{2.1}$$

Consequently,

$$(f(r) - f(s))(r-s) \ge -\kappa |r-s|^2 \quad (r,s \in \mathbb{R}). \tag{2.2}$$

Assume $F, y_d, z_d \in L^\infty(Q)$, $y_0 \in H \cap L^\infty(\Omega)$ and $z_0 \in L^\infty(\Omega)$. For constant finite bounds $u_a < u_b$, set

$$\mathcal{U}_{\mathrm{ad}} = \{u \in L^2(Q) : u_a \le u \le u_b \text{ a.e.}\}, \qquad U_* = \max(|u_a|, |u_b|).$$

For $u \in \mathcal{U}_{\mathrm{ad}}$, the state equations are

$$\begin{aligned} y_t - D\Delta y + f(y) + z &= F + u && \text{in } Q, && \text{(2.3a)}\\ z_t + bz - \varepsilon y &= 0 && \text{in } Q, && \text{(2.3b)}\\ y &= 0 && \text{on } \partial\Omega \times (0,T), && \text{(2.3c)}\\ y(0) = y_0, \qquad z(0) &= z_0. && && \text{(2.3d)} \end{aligned}$$

No boundary condition is prescribed for z. A weak solution satisfies

$$y \in L^2(0,T;V) \cap H^1(0,T;V^*) \cap L^\infty(Q), \qquad z \in H^1(0,T;H) \cap L^\infty(Q),$$

and, almost everywhere in time,

$$\begin{aligned} \langle y_t, v\rangle + D(\nabla y, \nabla v) + (f(y) + z, v) &= (F+u, v), && v \in V,\\ (z_t + bz - \varepsilon y, w) &= 0, && w \in H. \end{aligned}$$

The state traces are in H, with $y, z \in C([0,T];H)$.

2.1. **Uniform bounds and existence.** Choose $M \geq \|y_0\|_\infty$ sufficiently large and put

$$Z_* = \max\{\|z_0\|_\infty, \varepsilon M/b\}, \qquad f(M) \geq \|F\|_\infty + U_* + Z_*, \quad f(-M) \leq -\|F\|_\infty - U_* - Z_*. \tag{2.4}$$

Such an M exists because the cubic dominates the linear dependence of Z_* on M.

Theorem 2.1. *For every admissible control, (2.3) has a unique bounded weak solution. Uniformly over $\mathcal{U}_{\mathrm{ad}}$,*

$$|y| \leq M, \quad |z| \leq Z_* \quad \text{a.e. in } Q, \qquad z(t) = e^{-bt} z_0 + \varepsilon \int_0^t e^{-b(t-s)} y(s)\,\mathrm{d}s. \tag{2.5}$$

The solutions are uniformly bounded in the weak-solution spaces above.

Proof. We give the construction to specify the role of boundedness. Let c_M and c_Z be scalar clipping maps onto $[-M, M]$ and $[-Z_*, Z_*]$. Solve first

$$y_t - D\Delta y = F + u - f(c_M(y)) - c_Z(z), \qquad z_t = \varepsilon c_M(y) - bz.$$

The reaction maps are globally Lipschitz. The Dirichlet heat semigroup is contractive on H and on $L^\infty(\Omega)$. Its variation-of-constants formula, together with the integrating-factor formula for z, defines a contraction on a sufficiently short time interval in the product of bounded time-dependent H spaces; the contraction constant is bounded by the interval length times a fixed Lipschitz constant. Successive intervals construct a solution on $[0, T]$. Galerkin approximation of the linear heat equation, tested by y, gives $y \in L^2(V) \cap H^1(V^*)$; the ODE gives $z \in H^1(H)$. Bounded right-hand sides and semigroup contractivity also give boundedness on each finite interval.

Test the first truncated equation by $(y - M)_+ \in V$. On its support, $f(c_M(y)) = f(M)$ and $-c_Z(z) \leq Z_*$. Thus

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|(y - M)_+\|^2 + D\|\nabla(y - M)_+\|^2 \leq 0.$$

The initial positive part is zero. Testing with the negative excursion below $-M$ gives the lower bound. In the ODE, $\varepsilon c_M(y) - bZ_* \leq 0$ at the upper boundary, and the corresponding lower-bound inequality also holds. Positive-part tests therefore yield $|z| \leq Z_*$. All clipping maps can now be removed. These tests can be justified by smooth approximations of the positive-part function in the weak formulation.

Testing the original equations with y and z/ε cancels the coupling. Since $f(0) = 0$, (2.2) gives $(f(y), y) \geq -\kappa\|y\|^2$. Young's inequality and Gronwall's lemma bound y in $L^\infty(H) \cap L^2(V)$ and z in $L^\infty(H)$. The bounded reaction and the equations then bound y_t in $L^2(V^*)$ and z_t in $L^2(H)$. Integrating the ODE gives (2.5). Uniqueness follows from the difference estimate below with identical controls and zero initial difference. □

Set

$$c = 2\kappa + 1, \qquad E_T = e^{cT}, \qquad C_E = 1 + (1 + cT)E_T, \qquad C_0 = (TE_T)^{1/2}. \tag{2.6}$$

For a state pair (v, w) and an adjoint pair (r, s) define

$$\|(v,w)\|_{\mathcal{E}_s}^2 = \sup_{0\le t\le T} \left(\|v(t)\|^2 + \varepsilon^{-1}\|w(t)\|^2\right) + \int_0^T \left(D\|\nabla v\|^2 + \frac{b}{\varepsilon}\|w\|^2\right) \mathrm{d}t, \tag{2.7}$$

$$\|(r,s)\|_{\mathcal{E}_a}^2 = \sup_{0\le t\le T} \left(\|r(t)\|^2 + \varepsilon\|s(t)\|^2\right) + \int_0^T \left(D\|\nabla r\|^2 + \varepsilon b\|s\|^2\right) \mathrm{d}t. \tag{2.8}$$

