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SUPER QUASI-EINSTEIN WARPED PRODUCTS WITH AFFINE CONNECTIONS

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ABSTRACT. In this paper, we study super quasi-Einstein warped product spaces with semi-symmetric non-metric connection and quarter symmetric connection. Then we give the expressions of the Ricci tensors and scalar curvatures for the bases and fibers with these affine connections. Next we find the obstructions to the existence of super quasi-Einstein warped products with respect to affine connections. In the last section, we obtain an example of super quasi-Einstein space time.

1. Introduction

A Riemannian manifold $(M^n,\ g)$ with dimension $(n \geq 2)$ is said to be an Einstein manifold if it satisfies the condition $S(X,\ Y) = \frac{r}{n}g(X,\ Y)$, holds on M, where S and r denote the Ricci tensor and the scalar curvature of $(M^n,\ g)$ respectively. According to [1] the above equation is called the Einstein metric condition. Einstein manifolds play an important role in Riemannian Geometry, as well as in general theory of relativity.

The notion of quasi-Einstien manifold was defined in [7], [5]. A non-flat Riemannian manifold $(M^n,\ g)$, $(n \geq 2)$ is said to be a quasi Einstein manifold if the condition

$$S(X,Y) = \alpha g(X,Y) + \beta\rho(X)\rho(Y), \tag{1.1}$$

is fulfilled on M, where α and β are scalar functions of which $\beta \neq 0$ and ρ is non-zero 1-form such that $g(X,\xi) = \rho(X)$ for all $X \in \chi(M)$ and ξ is a unit vector field.

In [4], M. C. Chaki introduced super quasi-Einstein manifold, denoted by $S(QE)_n$ and gave an example of a 4-dimensional semi Riemannian super quasi-Einstein manifold, where the Ricci tensor S of type $(0,2)$ which is not identically zero satisfies the condition

$$S(X,Y) = \alpha g(X,\ Y) + \beta A(X)A(Y) + \gamma[A(X)B(Y) + A(Y)B(X)] + \delta D(X,Y), \tag{1.2}$$

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where $\alpha, \beta, \gamma, \delta$ are scalar functions such that β, γ, δ are nonzero and A, B are two nonzero 1-forms such that $g(X, \xi_1) = A(X)$ and $g(X, \xi_2) = B(X)$, ξ_1, ξ_2 being unit vectors which are orthogonal, i.e., $g(\xi_1, \xi_2) = 0$ and D is symmetric $(0, 2)$ tensor with zero trace which satisfies the condition $D(X, \xi_1) = 0, \ \forall \ X \in \chi(M)$.

Here $\alpha, \beta, \gamma, \delta$ are called the associated scalars, and A, B are called the associated main and auxiliary 1-forms respectively, ξ_1, ξ_2 are main and auxiliary generators and D is called the associated tensor of the manifold.

In [6] P. Debnath and A. Konar proved that 4-dimensional Lorentzian $S(QE)_4$ gives a general relativistic viscous fluid spacetime admitting heat flux and also they deduced the bounds of the cosmological constant in a viscous fluid super quasi Einstein spacetime. Many authors like C. Özgür [16] etc. have also studied super quasi Einstein manifolds.

The notion of warped product generalizes that of a surface of revolution. Warped products play important roles in the general theory of relativity [13]. It was introduced in [3] for studying manifolds of negative curvature. Let (B, g_B) and (F, g_F) be two Riemannian manifolds with dim $B = m > 0$, dim$F = k > 0$ and $f : B \longrightarrow (0, \infty)$, $f \in C^\infty(B)$. Consider the product manifold $B \times F$ with its projections $\pi : B \times F \to B$ and $\sigma : B \times F \to F$. The warped product $B \times_f F$ is the manifold $B \times F$ with the Riemannian structure such that $||X||^2 = ||\pi^*(X)||^2 + f^2(\pi(p))||\sigma^*(X)||^2$, for any vector field X on M. Thus we have $g_M = g_B + f^2 g_F$ holds on M. Here B is called the base of M and F the fiber. The function f is called the warping function of the warped product [13].

The idea of metric connection with torsion on Riemannian manifold was given by Hayden (1932) in [12]. In 1970, K. Yano [21] considered a semi-symmetric metric connection and studied some of its properties. Then in 1975, Golab [11] introduced the idea of a quarter-symmetric linear connection on differentiable manifold which is a generalization of semi-symmetric connection. Later in [18], Q. Qu and Y.Wang have generalized the results to warped product and multiply warped product with a special quarter-symmetric connection.

In [8] D. Dumitru has given a characterization of warped product on quasi-Einstein manifold. B. Pal, A. Bhattacharyya [17], Q. Qu [19] have also studied a characterization of warped product on different manifolds. S. Pahan, B. Pal, A. Bhattacharyya deal with super quasi-Einstein warped product spaces in [14]

In this paper I have studied super quasi-Einstein warped product spaces with respect to affine connections i.e. semi-symmetric non-metric connection and quarter-symmetric connection. I have computed the expressions of the Ricci tensors and scalar curvatures for the bases and fibers with respect to affine connections. In some cases I have investigated some obstructions to the existence of such manifolds with these affine connections and have given an example of super quasi-Einstein space time.

2. **Preliminaries:**

Let (M^n, g) be a Riemannian manifold with the Levi-civita connection ∇. A linear connection $\tilde{\nabla}$ on (M^n, g) is said to be semi-symmetric if its torsion tensor T can be written as

$$T(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y],$$

satisfies the condition

$$T(X,Y) = \pi(Y)X - \pi(X)Y,$$

where π is an $1-$ form on M^n with the associated vector field P defined by $\pi(X) = g(X,P)$, for all vector fields $X \in \chi(M^n)$.

A connection $\tilde{\nabla}$ is called semi-symmetric non-metric connection if $\tilde{\nabla} g \neq 0$.

The relation between semi-symmetric non-metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ of M^n and it is given by [20]

$$\tilde{\nabla}_X Y = \nabla_X Y + \pi(Y)X - g(X,Y)P, \tag{2.1}$$

where $g(X,P) = \pi(X)$.

It is easy to seen that

$$\tilde{\nabla}_X g(Y,Z) = -\pi(Y)g(X,Z) - \pi(Z)g(X,Z),$$

for all vector fields $X, Y, Z \in \chi(M^n)$.

Further, a relation between the curvature tensors R and $\tilde{R}$ of type (1,3) of the connections ∇ and $\tilde{\nabla}$ respectively is given by [20],

$$\begin{aligned} \tilde{R}(X,Y)Z = R(X,Y)Z + g(Z,\nabla_X P)Y - g(Z,\nabla_Y P)X, \\ +\pi(Z)[\pi(Y)X - \pi(X)Y], \end{aligned} \tag{2.2}$$

for any vector field X, Y, Z on M^n.

A linear connection $\breve{\nabla}$ on (M^n, g) is said to be quarter-symmetric connection if its torsion tensor T of the connection $\breve{\nabla}$ can be written as

$$T(X,Y) = \breve{\nabla}_X Y - \breve{\nabla}_Y X - [X,Y],$$

satisfies

$$T(X,Y) = \pi_2(Y)\phi X - \pi_2(X)\phi Y,$$

where π_2 is an $1-$ form on M^n with the associated vector field L defined by $\pi_2(X) = g(X, L)$, for all vector fields $X \in \chi(M^n)$.

A quarter-symmetric connection $\breve{\nabla}$ is called quarter-symmetric metric connection if $\breve{\nabla} g = 0$.