Lemma 2.2. *Suppose $v_t - D\Delta v + Av + w = g$ and $w_t + bw - \varepsilon v = r$, where $A \in L^\infty(Q)$ and $A \ge -\kappa$. Put*

$$B_s = \|v(0)\|^2 + \varepsilon^{-1}\|w(0)\|^2 + \int_0^T \left(D^{-1}\|g\|_{V^*}^2 + (\varepsilon b)^{-1}\|r\|^2\right) \mathrm{d}t.$$

Then $\|(v,w)\|_{\mathcal{E}_s}^2 \le C_E B_s$ and $\|v\|_Q^2 \le TE_T B_s$, $\|w\|_Q^2 \le \varepsilon T E_T B_s$. For the backward system

$$-r_t - D\Delta r + Ar - \varepsilon s = g, \qquad -s_t + bs + r = h,$$

the analogous estimates hold in $\mathcal{E}_a$ with

$$B_a = \|r(T)\|^2 + \varepsilon\|s(T)\|^2 + \int_0^T \left(D^{-1}\|g\|_{V^*}^2 + \varepsilon b^{-1}\|h\|^2\right) \mathrm{d}t.$$

In particular, $\|r\|_Q^2 \le TE_T B_a$.

Proof. Testing the forward equations by v and w/ε gives, for $E = \|v\|^2 + \varepsilon^{-1}\|w\|^2$,

$$\tfrac{1}{2}E' + D\|\nabla v\|^2 + \tfrac{b}{\varepsilon}\|w\|^2 \le \kappa\|v\|^2 + \|g\|_{V^*}\|\nabla v\| + \varepsilon^{-1}\|r\|\,\|w\|.$$

Apply $ab \le Da^2/2 + b^2/(2D)$ to the parabolic term, and the analogous inequality with b/ε to the recovery term. With $L = D\|\nabla v\|^2 + (b/\varepsilon)\|w\|^2$ and $G = D^{-1}\|g\|_{V^*}^2 + (\varepsilon b)^{-1}\|r\|^2$, this yields

$$E' + L \le 2\kappa E + G \le cE + G.$$

Hence $\sup_t E(t) \le E_T B_s$. Integration also gives $\int L \le B_s + cTE_T B_s$. Adding proves the energy estimate, and integration of E proves the two $L^2(Q)$ estimates. For the backward equations, introduce the reversed time variable $\theta = T - t$ and test the transformed equations by r and εs, respectively. The cross terms $-\varepsilon(s,r)$ and $\varepsilon(r,s)$ cancel. The same Young inequalities produce the stated weights and constants. □

Corollary 2.3. *For two controls u_1, u_2 and their states,*

$$\|(y_1 - y_2, z_1 - z_2)\|_{\mathcal{E}_s} \le \sqrt{C_E}\,\|u_1 - u_2\|_Q,$$

$$\|y_1 - y_2\|_Q \le C_0\|u_1 - u_2\|_Q, \qquad \|z_1 - z_2\|_Q \le \sqrt{\varepsilon}C_0\|u_1 - u_2\|_Q.$$

Proof. Subtract the equations and use $A = \int_0^1 f'(y_2 + \theta(y_1 - y_2))\,\mathrm{d}\theta \ge -\kappa$. The initial difference is zero. In the energy identity estimate the control term directly by $2(u_1 - u_2, y_1 - y_2) \le \|u_1 - u_2\|^2 + \|y_1 - y_2\|^2$. This gives $E' + L \le cE + \|u_1 - u_2\|^2$. The last part of the preceding proof applies unchanged. This direct

H estimate, rather than the dual-norm estimate, accounts for the constants in this corollary. □

3. Optimal controls and second-order information

We minimize

$$j(u) = \tfrac{1}{2}\,\|y(u) - y_d\|_Q^2 + \tfrac{1}{2}\,\|z(u) - z_d\|_Q^2 + \tfrac{\alpha}{2}\,\|u\|_Q^2\,, \qquad u \in \mathcal{U}_{\mathrm{ad}}. \tag{3.1}$$

Theorem 3.1. *Problem (3.1) has a global minimizer.*

Proof. Take a minimizing sequence u_n. The closed convex bounded set $\mathcal{U}_{\mathrm{ad}}$ is weakly compact in $L^2(Q)$, so a subsequence converges weakly to $\bar u \in \mathcal{U}_{\mathrm{ad}}$. Theorem 2.1 bounds y_n in $L^2(V) \cap H^1(V^*)$. Compactness of $V \hookrightarrow H$ and the time-compactness theorem [13] give $y_n \to \bar y$ strongly in $L^2(Q)$ after a further subsequence. The uniform bound M implies $\|f(y_n) - f(\bar y)\|_Q \le \max_{[-M,M]} |f'|\,\|y_n - \bar y\|_Q \to 0$. The ODE formula and Young's convolution inequality give

$$\|z_n - \bar z\|_Q \le \frac{\varepsilon}{b}\,\|y_n - \bar y\|_Q \to 0, \quad \bar z = e^{-bt} z_0 + \varepsilon \int_0^t e^{-b(t-s)} \bar y(s)\,\mathrm{d}s.$$

Passing to the limit in the time-integrated weak state equation identifies $(\bar y, \bar z)$ as the state of $\bar u$, including the initial trace through test functions not vanishing at $t = 0$. Strong convergence handles both tracking terms, and weak lower semicontinuity handles the control norm. Thus $j(\bar u) \le \liminf j(u_n)$. □

3.1. Linearization and adjoints.

For a bounded direction h, the linearized state (ξ, ζ) at u solves

$$\xi_t - D\Delta\xi + f'(y)\xi + \zeta = h, \qquad \zeta_t + b\zeta - \varepsilon\xi = 0, \qquad \xi(0) = \zeta(0) = 0, \tag{3.2}$$

with the homogeneous boundary condition for ξ. The adjoint is

$$-p_t - D\Delta p + f'(y)p - \varepsilon q = y - y_d, \tag{3.3a}$$

$$-q_t + bq + p = z - z_d, \tag{3.3b}$$

$$p|_{\partial\Omega} = 0, \qquad p(T) = q(T) = 0. \tag{3.3c}$$

Its solution belongs to the corresponding energy spaces; q has no boundary condition.