It is obvious that if $\phi X = X$ then the quarter-symmetric connection will reduce to a semi-symmetric connection. Thus, the quarter-symmetric connection generalizes the semi-symmetric connection which plays an important role in the geometry of Riemannian manifolds having a physical application i.e. the displacement on the earth surface facing one definite point is metric and semi-symmetric [10].

The relation between quarter-symmetric metric connection $\breve{\nabla}$ and the Levi-Civita connection ∇ of M^n and it is given by [15]

$$\breve{\nabla}_X Y = \nabla_X Y + \lambda_1 \pi_2(Y) X - \lambda_2 g(X, Y) L, \tag{2.3}$$

where $g(X, L) = \pi_2(X)$ and $\lambda_1 \neq 0, \lambda_2 \neq 0$ are scalar functions.

We can easily see that:

when $\lambda_1 = \lambda_2 = 1$, $\breve{\nabla}$ is a semi-symmetric metric connection;
when $\lambda_1 = \lambda_2 \neq 1$, $\breve{\nabla}$ is a quarter-symmetric metric connection;
when $\lambda_1 \neq \lambda_2$, $\breve{\nabla}$ is a quarter-symmetric non-metric connection;

Further, a relation between the curvature tensors R and $\breve{R}$ of type (1,3) of the connections ∇ and $\breve{\nabla}$ respectively is given by [15],

$$\begin{aligned} \breve{R}(X,Y)Z = R(X,Y)Z &+ \lambda_1 g(Z, \nabla_X P)Y - \lambda_2 g(Z, \nabla_Y P)X, \\ &+\lambda_2[g(X,Z)\nabla_Y P - g(Y,Z)\nabla_X P] \\ &+\lambda_1\lambda_2\pi_2(P)[g(X,Z)Y - g(Y,Z)X] \\ &+\lambda_2^2[g(Y,Z)\pi_2(X) - g(X,Z)\pi_2(Y)]P \\ &+\lambda_1^2\pi_2(Z)[\pi_2(Y)X - \pi_2(X)Y], \end{aligned} \tag{2.4}$$

for any vector field X, Y, Z on M.

3. Warped Product Manifolds with Semi-Symmetric Non-Metric Connection

In this section we have considered warped product manifolds with semi-symmetric non-metric connection and given expressions of curvature tensor, Ricci tensor and the scalar curvature admitting this connection where the associated vector field

$P \in \chi(B)$ and $P \in \chi(F)$.
Then, we have the following Propositions [20].

Proposition 3.1. *Let $M = B \times_f F$ be a warped product, $R, \tilde{R}$ be the Riemannian curvature tensors of M with respect to Levi-Civita connection and the semi-symmetric non-metric connection respectively. If $X, Y, Z \in \chi(B)$, $U, V, W \in \chi(F)$ and $P \in \chi(B)$, then*

(i) $\tilde{R}(X,Y)Z$ *is the lift of* $\tilde{R}^B(X,Y)Z$ *on* B,
(ii) $\tilde{R}(V,X)Y = -[\frac{H^f(X,Y)}{f} + g(Y, \nabla_X P) - \pi(X)\pi(Y)]V$,
(iii) $\tilde{R}(X,Y)V = 0$,
(iv) $\tilde{R}(V,W)X = 0$,
(v) $\tilde{R}(X,V)W = -g(V,W)[\frac{\nabla_X gradf}{f} + \frac{Pf}{f}X]$,
(vi) $\tilde{R}(U,V)W = R^F(U,V)W - \{\frac{|gradf|^2}{f^2} + \frac{Pf}{f}\}[g(V,W)U - g(U,W)V]$.

Proposition 3.2. *Let $M = B \times_f F$ be a warped product, $R, \tilde{R}$ be the Riemannian curvature tensors of M with respect to Levi-Civita connection and the semi-symmetric metric non-connection respectively. If $X, Y, Z \in \chi(B)$, $U, V, W \in \chi(F)$ and $P \in \chi(F)$, then*

(i) $\tilde{R}(X,Y)Z = R^B(X,Y)Z$,
(ii) $\tilde{R}(V,X)Y = -\frac{H^f(X,Y)}{f}]V - \pi(V)\frac{Yf}{f}X$,
(iii) $\tilde{R}(X,Y)V = \pi(V)[\frac{Xf}{f}Y - \frac{Yf}{f}X]$,
(iv) $\tilde{R}(V,W)X = \frac{Xf}{f}[\pi(W)V - \pi(V)W]$,
(v) $\tilde{R}(X,V)W = -g(V,W)\frac{(\nabla_X gradf)}{f} - g(W,\nabla_V P)X + \pi(V)\pi(W)X + \frac{Xf}{f}\pi(W)V$,
(vi) $\tilde{R}(U,V)W = R^F(U,V)W + g(W,\nabla_U P)V - g(W,\nabla_V P)U + \pi(W)[\pi(V)W - \pi(U)V] - \frac{|gradf|^2}{f^2}[g(V,W)U - g(U,W)V]$.

By Proposition 3.1 and Proposition 3.2, by a contraction of the curvature tensors we obtain the Ricci tensors of warped product with respect to semi-symmetric metric connection as follows.

Proposition 3.3. *Let $M = B \times_f F$ be a warped product, S and $\tilde{S}$ denote the Ricci tensors of M with respect to the Levi-Civita connection and the semi-symmetric non-metric connection respectively where $dimB = n_1$ and $dimF = n_2$. If $X, Y \in \chi(B)$, $V, W \in \chi(F)$ and $P \in \chi(B)$, then*

(i) $\tilde{S}(X,Y) = \tilde{S}^B(X,Y) + n_2[\frac{H^f(X,Y)}{f} + g(Y,\nabla_X P) - \pi(X)\pi(Y)]$,
(ii) $\tilde{S}(X,V) = \tilde{S}(V,X) = 0$,
(iii) $\tilde{S}(V,W) = \tilde{S}^F(V,W) + [(n_2-1)\frac{|gradf|^2}{f^2} + (n_1+n_2-1)\frac{Pf}{f} - \frac{\Delta f}{f}]g(V,W)$.

Proposition 3.4. *Let $M = B \times_f F$ be a warped product, S and $\tilde{S}$ denote the Ricci tensors of M with respect to the Levi-Civita connection and the semi-symmetric non-metric connection respectively where $dimB = n_1$ and $dimF = n_2$. If $X, Y \in \chi(B)$, $V, W \in \chi(F)$ and $P \in \chi(F)$, then*

(i) $\tilde{S}(X,Y) = S^B(X,Y) + n_2 \frac{H^f(X,Y)}{f}$,
(ii) $\tilde{S}(X,V) = (1-\bar{n})\pi(V)\frac{Xf}{f}$,
(iii) $\tilde{S}(V,X) = (\bar{n}-1)\pi(V)\frac{Xf}{f}$,
(iv) $\tilde{S}(V,W) = S^F(V,W) + g(V,W)[(n_2-1)\frac{|gradf|^2}{|f|^2} - \frac{\Delta f}{f} g(V,W) + (\bar{n}-1)g(W, \nabla_V P)$ $-(\bar{n}-1)\pi(V)\pi(W)$.

By Proposition 3.3 and Proposition 3.4 by a contraction of the Ricci tensors we obtain the scalar curvatures of warped product with respect to semi-symmetric metric connection as follows.