Lemma 3.2. *For every admissible control the adjoint exists uniquely and is bounded. A uniform bound is*

$$\|p\|_{L^\infty(Q)} \le B_p, \quad B_p = T e^{KT}\big(M + \|y_d\|_\infty + Z_* + \|z_d\|_\infty\big), \quad K = \kappa + \varepsilon + 1. \tag{3.4}$$

Proof. Reverse time. The bounded coefficient $f'(y)$ permits construction by linear Galerkin approximation, and Lemma 2.2 gives uniqueness and convergence in the weak spaces. Alternatively the heat and ODE integral equations give a bounded mild solution by iteration, which agrees with the weak solution. To bound it, let $P(s)$ and $Q(s)$ denote the spatial essential suprema of the absolute values of the

reversed unknowns. The parabolic comparison principle, using $f'(y) \geq -\kappa$, and the ODE integrating factor give the scalar comparison inequalities

$$P(s) \leq \int_0^s e^{\kappa(s-r)}(M + \|y_d\|_\infty + \varepsilon Q(r))\,\mathrm{d}r,$$

$$Q(s) \leq \int_0^s (P(r) + Z_* + \|z_d\|_\infty)\,\mathrm{d}r.$$

Compare these with the nonnegative solution of $P_0' = \kappa P_0 + \varepsilon Q_0 + M + \|y_d\|_\infty$, $Q_0' = P_0 + Z_* + \|z_d\|_\infty$, with zero initial values. Their sum satisfies $(P_0 + Q_0)' \leq K(P_0 + Q_0) + M + \|y_d\|_\infty + Z_* + \|z_d\|_\infty$. Gronwall gives (3.4). The comparison can also be obtained directly from positive-part tests and monotone iteration of the two integral inequalities. □

Proposition 3.3. *The reduced objective is twice continuously differentiable along bounded control directions. With (ξ, ζ) from (3.2),*

$$j'(u)h = (\alpha u + p, h)_{L^2(Q)}, \tag{3.5}$$

$$j''(u)[h,h] = \alpha \|h\|_Q^2 + \|\xi\|_Q^2 + \|\zeta\|_Q^2 - \int_Q f''(y)p\xi^2. \tag{3.6}$$

These formulas apply along every segment of admissible controls.

Proof. For completeness, bounded controls in a fixed L^∞ neighborhood admit a common invariant rectangle by the construction in Section 2. On that rectangle f, f', f'' are bounded, and f'' is Lipschitz. Subtraction of the mild equations and the heat-semigroup L^∞ estimate imply $\|y(u+h) - y(u)\|_\infty + \|z(u+h) - z(u)\|_\infty \leq C\|h\|_\infty$. Indeed, the sum of the two differences is bounded by $T\|h\|_\infty$ plus a constant times its time integral; Gronwall applies.

Subtract (3.2) from the state difference equations. The remainder has zero initial data, the same linear part, and first forcing $-[f(y+d) - f(y) - f'(y)d]$, where $d = y(u+h) - y(u)$. Taylor's formula bounds this forcing in L^∞ by $C\|h\|_\infty^2$. The same integral estimate bounds the state remainder by $C\|h\|_\infty^2$. Subtracting the linearized equations at two base controls and repeating this argument gives the second derivative. In the repeated direction h it is (χ, ω), with zero initial data and equations

$$\chi_t - D\Delta\chi + f'(y)\chi + \omega = -f''(y)\xi^2, \qquad \omega_t + b\omega - \varepsilon\chi = 0.$$

Taylor's integral remainder, with bounded Lipschitz f'', makes the remaining error $o(\|h\|_\infty^2)$ and gives continuous dependence of these derivatives. All equations also belong to the energy spaces, by their bounded forcing.

Differentiating (3.1) gives $j'(u)h = (y - y_d, \xi)_Q + (z - z_d, \zeta)_Q + \alpha(u, h)_Q$. Multiply (3.2) by p, q, integrate over space-time, and integrate the time derivatives by parts. The zero initial variations and zero terminal adjoints remove all endpoint terms. The coefficients of ξ, ζ are exactly the left sides of (3.3), so the two tracking terms equal $(p, h)_Q$.

Differentiating a second time produces the two squared linearized-state norms and $(y - y_d, \chi)_Q + (z - z_d, \omega)_Q$. Applying the same adjoint identity to (χ, ω) changes the latter expression to $-\int_Q pf''(y)\xi^2$, proving (3.6). This proof concerns

L^∞ directions; it does not assume an unproved twice-Fréchet differentiable map on all of $L^2(Q)$. Differences of admissible controls are bounded, which is sufficient below. □

Every local minimizer satisfies

$$(\alpha\bar{u} + \bar{p}, v - \bar{u})_Q \geq 0 \quad (v \in \mathcal{U}_{\mathrm{ad}}), \qquad \bar{u} = \Pi_{[u_a,u_b]}(-\bar{p}/\alpha). \tag{3.7}$$

The first assertion follows by differentiating along feasible segments; the second follows by pointwise projection onto a closed interval.

3.2. **An explicit sufficient convexity condition.** Let $M_2 = \max_{|s|\leq M} |f''(s)| = 6M + 2(1 + a)$.

Theorem 3.4. *If*

$$\mu := \alpha - M_2 B_p C_0^2 > 0, \tag{3.8}$$

then for all $u, v \in \mathcal{U}_{\mathrm{ad}}$,

$$(j'(u) - j'(v), u - v)_Q \geq \mu \|u - v\|_Q^2 . \tag{3.9}$$

The global minimizer is unique, and every solution of (3.7) *is that minimizer.*

Proof. The direct H-forcing estimate in Corollary 2.3, applied to (3.2), gives $\|\xi\|_Q \leq C_0 \|h\|_Q$. Thus (3.6) implies $j''(u)[h, h] \geq (\alpha - M_2 B_p C_0^2) \|h\|_Q^2$. For $h = u - v$, every control $v + \theta h$ is admissible. Integrate this Hessian inequality in $\theta \in [0, 1]$ to get (3.9). A second integration yields $j(v) \geq j(u) + j'(u)(v - u) + (\mu/2) \|v - u\|_Q^2$. Combined with the variational inequality this proves global minimality and uniqueness. □

Remark 3.5. Condition (3.8) is a sufficient, generally conservative condition for global strong convexity. It bounds the possibly negative contribution in (3.6) uniformly over all admissible controls. From (2.6) and (3.4), its threshold is $M_2 B_p C_0^2 = M_2\big(M + \|y_d\|_\infty + Z_* + \|z_d\|_\infty\big) T^2 e^{(K+c)T}$. For fixed model parameters and uniformly bounded data, this threshold decreases as the time horizon is shortened. Larger state or target bounds and longer time horizons can make the condition substantially more restrictive.

Consequently, the control reliability result is most directly applicable when the prescribed regularization is sufficiently strong or the time horizon and data bounds make this threshold sufficiently small. Increasing α solely to satisfy (3.8) changes the balance between tracking accuracy and control cost, and therefore changes the optimization problem. The theorem should not be interpreted as an unconditional control-error bound for small regularization parameters or long time horizons. Moreover, the factor μ^{-1} in (7.5) can make the bound large when μ is positive but close to zero.

Failure of (3.8) does not imply nonexistence, nonuniqueness, or loss of positive definiteness of the actual reduced Hessian. Existence follows independently from Theorem 3.1, and the state and adjoint estimates in Theorems 6.1 and 7.1 remain valid at a fixed admissible control. A local control estimate may instead use (3.9) on a specified convex neighborhood containing the relevant controls. Coercivity only on a critical cone does not, by itself, justify replacing that neighborhood

condition; such a replacement requires a separate local analysis, which is beyond the scope of this paper.

4. A fully discrete scheme and its exact discrete adjoint

Let $0 = t_0 < \cdots < t_N = T$ be a uniform partition with $\tau = T/N$. We use a fixed conforming spatial triangulation, continuous piecewise affine $V_h \subset V$, and continuous piecewise affine $W_h \subset H^1(\Omega)$ with no boundary restriction. In particular $V_h \subset W_h$. The extra conformity of W_h is a discretization choice, not an asserted gradient estimate for the ODE. Let $\mathcal{U}_h$ consist of functions constant on each space-time control cell, with values in $[u_a, u_b]$. Define F^n, y_d^n, z_d^n as time averages on $I_n = (t_{n-1}, t_n)$.

For $U \in \mathcal{U}_h$ and fixed initial approximations $Y^0 \in V_h$, $Z^0 \in W_h$, solve

$$(\delta_t Y^n, v_h) + D(\nabla Y^n, \nabla v_h) + (f(Y^n) + Z^n, v_h) = (F^n + U^n, v_h), \tag{4.1a}$$

$$(\delta_t Z^n + bZ^n - \varepsilon Y^n, w_h) = 0, \tag{4.1b}$$

for all $v_h \in V_h, w_h \in W_h$, with $\delta_t Y^n = (Y^n - Y^{n-1})/\tau$. The objective is

$$j_h(U) = \frac{\tau}{2} \sum_{n=1}^{N} \left(\|Y^n - y_d^n\|^2 + \|Z^n - z_d^n\|^2 + \alpha \|U^n\|^2 \right). \tag{4.2}$$

All spatial integrals are exact in this definition. Quadrature or solver errors can be retained later through the continuous residuals.

Proposition 4.1. *If $\tau\kappa < 1$, the state scheme has a unique solution for each control, and the discrete problem has a minimizer. Its discrete adjoints satisfy $P^{N+1} = Q^{N+1} = 0$ and, for $n = N, \dots, 1$,*

$$((P^n - P^{n+1})/\tau, v_h) + D(\nabla P^n, \nabla v_h) + (f'(Y^n)P^n - \varepsilon Q^n, v_h) = (Y^n - y_d^n, v_h), \tag{4.3a}$$

$$((Q^n - Q^{n+1})/\tau + bQ^n + P^n, w_h) = (Z^n - z_d^n, w_h). \tag{4.3b}$$

Moreover,

$$\tau \sum_n (\alpha U^n + P^n, V^n - U^n) \geq 0 \quad (V \in \mathcal{U}_h). \tag{4.4}$$

On each spatial control cell K, $U^n|_K = \Pi_{[u_a,u_b]}(-|K|^{-1} \int_K P^n/\alpha)$.

Proof. Since $Y^n \in W_h$, the ODE can be solved exactly in that space:

$$Z^n = (Z^{n-1} + \tau\varepsilon Y^n)/(1 + \tau b).$$

After substitution, the left-hand operator for Y^n is the derivative of

$$\frac{1}{2}\left(\tau^{-1} + \frac{\tau\varepsilon}{1+\tau b}\right)\|v\|^2 + \frac{D}{2}\|\nabla v\|^2 + \int_\Omega \mathcal{F}(v) - (\tau^{-1}Y^{n-1} + F^n + U^n - Z^{n-1}/(1+\tau b), v),$$

where $\mathcal{F}' = f$. Its Hessian is bounded below by $(\tau^{-1} - \kappa)\|w\|^2 + D\|\nabla w\|^2$. The quartic leading term also gives coercivity. Finite-dimensional minimization and strict convexity yield a unique state. The invertible Jacobian gives continuously

differentiable dependence on U. The discrete control box is compact, so j_h attains a minimum.