Proposition 3.5. *Let $M = B \times F$ be a warped product, r and $\tilde{r}$ denote the scalar curvature of M with respect to the Levi-Civita connection and the semi-symmetric non-metric connection respectively and $P \in \chi(B)$, then*

$$\tilde{r} = \tilde{r}^B + \frac{r^F}{f^2} + n_2(n_2-1)\frac{|gradf|^2}{f^2} + n_2(n-1)\frac{Pf}{f}$$
$$-2n_2\frac{\Delta f}{f} - n_2\pi(P) - n_2 divP.$$

Proposition 3.6. *Let $M = B \times F$ be a warped product, r and $\tilde{r}$ denote the scalar curvature of M with respect to the Levi-Civita connection and the semi-symmetric non-metric connection respectively and $P \in \chi(F)$, then*

$$\tilde{r} = r^B + \frac{r^F}{f^2} + (\bar{n}-1)divP - (\bar{n}-1)\pi(P)$$
$$+n_2(n_2-1)\frac{|gradf|^2}{f^2} - 2n_2\frac{\Delta f}{f}.$$

4. Super Quasi Einstein Warped Products with Affine Connections

In this section, we have given Ricci curvature and scalar curvature of super quasi-Einstein warped products with a semi-symmetric non metric connection.

Theorem 4.1. *Let $M = B \times_f F$ be a super quasi-Einstein warped product space, dim $B = n_1$, dim $F = n_2$, dim $M = \bar{n} = n_1 + n_2$, $X, Y \in \Gamma(TB)$, $V, W \in \Gamma(TF)$, $P \in \Gamma(TB)$ Then the following conditions hold.*

i) When $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$ *then*

$$\begin{cases} \tilde{S}_B(X,Y) &= \alpha g_B(X,Y) + \beta g_B(X,\xi_1) g_B(Y,\xi_1) + \delta D_B(X,Y) \\ &+ \gamma[g_B(X,\xi_1)g_B(Y,\xi_2) + g_B(X,\xi_2)g_B(Y,\xi_1)] - n_2[\frac{H^f(X,Y)}{f} \\ &+ g(Y,\nabla_X P) - \pi(X)\pi(Y)] \\ \tilde{S}_F(V,W) &= g_F(V,W)[\alpha f^2 + f\Delta f + (1-n_2)|gradf|^2 + (1-\bar{n})fPf] \\ &+ \delta D_F(V,W) \end{cases} \tag{4.1}$$

ii) When $\xi_1 \in \Gamma(TF), \xi_2 \in \Gamma(TF)$ *then*

$$\begin{cases} \tilde{S}_B(X,Y) &= \alpha g_B(X,Y) + \delta D_B(X,Y) - n_2[\frac{H^f(X,Y)}{f} + g(Y,\nabla_X P) - \pi(X)\pi(Y)] \\ \tilde{S}_F(V,W) &= g_F(V,W)[\alpha f^2 + f\Delta f + (1-n_2)|gradf|^2 + (1-\bar{n})fPf] \\ &+ \delta D_F(V,W) + \beta f^4 g_B(V,\xi_1)g_B(W,\xi_1) + \gamma f^4[g_B(V,\xi_1)g_B(W,\xi_2) \\ &+ g_B(V,\xi_2)g_B(W,\xi_1)] \end{cases} \tag{4.2}$$

iii) When $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TF)$ *then*

$$\begin{cases} \tilde{S}_B(X,Y) &= \alpha g_B(X,Y) + \beta g_B(X,\xi_1)g_B(Y,\xi_1) + \delta D_B(X,Y) \\ &- n_2[\frac{H^f(X,Y)}{f} + g(Y,\nabla_X P) - \pi(X)\pi(Y)] \\ \tilde{S}_F(V,W) &= g_F(V,W)[\alpha f^2 + f\Delta f + (1-n_2)|gradf|^2 + (1-\bar{n})fPf] \\ &+ \delta D_F(V,W) \end{cases} \tag{4.3}$$

Proof. i) Since, M is a super quasi-Einstein manifold and For X, Y on $\Gamma(TB)$, we have $\tilde{S}_M(X,Y) = \alpha g_B(X,Y) + \beta g_B(X,\xi_1)g_B(X,\xi_1) + \gamma[g_B(X,\xi_1)g_B(X,\xi_2) + g_B(Y,\xi_1)g_B(X,\xi_2)] + \delta D_B(X,Y)$. From Proposition 3.4 we get the equation $\tilde{S}(X,Y) = \tilde{S}^B(X,Y) + n_2[\frac{H^f(X,Y)}{f} + g(Y,\nabla_X P) - \pi(X)\pi(Y)]$. By applying the above two equations we get the result of equation (4.1) (i).
Since, M is a super quasi-Einstein manifold and For V, W on $\Gamma(TF)$, we have $\tilde{S}_M(V,W) = f^2\alpha g_F(V,W) + \delta D_F(V,W)$. From Proposition 3.4 we get the equation $\tilde{S}(V,W) = \tilde{S}^F(V,W) + [(n_2-1)\frac{|gradf|^2}{f^2} + (\bar{n}-1)\frac{Pf}{f} - \frac{\Delta f}{f}]f^2 g(V,W)$. By applying the above two equations we get the second result of equation (4.1) (i).

ii) Since, M is a super quasi-Einstein manifold and For X, Y on $\Gamma(TB)$, we have $\tilde{S}_M(X,Y) = \alpha g_B(X,Y) + \delta D_B(X,Y)$. From Proposition 3.4 we get the equation $\tilde{S}(X,Y) = \tilde{S}^B(X,Y) + n_2[\frac{H^f(X,Y)}{f} + g(Y,\nabla_X P) - \pi(X)\pi(Y)]$. By applying the above two equations we get the result of equation (4.1) (ii).
Since, M is a super quasi-Einstein manifold and For V, W on $\Gamma(TF)$, we have $\tilde{S}_M(V,W) = f^2\alpha g_F(V,W) + f^4\beta g_F(V,\xi_1)g_F(W,\xi_1) + f^4\gamma[g_F(V,\xi_1)g_F(W,\xi_2) + g_F(W,\xi_1)g_F(V,\xi_2)] + \delta D_F(X,Y)$. From Proposition 3.4 we get the equation $\tilde{S}(V,W) = \tilde{S}^F(V,W) + [(n_2-1)\frac{|gradf|^2}{f^2} + (\bar{n}-1)\frac{Pf}{f} - \frac{\Delta f}{f}]f^2 g(V,W)$. By applying the above two equations we get the second result of equation (4.1) (ii).

iii) Since, M is a super quasi-Einstein manifold and For X, Y on $\Gamma(TB)$, we have $\tilde{S}_M(X,Y) = \alpha g_B(X,Y) + \beta g_B(X,\xi_1) g_B(X,\xi_1) + \delta D_B(X,Y)$. From Proposition 3.4 we get the equation $\tilde{S}(X,Y) = \tilde{S}^B(X,Y) + n_2[\frac{H^f(X,Y)}{f} + g(Y, \nabla_X P) - \pi(X)\pi(Y)]$. By applying the above two equations we get the result of equation (4.1) (iii).
Since, M is a super quasi-Einstein manifold and For V, W on $\Gamma(TF)$, we have $\tilde{S}_M(X,Y) = f^2 \alpha g_F(X,Y) + \delta D_F(X,Y)$. From Proposition **??** we get the equation $\tilde{S}(V,W) = \tilde{S}^F(V,W) + [(n_2-1)\frac{|gradf|^2}{f^2} + (\bar{n}-1)\frac{Pf}{f} - \frac{\Delta f}{f}] f^2 g(V,W)$. By applying the above two equations we get the second result of equation (4.1) (iii).