To derive the adjoint, subtract from j_h the sum of the two state residuals paired with $\tau P^n, \tau Q^n$. The coefficient of a variation in Y^n in the time-difference part is $-P^n + P^{n+1}$ for $n < N$, and $-P^N$ at $n = N$. The analogous expression for Z^n is $-Q^n + Q^{n+1}$. The derivative of the reaction is $f'(Y^n)$, at the same time level as the state constraint. The two off-diagonal derivatives are $-\varepsilon Q^n$ and P^n after rearrangement. Dividing the stationarity equations by τ gives (4.3). Differentiation with respect to U^n gives $\alpha U^n + P^n$, proving (4.4). Restricting the variation to one constant control cell yields the cell-average projection formula. The backward linear systems are invertible by the same weighted coercivity and $\tau\kappa < 1$. □

Remark 4.2. The proof uses a uniform time step and fixed spaces. For variable steps the Lagrange multipliers must be scaled consistently; simply reusing (4.3) without rederivation can give incorrect factors. Continuous residual reliability below does not require a particular discrete solver or mesh, but the discrete optimality statement above concerns precisely this scheme.

5. Continuous reconstructions and computable residuals

On I_n define

$$Y(t) = \frac{t_n - t}{\tau} Y^{n-1} + \frac{t - t_{n-1}}{\tau} Y^n, \qquad Z(t) = \frac{t_n - t}{\tau} Z^{n-1} + \frac{t - t_{n-1}}{\tau} Z^n,$$
$$P(t) = \frac{t_n - t}{\tau} P^n + \frac{t - t_{n-1}}{\tau} P^{n+1}, \qquad Q_h(t) = \frac{t_n - t}{\tau} Q^n + \frac{t - t_{n-1}}{\tau} Q^{n+1}.$$

Here Q_h denotes the reconstructed recovery adjoint and is not the space-time domain Q. In particular $P(T) = Q_h(T) = 0$. The control remains $U(t) = U^n$. All four reconstructions are continuous in time with values in H; Y, P are conforming in V. They are bounded for any fixed finite mesh.

The estimates below also apply to any bounded conforming reconstructions with these time regularities and any $U \in \mathcal{U}_{\mathrm{ad}}$. Define

$$\langle R_y, v\rangle = (F + U - Y_t - f(Y) - Z, v) - D(\nabla Y, \nabla v), \tag{5.1a}$$
$$R_z = \varepsilon Y - Z_t - bZ, \tag{5.1b}$$
$$\langle R_p, v\rangle = (Y - y_d + P_t - f'(Y)P + \varepsilon Q_h, v) - D(\nabla P, \nabla v), \tag{5.1c}$$
$$R_q = Z - z_d + (Q_h)_t - bQ_h - P. \tag{5.1d}$$

Set $r_y(t) = \|R_y(t)\|_{V^*}$, $r_p(t) = \|R_p(t)\|_{V^*}$, and use $\|R_z(t)\|$, $\|R_q(t)\|$ in H. The data in these definitions are the original data. Thus data averaging, time reconstruction, spatial approximation and algebraic errors are all retained. No Galerkin orthogonality is assumed in defining the residuals.

5.1. Flux majorants in any stated spatial dimension.

Proposition 5.1. *For $g \in H$, $V_0 \in V$ and $R(v) = (g, v) - D(\nabla V_0, \nabla v)$, every $\sigma \in H(\mathrm{div}; \Omega)$ gives*

$$\|R\|_{V^*} \le \|\sigma - D\nabla V_0\| + C_P \|g + \mathrm{div}\,\sigma\|. \tag{5.2}$$

Proof. The zero trace of v implies $(\sigma, \nabla v) = -(\operatorname{div}\sigma, v)$, with no boundary flux term. Therefore

$$R(v) = (g + \operatorname{div}\sigma, v) + (\sigma - D\nabla V_0, \nabla v).$$

Cauchy–Schwarz and Poincaré give (5.2) after division by $\|v\|_V$ and taking the supremum. □

Apply this formula with the first and third right-hand densities in (5.1). Fluxes from a conforming Raviart–Thomas space are admissible choices. For example, one may minimize $\|\sigma - D\nabla V_0\|^2 + C_P^2 \|g + \operatorname{div}\sigma\|^2$ in that finite-dimensional space and then evaluate the sum in (5.2). Even $\sigma = 0$ is valid, although often inefficient. The estimate requires a conforming flux, not an unqualified broken element gradient. All norms must be integrated exactly or bounded above with a controlled quadrature error for a mathematically certified majorant.

5.2. **An exact dual norm on an interval.**

Proposition 5.2. *If $\Omega = (0, L)$, put $G(x) = \int_0^x g(s)\,\mathrm{d}s$, $H_0 = G + D(V_0)_x$ and $\overline{H}_0 = L^{-1}\int_0^L H_0$. Then*

$$\|R\|_{V^*}^2 = \int_0^L |H_0 - \overline{H}_0|^2\,\mathrm{d}x. \tag{5.3}$$

Proof. Integration by parts gives $R(v) = -\int_0^L H_0 v_x$. Derivatives of $H_0^1(0, L)$ functions are exactly the zero-mean functions in $L^2(0, L)$: the converse follows by integrating a zero-mean function from 0 to x. Hence the operator norm is the norm of the orthogonal projection of H_0 onto that zero-mean subspace. It is $H_0 - \overline{H}_0$. This argument remains valid when $(V_0)_x$ is piecewise constant and has jumps; no jump contribution is discarded. □

6. State reliability at a reconstructed control

Let (y_U, z_U) be the exact state for U. Define

$$A_s = \|y_0 - Y(0)\|^2 + \varepsilon^{-1}\|z_0 - Z(0)\|^2 + \int_0^T \left(D^{-1} r_y^2 + (\varepsilon b)^{-1}\|R_z\|^2\right)\,\mathrm{d}t. \tag{6.1}$$

Throughout the estimates, dual norms may be replaced by the flux majorants in Section 5.

Theorem 6.1. *For any admissible reconstructed control and bounded conforming states,*

$$\|(y_U - Y, z_U - Z)\|_{\mathcal{E}_s} \le \sqrt{C_E A_s}, \tag{6.2}$$

$$\|y_U - Y\|_Q^2 \le T E_T A_s, \qquad \|z_U - Z\|_Q^2 \le \varepsilon T E_T A_s.$$

The constants are independent of the mesh and time step.

Proof. Set $e = y_U - Y$, $d = z_U - Z$. Subtract the reconstruction identity from the exact equation to obtain

$$e_t - D\Delta e + f(y_U) - f(Y) + d = R_y, \qquad d_t + bd - \varepsilon e = R_z.$$

The residual signs follow directly from (5.1). The coefficient $A(x,t) = \int_0^1 f'(Y + \theta e)\,\mathrm{d}\theta$ is bounded for the fixed reconstruction and satisfies $A \geq -\kappa$, regardless of its amplitude. Thus the system is precisely the forward system in Lemma 2.2. Its initial energy and forcing expression are (6.1). Applying that lemma proves all assertions. No upper bound for f' enters the reliability constant in this step. □

Remark 6.2. The recovery residual has no factor of the spatial mesh size. The energy norm controls z in H, not in V. A simple example shows why that residual cannot just be omitted. Take $F = 0$, $u_a \leq 1 \leq u_b$, $y_0 = 0$, $z_0 = 1$, and reconstructions $U = 1$, $Y = 0$, $Z = 1$. The initial errors and parabolic residual are zero, but $R_z = -b \neq 0$. The exact state cannot equal these reconstructions because the ODE would read $b = 0$. An estimator retaining only the initial errors and the parabolic residual would incorrectly report zero error.

7. Adjoint and optimal-control reliability

Let (p_U, q_U) be the exact adjoint associated with (y_U, z_U). Put

$$M_R = \max\{M, \|Y\|_{L^\infty(Q)}\}, \qquad L_R = 6M_R + 2(1+a), \qquad B_R = 1 + L_R \|P\|_{L^\infty(Q)}.$$

These are known quantities. Define

$$A_{a,0} = \|P(T)\|^2 + \varepsilon \|Q_h(T)\|^2 + \int_0^T \left(2D^{-1} r_p^2 + 2\varepsilon b^{-1} \|R_q\|^2\right) \mathrm{d}t, \tag{7.1}$$

$$A_a = A_{a,0} + 2TE_T \left(C_P^2 D^{-1} B_R^2 + \varepsilon^2 / b\right) A_s. \tag{7.2}$$

The terminal terms are zero for the stated discrete reconstruction, but are retained for general approximate adjoints.

Theorem 7.1. *The adjoint error at U satisfies*

$$\|(p_U - P, q_U - Q_h)\|_{\mathcal{E}_a} \leq \sqrt{C_E A_a}, \qquad \|p_U - P\|_Q \leq \sqrt{TE_T A_a}. \tag{7.3}$$

Proof. Set $e_p = p_U - P$, $e_q = q_U - Q_h$, $e_y = y_U - Y$, $e_z = z_U - Z$. Subtraction gives

$$-(e_p)_t - D\Delta e_p + f'(y_U)e_p - \varepsilon e_q = R_p + e_y - [f'(y_U) - f'(Y)]P,$$
$$-(e_q)_t + be_q + e_p = R_q + e_z.$$

Both the tracking difference e_y and the coefficient difference are essential. Since $|y_U| \leq M$ and $|Y| \leq M_R$, the mean value theorem implies

$$\|e_y - [f'(y_U) - f'(Y)]P\|_{V^*} \leq C_P B_R \|e_y\|.$$

Thus the squared dual norm of the first forcing is at most $2r_p^2 + 2C_P^2 B_R^2 \|e_y\|^2$. The squared H norm of the second is at most $2\|R_q\|^2 + 2\|e_z\|^2$. Apply the backward part of Lemma 2.2, using $f'(y_U) \geq -\kappa$. Insert the state bounds from Theorem 6.1. The resulting expression is exactly (7.2). The same lemma gives the $L^2(Q)$ estimate. □

7.1. **A normal-cone residual for the control.** For $G_U = \alpha U + P$, define the pointwise function

$$\rho_U = \begin{cases} \min\{G_U, 0\}, & U = u_a, \\ G_U, & u_a < U < u_b, \\ \max\{G_U, 0\}, & U = u_b. \end{cases} \qquad \eta_u = \|\rho_U\|_Q . \tag{7.4}$$

All equalities and inequalities defining these sets are almost-everywhere statements. For an implemented box projection, active values can be stored exactly at their bound. Numerical tolerances must not silently redefine these sets when a rigorous bound is sought.