From the above theorem we can give the scalar curvatures of M, B, and F. □

Corollary 4.2. *Taking the traces of theorem 4.1 we get the scalar curvature of M, B and F of different cases and $P \in \Gamma(TB)$. For the first case, we get*
i) When $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$ then

$$\left\{ \begin{array}{lll} \tilde{r}_M & = & \bar{n}\alpha + \beta, \\ \tilde{r}_B & = & n_1\alpha - n_2\frac{\Delta f}{f} + n_2 div_B P + n_2\pi(P) + \beta, \\ \tilde{r}_F & = & n_2[\alpha f^2 + f\Delta f + (1-n_2)|gradf|^2 + (1-\bar{n})fPf] \end{array} \right. \tag{4.4}$$

ii) When $\xi_1 \in \Gamma(TF), \xi_2 \in \Gamma(TF)$ then

$$\left\{ \begin{array}{lll} \tilde{r}_M & = & \bar{n}\alpha + \beta, \\ \tilde{r}_B & = & n_1\alpha - n_2\frac{\Delta f}{f} + n_2 div_B P + n_2\pi(P), \\ \tilde{r}_F & = & n_2[\alpha f^2 + f\Delta f + (1-n_2)|gradf|^2 + (1-\bar{n})fPf] + \beta f^4 \end{array} \right. \tag{4.5}$$

iii) When $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TF)$ then

$$\left\{ \begin{array}{lll} \tilde{r}_M & = & \bar{n}\alpha + \beta, \\ \tilde{r}_B & = & n_1\alpha - n_2\frac{\Delta f}{f} + n_2 div_B P + n_2\pi(P), \\ \tilde{r}_F & = & n_2[\alpha f^2 + f\Delta f + (1-n_2)|gradf|^2 + (1-\bar{n})fPf] \end{array} \right. \tag{4.6}$$

Theorem 4.3. *Let $M = B \times_f F$ be a super quasi-Einstein warped product space, $dimB = n_1$, $dimF = n_2$, $dimM = \bar{n} = n_1 + n_2$, $X, Y \in \Gamma(TB)$, $V, W \in \Gamma(TF)$, $P \in \Gamma(TF)$ Then the following conditions hold.*

i) When $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$ then

$$\left\{ \begin{array}{lll} \tilde{S}_B(X,Y) & = & \alpha g_B(X,Y) + \beta g_B(X,\xi_1) g_B(Y,\xi_1) + \delta D_B(X,Y) - n_2\frac{H^f(X,Y)}{f} \\ & + & \gamma[g_B(X,\xi_1)g_B(Y,\xi_2) + g_B(X,\xi_2)g_B(Y,\xi_1)] \\ \tilde{S}_F(V,W) & = & g_F(V,W)[\alpha f^2 + f\Delta f + (1-n_2)|gradf|^2] + \delta D_F(V,W) \\ & + & (\bar{n}-1)\pi(V)\pi(W) - (\bar{n}-1)g(W, \nabla_V P) \end{array} \right. \tag{4.7}$$

ii) When $\xi_1 \in \Gamma(TF), \xi_2 \in \Gamma(TF)$ *then*

$$\begin{cases} \tilde{S}_B(X,Y) &=& \alpha g_B(X,Y) + \delta D_B(X,Y) - n_2 \frac{H^f(X,Y)}{f} \\ \tilde{S}_F(V,W) &=& g_F(V,W)[f^2\alpha + f\Delta f + (1-n_2)|gradf|^2] + \delta D_F(V,W) \\ &+& (1-\bar{n})g(W,\nabla_V P) + (\bar{n}-1)\pi(V)\pi(W) + \beta f^4 g_B(V,\xi_1)g_B(W,\xi_1) \\ &+& \gamma f^4[g_B(V,\xi_1)g_B(W,\xi_2) + g_B(V,\xi_2)g_B(W,\xi_1)]. \end{cases} \tag{4.8}$$

iii) When $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TF)$ *then*

$$\begin{cases} \tilde{S}_B(X,Y) &=& \alpha g_B(X,Y) + \beta g_B(X,\xi_1)g_B(Y,\xi_1) + \delta D_B(X,Y) - n_2 \frac{H^f(X,Y)}{f} \\ \tilde{S}_F(V,W) &=& g_F(V,W)[\alpha f^2 + f\Delta f + (1-n_2)|gradf|^2] + \delta D_F(V,W) \\ &+& (1-\bar{n})g(W,\nabla_V P) - (1-\bar{n})\pi(V)\pi(W) \end{cases} \tag{4.9}$$

Corollary 4.4. *Taking the traces of theorem 4.3 we get the scalar curvature of* M, B *and* F *of different cases and* $P \in \Gamma(TF)$. *We get*
i) When $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$ *then*

$$\begin{cases} \tilde{r}_M &=& \bar{n}\alpha + \beta, \\ \tilde{r}_B &=& n_1\alpha - n_2\frac{\Delta f}{f} + \beta, \\ \tilde{r}_F &=& n_2[\alpha f^2 + f\Delta f + (1-n_2)|gradf|^2] - (\bar{n}-1)div_B P + (\bar{n}-1)\pi(P) \end{cases} \tag{4.10}$$

ii) When $\xi_1 \in \Gamma(TF), \xi_2 \in \Gamma(TF)$ *then*

$$\begin{cases} \tilde{r}_M &=& \bar{n}\alpha + \beta, \\ \tilde{r}_B &=& n_1\alpha - n_2\frac{\Delta f}{f}, \\ \tilde{r}_F &=& n_2[\alpha f^2 + f\Delta f + (1-n_2)|gradf|^2] - (\bar{n}-1)div_B P + (\bar{n}-1)\pi(P) + \beta f^4 \end{cases} \tag{4.11}$$

iii) When $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TF)$ *then*

$$\begin{cases} \tilde{r}_M &=& \bar{n}\alpha + \beta, \\ \tilde{r}_B &=& n_1\alpha - n_2\frac{\Delta f}{f} + \beta, \\ \tilde{r}_F &=& n_2[\alpha f^2 + f\Delta f + (1-n_2)|gradf|^2] - (\bar{n}-1)div_B P + (\bar{n}-1)\pi(P) \end{cases} \tag{4.12}$$

Similarly for Ricci curvature and scalar curvature of super quasi-Einstein warped products with quarter symmetric connection we have the following corollaries.