Lemma 7.2. *For every $v \in \mathcal{U}_{\mathrm{ad}}$,*

$$(G_U, U - v)_Q \le (\rho_U, U - v)_Q \le \eta_u \|U - v\|_Q .$$

Proof. On the lower active set, $G_U - \rho_U = \max(G_U, 0) \ge 0$ and $U - v \le 0$. On the upper active set, $G_U - \rho_U = \min(G_U, 0) \le 0$ and $U - v \ge 0$. On the inactive set the difference is zero. Integrating gives the first inequality. Cauchy–Schwarz gives the second. Notice that a nonzero gradient on an active set can be compatible with exact optimality; this is why its unrestricted norm is not used. □

Theorem 7.3. *Assume that the sufficient strong-convexity condition (3.8) holds, and let $(\bar y, \bar z, \bar p, \bar q, \bar u)$ denote the unique optimality system. For any bounded conforming reconstruction with $U \in \mathcal{U}_{\mathrm{ad}}$, set*

$$B_u = \mu^{-1} \left(\eta_u + \sqrt{T E_T A_a} \right), \tag{7.5}$$

$$K_a = \sqrt{C_E}\, C_0 \left(C_P^2 D^{-1} (1 + M_2 B_p)^2 + \varepsilon^2 / b \right)^{1/2} .$$

Then

$$\|\bar u - U\|_Q \le B_u, \tag{7.6}$$

$$\|(\bar y - Y, \bar z - Z)\|_{\mathcal{E}_s} \le \sqrt{C_E A_s} + \sqrt{C_E} B_u, \tag{7.7}$$

$$\|(\bar p - P, \bar q - Q_h)\|_{\mathcal{E}_a} \le \sqrt{C_E A_a} + K_a B_u. \tag{7.8}$$

Every quantity on the right can be bounded using the data and the reconstructions. In particular, the theorem applies to the fully discrete scheme of Section 4.

Proof. Let $e_u = U - \bar u$. Strong monotonicity and the continuous variational inequality give

$$\begin{aligned} \mu \|e_u\|_Q^2 &\le (j'(U) - j'(\bar u), e_u)_Q \le (j'(U), e_u)_Q \\ &= (\alpha U + P, e_u)_Q + (p_U - P, e_u)_Q \\ &\le (\eta_u + \|p_U - P\|_Q) \|e_u\|_Q . \end{aligned}$$

If $e_u = 0$ the claim is immediate; otherwise divide and apply Theorem 7.1. This proves (7.6). The triangle inequality in $\mathcal{E}_s$, followed by Corollary 2.3 and Theorem 6.1, proves (7.7).

For the last assertion compare exact adjoints at two controls u_1, u_2. Their difference solves the backward system with coefficient $f'(y_1)$ and forcing

$$(y_1 - y_2) - [f'(y_1) - f'(y_2)]p_2, \qquad z_1 - z_2.$$

Both states lie in $[-M, M]$, and $\|p_2\|_\infty \le B_p$. The first forcing therefore has dual norm at most $C_P(1 + M_2 B_p)\,\|y_1 - y_2\|$. Lemma 2.2 with zero terminal data and Corollary 2.3 imply

$$\|(p_1 - p_2, q_1 - q_2)\|_{\mathcal{E}_a} \le K_a \,\|u_1 - u_2\|_Q \,.$$

Apply this with $u_1 = \bar{u}, u_2 = U$ and add the reconstruction error (7.3). This gives (7.8). □

Remark 7.4. The state reliability constant is independent of reconstruction amplitudes. The adjoint majorant explicitly contains $\|Y\|_\infty$ and $\|P\|_\infty$. For a family of meshes, a single uniform adjoint constant requires uniform bounds for these amplitudes; no discrete maximum principle has been assumed. The factors involving $D^{-1}, \varepsilon^{-1}, b^{-1}$ and μ^{-1} also show that the result is not uniform in singular parameter limits.

8. Memory interpretation and a manufactured benchmark

8.1. **Exact elimination and the recovery defect.** By (2.5), the excitation equation is equivalent to

$$y_t - D\Delta y + f(y) + \varepsilon \int_0^t e^{-b(t-s)} y(s)\,\mathrm{d}s = F + u - e^{-bt} z_0. \tag{8.1}$$

Conversely, a solution of (8.1), together with the expression for z in (2.5), satisfies both equations in (2.3). For the adjoint,

$$q(t) = \int_t^T e^{-b(s-t)} \big(z(s) - z_d(s) - p(s)\big)\,\mathrm{d}s. \tag{8.2}$$

Substitution into the p equation gives the backward exponential-memory term

$$+\varepsilon \int_t^T e^{-b(s-t)} p(s)\,\mathrm{d}s$$

on the left-hand side and

$$y - y_d + \varepsilon \int_t^T e^{-b(s-t)} (z(s) - z_d(s))\,\mathrm{d}s$$

on the right-hand side. In particular, tracking z leaves a genuine additional term in the reduced adjoint.

Proposition 8.1. *For any reconstructed Y, Z, define $\mathcal{Z}[Y](t) = e^{-bt} z_0 + \varepsilon \int_0^t e^{-b(t-s)} Y(s)\,\mathrm{d}s$. Then*

$$\mathcal{Z}[Y](t) - Z(t) = e^{-bt}(z_0 - Z(0)) + \int_0^t e^{-b(t-s)} R_z(s)\,\mathrm{d}s, \tag{8.3}$$

and

$$\|\mathcal{Z}[Y] - Z\|_Q \le \left(\frac{1 - e^{-2bT}}{2b}\right)^{1/2} \|z_0 - Z(0)\| + \frac{1 - e^{-bT}}{b}\,\|R_z\|_Q \,.$$

Proof. Differentiate $d = \mathcal{Z}[Y] - Z$. The defining formula for $\mathcal{Z}[Y]$ gives $d_t + bd = \varepsilon Y - Z_t - bZ = R_z$, with the stated initial value. The integrating factor yields (8.3). The first term has the displayed exact time norm; Young's inequality for convolution bounds the second using $\left\|e^{-bt}\right\|_{L^1(0,T)} = (1 - e^{-bT})/b$. □

This identity explains what the recovery residual measures in the scalar-memory formulation. A history approximation or an inexact update of Z is admissible in the reliability theorem, provided its full R_z is evaluated. The formula does not extend the theory to general or weakly singular memory kernels.

8.2. **A nonzero manufactured optimality system.** Take $\Omega = (0,1)$, $T = 0.2$, $D = 0.1$, $a = 1/4$, $\varepsilon = b = 1$, $\alpha = 10$ and $[u_a, u_b] = [-1, 1]$. Let $A = 0.1$, $B = 0.02$, and $s(x) = \sin(\pi x)$. Define

$$\begin{aligned} y_*(x,t) &= A(1 - e^{-t})s(x), \qquad & z_*(x,t) &= A(1 - e^{-t} - te^{-t})s(x), \\ p_*(x,t) &= B(T - t)s(x), & q_* &= 0, \qquad u_* = -p_*/\alpha. \end{aligned} \tag{8.4}$$

Choose the data

$$\begin{aligned} F &= (y_*)_t - D(y_*)_{xx} + f(y_*) + z_* - u_*, \\ y_d &= y_* + (p_*)_t + D(p_*)_{xx} - f'(y_*)p_*, \\ z_d &= z_* - p_*, \qquad y_0 = z_0 = 0. \end{aligned} \tag{8.5}$$

The sign in z_d follows from the adjoint equation $p_* = z_* - z_d$.

Proposition 8.2. *The functions* (8.4) *form the unique global optimality system for the data* (8.5).

Proof. Differentiation gives $(z_*)_t + z_* = y_*$. The definition of F proves the parabolic equation. Both the boundary conditions and the initial and terminal conditions hold. The definitions of y_d, z_d prove both adjoint equations. Since $|u_*| \le BT/\alpha = 0.0004 < 1$, the projection equation reduces to $\alpha u_* + p_* = 0$.

It remains to verify global optimality rather than merely stationarity. Elementary bounds give $|y_*| \le 0.02$, $|z_*| \le 0.002$, $|p_*| \le 0.004$, $|(y_*)_t| \le 0.1$ and $|(p_*)_t| \le 0.02$. On $[0, 0.02]$, $|f(y_*)| \le 0.005$ and $|f'(y_*)| \le 0.25$. Using $\pi^2 < 10$ yields

$$\|F\|_\infty \le 0.128, \qquad \|y_d\|_\infty \le 0.045, \qquad \|z_d\|_\infty \le 0.006.$$

For all admissible controls, choose $M = Z_* = 2$. Since $f(2) = 3.5 > 3.128$ and $f(-2) = -13.5 < -3.128$, (2.4) holds. Here $\kappa = 13/48$, $M_2 = 14.5$, $C_0^2 = 0.2e^{(37/24)0.2}$, and

$$B_p \le 0.2e^{(109/48)0.2}(4.051).$$

These values give $M_2 B_p C_0^2 < 5.1$, so $\mu > 4.9$. Theorem 3.4 turns the verified variational inequality into unique global optimality. □

8.3. **Reproducible reconstruction check.** A companion script evaluates the residuals on interpolants of (8.4). It uses continuous piecewise affine interpolation in space and time for Y, Z, P, sets $Q_h = 0$, and takes cell averages of u_* for U. The primitive formula (5.3) evaluates the parabolic dual norms.

This procedure checks the reconstruction identities and evaluates nonzero residuals without solving the discrete nonlinear optimization problem. In particular, the interpolated states and adjoints are not asserted to satisfy (4.1)–(4.4). The calculation is therefore a consistency check of the estimator, rather than a numerical assessment of the fully discrete optimization scheme or an AMR algorithm.

TABLE 1. Manufactured reconstruction check; $N_t = N_x = n$. These are interpolants, not optimized discrete solutions.

n	$\sqrt{A_s}$	$\|u_* - U\|_Q$	B_u	$B_u/\|u_* - U\|_Q$
8	6.348e-04	9.421e-06	9.246e-05	9.81
16	2.604e-04	4.723e-06	3.972e-05	8.41
32	1.219e-04	2.363e-06	1.892e-05	8.01
64	5.989e-05	1.182e-06	9.339e-06	7.90

The benchmark reports the exact-control error $\|u_* - U\|_Q$, the control majorant B_u from Theorem 7.3, and the ratio $B_u/\|u_* - U\|_Q$. This ratio indicates the sharpness of the bound for the chosen interpolants; it does not establish local efficiency or the computational benefit of adaptive refinement.

Residual integrations use high-order Gaussian quadrature; the reported numbers are numerical evaluations, not interval-arithmetic certificates. The sufficient-convexity test above is analytic and does not depend on these integrations. The state and adjoint majorants apply to conforming reconstructions of solutions of the finite-element optimality system under the stated assumptions; the combined control estimate additionally requires (3.8).

A practical assessment of estimator-driven AMR would require solving the discrete optimality system on successive meshes, specifying and implementing a marking and refinement strategy, and comparing adaptive and uniform refinement at comparable computational cost. Errors, estimator values, numbers of degrees of freedom, and solver work would then need to be reported. Such an experiment is not included here, so the present benchmark makes no claim about adaptive efficiency, convergence rates, or computational savings.

9. Conclusion

The coupled FitzHugh–Nagumo control problem admits residual reliability estimates that distinguish the diffusive excitation from the nondiffusive recovery variable. A one-sided bound for the cubic and two different energy weights give state and adjoint estimates with explicit constants. The control estimate follows from a normal-cone residual under an explicit sufficient strong-convexity condition. This condition can be restrictive for small regularization parameters, large data bounds, or long time horizons; the state and adjoint residual estimates

remain available independently of it. The fully discrete adjoint is the exact derivative of the specified discrete objective and constraints.

Elimination of the recovery variable connects its residual with an exponential-memory defect, while retaining a direct parabolic–ODE implementation. The present results establish upper bounds for fixed positive parameters. Local efficiency, adaptive convergence, less restrictive local second-order hypotheses and parameter-robust estimates remain separate problems. The manufactured benchmark checks the consistency of the reconstructions and estimator. It does not replace experiments with computed finite-element optimal controls, and the practical effectiveness of estimator-driven AMR remains to be assessed.

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