Corollary 4.5. *Let $M = B \times_f F$ be a super quasi-Einstein warped product space, $dimB = n_1$, $dimF = n_2$, $dimM = \bar{n} = n_1 + n_2$, $X, Y \in \Gamma(TB)$, $V, W \in \Gamma(TF)$, $P \in \Gamma(TB)$ Then the following conditions hold.*

i) When $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$ then

$$\begin{cases} \breve{S}_B(X,Y) &= \alpha g_B(X,Y) + \beta g_B(X,\xi_1)g_B(Y,\xi_1) + \gamma[g_B(X,\xi_1)g_B(Y,\xi_2) \\ &+ g_B(X,\xi_2)g_B(Y,\xi_1)] + \delta D_B(X,Y) - n_2[\frac{H^f(X,Y)}{f} + \lambda_1 g(Y,\nabla_X P) \\ &+ \lambda_2\frac{Pf}{f}g(X,Y) - \lambda_2\pi_2(X)\pi_2(Y) + \lambda_1\lambda_2\pi_2(P)g(X,Y) + \lambda_2\pi_2(X)] \\ \breve{S}_F(V,W) &= g_F(V,W)\{\alpha f^2 - f\Delta f - (n_2-1)|gradf|^2 - [(\bar{n}-1)\lambda_1 \\ &+ (n_2-1)\lambda_2]fPf - f^2\lambda_2 div_B P - f^2[(\bar{n}-1)\lambda_1\lambda_2 \\ &- \lambda_2^2]\pi_2(P)\} + \delta D_F(V,W) \end{cases} \tag{4.13}$$

ii) When $\xi_1 \in \Gamma(TF), \xi_2 \in \Gamma(TF)$ then

$$\begin{cases} \breve{S}_B(X,Y) &= \alpha g_B(X,Y) + \delta D_B(X,Y) - n_2[\frac{H^f(X,Y)}{f} + \lambda_1 g(Y,\nabla_X P) - \lambda_2\pi_2(X)\pi_2(Y) \\ &+ \lambda_2\frac{Pf}{f}g(X,Y) + \lambda_1\lambda_2\pi_2(P)g(X,Y) + \lambda_2\pi_2(X)] \\ \breve{S}_F(V,W) &= g_F(V,W)\{\alpha f^2 - f\Delta f - (n_2-1)|\nabla f|^2 - [(\bar{n}-1)\lambda_1 + (n_2-1)\lambda_2]fPf \\ &- f^2\lambda_2 div_B P - f^2[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]\pi_2(P)\} + \beta f^4 g_B(V,\xi_1)g_B(W,\xi_1) \\ &+ \delta D_F(V,W) + \gamma f^4[g_B(V,\xi_1)g_B(W,\xi_2) + g_B(V,\xi_2)g_B(W,\xi_1)] \end{cases} \tag{4.14}$$

iii) When $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TF)$ then

$$\begin{cases} \breve{S}_B(X,Y) &= \alpha g_B(X,Y) + \beta g_B(X,\xi_1)g_B(Y,\xi_1) + \delta D_B(X,Y) - n_2[\frac{H^f(X,Y)}{f} \\ &+ \lambda_1 g(Y,\nabla_X P) - \lambda_2\pi_2(X)\pi_2(Y) + \lambda_2\frac{Pf}{f}g(X,Y) + \lambda_1\lambda_2\pi_2(P)g(X,Y) \\ &+ \lambda_2\pi_2(X)] \\ \breve{S}_F(V,W) &= g_F(V,W)\{\alpha f^2 - f\Delta f - (n_2-1)|\nabla f|^2 - [(\bar{n}-1)\lambda_1 + (n_2-1)\lambda_2]fPf \\ &- f^2\lambda_2 div_B P - f^2[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]\pi_2\} + \delta D_F(V,W) \end{cases} \tag{4.15}$$

From we have the scalar curvatures of M, B, and F.

Corollary 4.6. *Taking the traces of corollary 4.5 we get the scalar curvature of M, B and F of different cases and $P \in \Gamma(TB)$. For the first case, we get*
i) When $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$ then

$$\begin{cases} \breve{r}_M &= \bar{n}\alpha + \beta, \\ \breve{r}_B &= n_1\alpha + \beta - n_2\frac{\Delta f}{f} - n_2\lambda_1 div_B P \\ &+ \lambda_1^2 n_2\pi_2(P) - \lambda_1\lambda_2 n_1 n_2\pi_2(P), \\ \breve{r}_F &= n_2[\alpha f^2 - f\Delta f - (n_2-1)|\nabla f|^2 - [(\bar{n}-1)\lambda_1 + (n_2-1)\lambda_2]fPf \\ &- f^2\lambda_2 div_B P - f^2[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]\pi_2(P)] \end{cases} \tag{4.16}$$

ii) When $\xi_1 \in \Gamma(TF), \xi_2 \in \Gamma(TF)$ then

$$\begin{cases} \breve{r}_M &= \bar{n}\alpha + \beta, \\ \breve{r}_B &= n_1\alpha - n_2\frac{\Delta f}{f} - n_2\lambda_1 div_B P + \lambda_1^2 n_2\pi_2(P) - \lambda_1\lambda_2 n_1 n_2\pi_2(P), \\ \breve{r}_F &= n_2[\alpha f^2 - f\Delta f - (n_2-1)|gradf|^2 - [(\bar{n}-1)\lambda_1 + (n_2-1)\lambda_2]fPf \\ &- f^2\lambda_2 div_B P - f^2[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]\pi_2(P)] + \beta f^4 + \gamma f^4 \end{cases} \tag{4.17}$$

iii) When $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TF)$ then

$$\begin{cases} \breve{r}_M &= \bar{n}\alpha + \beta, \\ \breve{r}_B &= n_1\alpha - n_2\frac{\Delta f}{f} - n_2\lambda_1 div_B P + \lambda_1^2 n_2\pi_2(P) + \beta - \lambda_1\lambda_2 n_1 n_2\pi_2(P), \\ \breve{r}_F &= n_2[\alpha f^2 - f\Delta f - (n_2-1)|gradf|^2 - [(\bar{n}-1)\lambda_1 + (n_2-1)\lambda_2]fPf \\ &- f^2\lambda_2 div_B P - f^2[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]\pi_2(P)] \end{cases} \tag{4.18}$$

Corollary 4.7. *Let $M = B \times_f F$ be a super quasi-Einstein warped product space, $dimB = n_1$, $dimF = n_2$, $dimM = \bar{n} = n_1 + n_2$, $X, Y \in \Gamma(TB)$, $V, W \in \Gamma(TF)$, $P \in \Gamma(TF)$ Then the following conditions hold.*

i) When $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$ then

$$\begin{cases} \breve{S}_B(X,Y) &= \alpha g_B(X,Y) + \beta g_B(X,\xi_1)g_B(Y,\xi_1) + \delta D_B(X,Y) - n_2\frac{H^f(X,Y)}{f} \\ &- \lambda_2 g(X,Y)div_F P + \gamma[g_B(X,\xi_1)g_B(Y,\xi_2) + g_B(X,\xi_2)g_B(Y,\xi_1)] \\ &- [(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]\pi_2(P)g(X,Y) \\ \breve{S}_F(V,W) &= g_F(V,W)\{\alpha f^2 - f\Delta f - (n_2-1)|gradf|^2\delta D_F(V,W) \\ &+ \lambda f^2 g(V,W)div_F P - f^2[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]\pi_2(P)\} - [(\bar{n}-1)\lambda_1 \\ &- \lambda_2]g(W,\nabla_V P) - [\lambda_2^2 + (1-\bar{n}\lambda_1^2]\pi_2(V)\pi_2(W) \end{cases} \tag{4.19}$$

ii) When $\xi_1 \in \Gamma(TF), \xi_2 \in \Gamma(TF)$ then

$$\begin{cases} \breve{S}_B(X,Y) &= \alpha g_B(X,Y) + +\delta D_B(X,Y) - n_2\frac{H^f(X,Y)}{f} \\ &- [(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]\pi_2(P)g(X,Y) - \lambda_2 g(X,Y)div_F P \\ \breve{S}_F(V,W) &= g_F(V,W)\{\alpha f^2 - f\Delta f - (n_2-1)|gradf|^2 - f^2[(\bar{n}-1)\lambda_1\lambda_2 \\ &- \lambda_2^2]\pi_2(P)\} - [(\bar{n}-1)\lambda_1 - \lambda_2]g(W,\nabla_V P) + \lambda f^2 g(V,W)div_F P \\ &- [\lambda_2^2 + (1-\bar{n})\lambda_1^2]\pi_2(V)\pi_2(W) + \beta f^4 g_B(V,\xi_1)g_B(W,\xi_1) \\ &+ \gamma f^4[g_B(V,\xi_1)g_B(W,\xi_2) + g_B(V,\xi_2)g_B(W,\xi_1)] + \delta D_F(V,W) \end{cases} \tag{4.20}$$

iii) When $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TF)$ *then*

$$\begin{cases} \breve{S}_B(X,Y) &= \alpha g_B(X,Y) + \beta g_B(X,\xi_1) g_B(Y,\xi_1) + \delta D_B(X,Y) - n_2 \frac{H^f(X,Y)}{f} \\ &- [(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]\pi_2(P) g(X,Y) - \lambda_2 g(X,Y) div_F P \\ \breve{S}_F(V,W) &= g_F(V,W)\{\alpha f^2 - f\Delta f - (n_2-1)|gradf|^2 - f^2[(\bar{n}-1)\lambda_1\lambda_2 \\ &- \lambda_2^2]\pi_2(P)\} + \delta D_F(V,W) - [(\bar{n}-1)\lambda_1 - \lambda_2] g(W, \nabla_V P) \\ &+ +\lambda f^2 g(V,W) div_F P - [\lambda_2^2 + (1-\bar{n}\lambda_1^2]\pi_2(V)\pi_2(W) \end{cases} \tag{4.21}$$

Corollary 4.8. *Taking the traces of corollary 4.7 we get the scalar curvature of* M, B *and* F *of different cases* $P \in \Gamma(TF)$. *We get*
i) When $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$ *then*

$$\begin{cases} \breve{r}_M &= \bar{n}\alpha + \beta, \\ \breve{r}_B &= \alpha n_1 + \beta - n_2 \frac{\Delta f}{f} - n_1[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]\pi_2(P) - \lambda_2 n_1 div_F P, \\ \breve{r}_F &= n_2\{\alpha f^2 - f\Delta f - (n_2-1)|gradf|^2 - \{f^2[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2] \\ &+ [\lambda_2^2 + (1-\bar{n}\lambda_1^2]\pi_2(P)\} - [(\bar{n}-1)\lambda_1 - \lambda_2] div_F P + \lambda f^2 n_2 div_F P \end{cases} \tag{4.22}$$

ii) When $\xi_1 \in \Gamma(TF), \xi_2 \in \Gamma(TF)$ *then*

$$\begin{cases} \breve{r}_M &= \bar{n}\alpha + \beta, \\ \breve{r}_B &= \alpha n_1 - n_2 \frac{\Delta f}{f} - n_1[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]\pi_2(P) - \lambda_2 n_1 div_F P, \\ \breve{r}_F &= n_2\{\alpha f^2 - f\Delta f - (n_2-1)|gradf|^2 - \{f^2[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2] \\ &+ [\lambda_2^2 + (1-\bar{n}\lambda_1^2]\pi_2(P)\} - [(\bar{n}-1)\lambda_1 - \lambda_2] div_F P + \lambda f^2 n_2 div_F P + \beta f^4 \end{cases} \tag{4.23}$$

iii) When $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TF)$ *then*

$$\begin{cases} \breve{r}_M &= \bar{n}\alpha + \beta, \\ \breve{r}_B &= \alpha n_1 + \beta - n_2 \frac{\Delta f}{f} - n_1[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]\pi_2(P) - \lambda_2 n_1 div_F P, \\ \breve{r}_F &= n_2\{\alpha f^2 - f\Delta f - (n_2-1)|gradf|^2 - \{f^2[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2] \\ &+ [\lambda_2^2 + (1-\bar{n}\lambda_1^2]\pi_2(P)\} - [(\bar{n}-1)\lambda_1 - \lambda_2] div_F P + \lambda f^2 n_2 div_F P \end{cases} \tag{4.24}$$

5. Obstructions To The Existence Of Super Quasi-Einstein Warped Products with Affine Connections:

In this section, we prove some obstructions to the existence of super quasi-Einstein warped products with a semi-symmetric non-metric connection. We consider the following six cases:

When $\underline{P \in \Gamma(TB), \xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)}$**:**

From the equation (4.1), we consider

$$\alpha f^2 + f\Delta f + (1-n_2)|\nabla f|^2 + (1-\bar{n})fPf = \alpha_1.$$

Then the equation (4.4) becomes

$$\begin{cases} \tilde{S}_B(X,Y) &= \alpha g_B(X,Y) + \beta g_B(X,\xi_1)g_B(X,\xi_1) + \gamma[g_B(X,\xi_1)g_B(Y,\xi_2) \\ &+ \ g_B(X,\xi_2)g_B(Y,\xi_1)] + \delta D_B(X,Y) - n_2[\frac{H^f(X,Y)}{f} \\ &+ \ g(Y,\nabla_X P) - \pi(X)\pi(Y)] \\ \tilde{S}_F(X,Y) &= \alpha_1 g_F(X,Y), \\ \alpha_1 &= \alpha f^2 + f\Delta f + (1-n_2)|\nabla f|^2 + (1-\bar{n})fPf. \end{cases} \tag{5.1}$$

Theorem 5.1. *Let $M = B\times_f F$ be a super quasi-Einstein warped product with respect to semi-symmetric non-metric connection with B compact and connected, $dimB = n_1 \geq 1$, $dimF = n_2 \geq 2$, $dimM = n_1 + n_2 = \bar{n}$, $P \in \Gamma(TB)$ and $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$. If $n_1 = 1$, $divP = c_1$, and $\pi(P) = c_2$ are both constants, then M is a simply Riemannian product.*

Proof. Since $n_1 = 1$, $\tilde{r}_B = 0$, by the second equation of (4.4) we get

$$0 = \alpha - n_2\frac{\Delta f}{f} - n_2 c_1 + n_2 c_2 + \beta,$$

$$\therefore \frac{\Delta f}{f} = -\frac{-\alpha - \beta + n_2 c_1 - n_2 c_2}{n_2} = c_0,$$

$\therefore \Delta f = c_0 f$. Hence, the Laplacian has constant in sign and so M is a simply Riemannian product. $\square$

Theorem 5.2. *Let $M = B\times_f F$ be a super quasi-Einstein warped product with respect to semi-symmetric non-metric connection with B compact and connected, $dimB = n_1 \geq 2$, $dimF = n_2 \geq 2$, $dimM = n_1 + n_2 = \bar{n}$, $P \in \Gamma(TB)$ and $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$. If $\tilde{r_M} \leq \tilde{r_B}$, $divP \leq 0$, $\alpha > 0$ and $\pi(P) \geq 0$, then M is a simply Riemannian product.*

Proof. From the equation (4.4),

$$\tilde{r}_B = n_1\alpha - n_2\frac{\Delta f}{f} - n_2 divP + n_2\pi(P) + \beta,$$

$$\Rightarrow \tilde{r}_M - \tilde{r}_B = n_2[\alpha - \frac{\Delta f}{f} + divP - \pi(P)].$$

By using the condition we can easily shown that f is constant. $\square$

Theorem 5.3. *Let $M = B\times_f F$ be a super quasi-Einstein warped product with respect to semi-symmetric non-metric connection with B compact and connected, $dimB = n_1 \geq 2$, $dimF = n_2 \geq 2$, $dimM = n_1 + n_2 = \bar{n}$, $P \in \Gamma(TB)$ and $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$. If $Pf \leq 0$, $\alpha > 0$ and $|\nabla f|^2 \geq \frac{\alpha_1}{1-n_2}$, then M is a simply Riemannian product.*

Proof. From the third equation of (5.1) we get M is a simply Riemannian product. □

Theorem 5.4. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with respect to semi-symmetric non-metric connection with B compact and connected, $dimB = n_1 \geq 2$, $dimF = n_2 \geq 2$, $dimM = n_1 + n_2 = \bar{n}$, $P \in \Gamma(TB)$ and $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$. If $\tilde{r_B} \leq 0$, $divP \leq 0$, $\alpha > 0$, $\beta > 0$ and $\pi(P) \geq 0$, then M is a simply Riemannian product.*

Proof. From the third equation of (4.4) we have $\tilde{r}_F = n_2\alpha_1 \geq n_2\alpha$.
So, we get $\tilde{r}_F + \tilde{r}_B \geq \tilde{r}_M - n_2\frac{\Delta f}{f} - n_2 divP + n_2\pi(P)$,
By using the condition we can easily shown that f is constant. □

Theorem 5.5. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with respect to semi-symmetric non-metric connection with B compact and connected, $dimB = n_1 \geq 2$, $dimF = n_2 \geq 2$, $dimM = n_1 + n_2 = \bar{n}$, $P \in \Gamma(TB)$ and $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$. If $\tilde{r_M} \geq \tilde{\beta} + \frac{n_1\alpha_1}{f^2}$, $divP \leq 0$, and $\pi(P) \geq 0$, $n_2 \geq n_1$, $Pf < 0$, then M is a simply Riemannian product.*

Proof. From the third equation of theorem 5.1 and the second equation of (4.4) we get

$$f^2\tilde{r}_B - n_1\alpha_1 - \beta f^2 = (n_2 - n_1)f\Delta f + n_1(n_2 - 1)|\nabla f|^2 + n_1(\bar{n} - 1)fPf - n_2 f^2 divP + n_2 f^2\pi(P).$$

Using the conditions we get f is constant. □

Theorem 5.6. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with respect to semi-symmetric non-metric connection with B compact and connected, $dimB = n_1 \geq 2$, $dimF = n_2 \geq 2$, $dimM = n_1 + n_2 = \bar{n}$, $P \in \Gamma(TB)$ and $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$. If $\alpha \leq 0$, $Pf \geq 0$, and $\tilde{r_F} \geq 0$, then M is a simply Riemannian product.*

Proof. From the third equation of (4.4) we have $\tilde{r}_F = n_2\alpha_1 \geq 0$. By the third equation of theorem 5.1 and using the condition we can easily shown that f is constant. □

Theorem 5.7. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with respect to semi-symmetric non-metric connection with B compact and connected, $dimB = n_1 \geq 2$, $dimF = n_2 \geq 2$, $dimM = n_1 + n_2 = \bar{n}$, $P \in \Gamma(TB)$ and $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$. If $\alpha \geq 0$, $Pf \geq 0$ then M is a simply Riemannian product.*

Proof. Let $z \in B$ such that $f(z)$ is maximum of f on B. Then $\mathrm{grad_B f(z)} = 0$ and $\Delta_B f(z) = 0$. So, $Pf(Z) = 0$. From the equation (5.1) in the point z we obtain

$$\alpha f^2(z) + f(z)\Delta_B f(z) = \alpha_1,$$

Again also we have

$$\alpha f^2(z) + f(z)\Delta_B f(z) = \alpha_1 = \alpha f^2 + f\Delta f + (1 - n_2)|\nabla f|^2 + (1 - \bar{n})fPf.$$

Hence by using the conditions we get $f\Delta_B f \geq 0$. So, f is constant. □

Theorem 5.8. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with respect to semi-symmetric non-metric connection with B compact and connected, $dimB = n_1 \geq 2$, $dimF = n_2 \geq 2$, $dimM = n_1 + n_2 = \bar{n}$, $P \in \Gamma(TB)$ and $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$. If $\tilde{r_M} \geq \tilde{r_B} + \tilde{r_F}$, $divP \leq 0$, and $\pi(P) \geq 0$, $\alpha_1 \geq \alpha$, then M is a simply Riemannian product.*

Proof. From the third equation of (4.4) we have $\tilde{r}_F = n_2\alpha_1 \geq n_2\alpha_1$. By the third equation of theorem 5.1 and using the condition we can easily shown that f is constant. □

Similarly for quarter symmetric connection we have the following corollaries.

Corollary 5.9. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with B compact and connected, $dimB = n_1 \geq 1$, $dimB = n_1 \geq 2$, $dimM = n_1+n_2 = \bar{n}$, $P \in \Gamma(TB)$ and $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$. If $n_1 = 1$, $divP = c_1$, $\frac{Pf}{f} = c_2$ and $\pi(P) = c_3$, $\lambda_1 = \lambda_2$, $\lambda_1 > 0, \lambda_2 > 0$ where c_1, C_2 are constants then M is a simply Riemannian product.*

Corollary 5.10. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with B compact and connected, $dimB = n_1 \geq 2$, $dimF = n_2 \geq 1$, $dimM = n_1+n_2 = \bar{n}$, $P \in \Gamma(TB)$ and $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$. If $n_2 = 1$,$\alpha \leq 0$, $Pf \leq 0$, $div_BP \geq 0$, $\lambda_1, \lambda_2 \geq 0$, $\lambda_2 = (\bar{n} - 1)\lambda_1$ then M is a simply Riemannian product.*

Corollary 5.11. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with B compact and connected, $dimB = n_1 \geq 2$, $dimF = n_2 \geq 2$, $dimM = n_1+n_2 = \bar{n}$, $P \in \Gamma(TB)$ and $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$. Then M is a simply Riemannian product if any one of the following relation hold.*

a) $\breve{r}_F \geq 0$, $\alpha \leq 0$, $Pf \geq 0$, $div_BP \geq 0$, $\lambda_1, \lambda_2 \geq 0$, $\lambda_2 = (\bar{n} - 1)\lambda_1$,
b) $\breve{r}_M \leq \breve{r}_F$, $\alpha \geq 0$, $Pf \geq 0$, $div_BP \leq 0$, $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 = n_1\lambda_2$,
c) $\frac{Pf}{f} \geq 0$, $\alpha \leq 0$, $div_BP \geq 0$, $\lambda_1, \lambda_2 \geq 0$, $\lambda_2 = (\bar{n} - 1)\lambda_1$, $|\nabla f|^2 \leq \frac{\alpha_3}{1-n_2}$,
d) $\frac{Pf}{f} \leq 0$, $div_BP \geq 0$, $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 = n_1\lambda_2$, $\breve{r}_B \leq 0$, $\alpha \geq 0$, $\beta \geq 0$,
e) $\frac{Pf}{f} \geq 0$, $div_BP \geq 0$, $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 = n_1\lambda_2$, $0 \geq \breve{r}_B$, $\alpha \geq 0$, $\beta \geq 0$,
f) $Pf \leq 0$, $div_BP \leq 0$, $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 = \lambda_2$, $\breve{r}_B \geq \frac{\alpha_3 n_1}{f^2} + \beta$, $n_2 \geq n_1$.

When $P \in \Gamma(TB), \xi_1 \in \Gamma(TF), \xi_2 \in \Gamma(TF)$:

Theorem 5.12. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with respect to semi-symmetric non-metric connection with B compact and connected, $dimB = n_1 \geq 1$, $dimF = n_2 \geq 2$, $dimM = n_1 + n_2 = \bar{n}$, $P \in \Gamma(TB)$ and $\xi_1 \in \Gamma(TF), \xi_2 \in \Gamma(TF)$. If $n_1 = 1$, $divP = c_1$, and $\pi(P) = c_2$ are both constants, then M is a simply Riemannian product.*

Proof. Since $n_1 = 1$, $\tilde{r}_B = 0$, by the second equation of (4.5) we get

$$0 = \alpha - n_2 \frac{\Delta f}{f} - n_2 c_1 + n_2 c_2,$$

$$\therefore \frac{\Delta f}{f} = \frac{\alpha - n_2 c_1 + n_2 c_2}{n_2} = c_3,$$

$\therefore \Delta f = c_3 f$. Hence, the Laplacian has constant in sign and so M is a simply Riemannian product. □

Theorem 5.13. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with respect to semi-symmetric non-metric connection with B compact and connected, $dimB = n_1 \geq 2$, $dimF = n_2 \geq 2$, $dimM = n_1 + n_2 = \bar{n}$, $P \in \Gamma(TB)$ and $\xi_1 \in \Gamma(TF), \xi_2 \in \Gamma(TF)$. If $\beta \neq 0$ then M is a simply Riemannian product.*

Proof. Consider the second equation of (4.1) that V, W are orthogonal vector fields tangent to F such that $g_F(V, \xi_1) \neq 0$, $g_F(W, \xi_1) \neq 0$, $D_F(V, W) = 0$ or $g_F(V, \xi_1) \neq 0$, $g_F(W, \xi_2) \neq 0$, $D_F(V, W) = 0$ or $g_F(V, \xi_2) \neq 0$, $g_F(W, \xi_1) \neq 0$, $D_F(V, W) = 0$. Then we have

$$\tilde{S}_F(V, W) = \beta f^4 g_F(V, \xi_1) g_F(W, \xi_1),$$

or

$$\tilde{S}_F(V, W) = \gamma f^4 g_F(V, \xi_1) g_F(W, \xi_2),$$

or

$$\tilde{S}_F(V, W) = \gamma f^4 g_F(V, \xi_2) g_F(W, \xi_1).$$

Taking in consideration the different domains of definition of the functions that appear in the above equations, we get f is a constant. □

Similarly for quarter symmetric connection we have the following corollaries.

Corollary 5.14. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with B compact and connected, $dimB = n_1 \geq 1$, $dimF = n_2 \geq 2$, $dimM = n_1 + n_2 = \bar{n}$, $P \in \Gamma(TB)$ and $\xi_1 \in \Gamma(TF), \xi_2 \in \Gamma(TF)$. If $n_1 = 1$, $div_B P = c_1$, $\pi(P) = c_2$, $\frac{Pf}{f} = c_3$, $\lambda_1 = \lambda_2 > 0$ where c_1, c_2, c_3 are constants then M is a simply Riemannian product.*

Corollary 5.15. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with B compact and connected, $dimB = n_1 \geq 2$, $dimF = n_2 \geq 2$, $dimM = n_1 + n_2 = \bar{n}$, $P \in \Gamma(TB)$ and $\xi_1 \in \Gamma(TF), \xi_2 \in \Gamma(TF)$. If $\beta \neq 0$ then M is a simply Riemannian product.*

When $P \in \Gamma(TB), \xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TF)$ **:**

Theorem 5.16. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with respect to semi-symmetric non-metric connection with B compact and connected, $dimB = n_1 \geq 1$, $dimF = n_2 \geq 2$, $dimM = n_1 + n_2 = \bar{n}$, $P \in \Gamma(TB)$ and $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TF)$. If $n_1 = 1$, $divP = c_1$, and $\pi(P) = c_2$ are both constants, then M is a simply Riemannian product.*

Proof. Since $n_1 = 1$, $\tilde{r}_B = 0$, by the second equation of (4.6) we get

$$0 = \alpha - n_2\frac{\Delta f}{f} - n_2c_1 + n_2c_2,$$

$$\therefore \frac{\Delta f}{f} = \frac{\alpha - n_2c_1 + n_2c_2}{n_2} = c_3,$$

$\therefore \Delta f = c_3 f$. Hence, the Laplacian has constant in sign and so M is a simply Riemannian product. □

Similarly for quarter symmetric connection we have the following corollaries.

Corollary 5.17. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with B compact and connected, $dimB = n_1 \geq 1$, $dimF = n_2 \geq 2$, $dimM = n_1 + n_2 = \bar{n}$, $P \in \Gamma(TB)$ and $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TF)$. If $n_1 = 1$, $div_BP = c_1$, $\pi(P) = c_2$, $\frac{Pf}{f} = c_3$, $\lambda_1 = \lambda_2 > 0$ where c_1, c_2, c_3 are constants then M is a simply Riemannian product.*

<u>**When** $P \in \Gamma(TF), \xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$:</u>

From the equation of (4.7), we consider

$$\alpha f^2 + f\Delta f + (1 - n_2)|\nabla f|^2 = \alpha_2.$$

Then the equation (4.7) becomes

$$\begin{cases} \tilde{S}_B(X,Y) &= \alpha g_B(X,Y) + \beta g_B(X,\xi_1)g_B(X,\xi_1) - n_2\frac{H^f(X,Y)}{f} \\ &+ \gamma[g_B(X,\xi_1)g_B(Y,\xi_2) + g_B(X,\xi_2)g_B(Y,\xi_1)] + \delta D_B(X,Y) \\ \tilde{S}_F(X,Y) &= \alpha_2 g_F(X,Y) + (1 - \bar{n}g(W,\nabla_V P)) - (1-\bar{n})\pi(V)\pi(W), \\ \alpha_2 &= \alpha f^2 + f\Delta f + (1-n_2)|\nabla f|^2. \end{cases} \tag{5.2}$$

Theorem 5.18. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with respect to semi-symmetric non-metric connection with B compact and connected, $dimB = n_1 \geq 1$, $dimF = n_2 \geq 2$, $dimM = n_1 + n_2 = \bar{n}$, $P \in \Gamma(TF)$ and $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$. If $n_1 = 1$, $divP = c_1$, and $\pi(P) = c_2$ are both constants, then M is a simply Riemannian product.*

Proof. Since $n_1 = 1$, $\tilde{r}_B = 0$, by the second equation of (4.10) we get

$$0 = \alpha - n_2 \frac{\Delta f}{f} + \beta,$$
$$\therefore \frac{\Delta f}{f} = \frac{-\alpha - \beta}{n_2} = c_4,$$

$\therefore \Delta f = c_4 f$. Hence, the Laplacian has constant in sign and so M is a simply Riemannian product. □

Theorem 5.19. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with respect to semi-symmetric non-metric connection with B compact and connected, $dimB = n_1 \geq 2$, $dimF = n_2 \geq 2$, $dimM = n_1 + n_2 = \bar{n}$, $P \in \Gamma(TF)$ and $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$. If $\tilde{r_M} \geq \tilde{r_B} + \tilde{r_F}$, $divP = 0$, and $\pi(P) = 0$, $\alpha_2 \geq \alpha$, then M is a simply Riemannian product.*

Proof. From the third equation of (4.10) we have $\tilde{r}_F = n_2\alpha_2 \geq n_2\alpha$.
So, we get $\tilde{r}_F + \tilde{r}_B \geq \tilde{r}_M - n_2\frac{\Delta f}{f}$,
By using the condition we can easily shown that f is constant. □

Theorem 5.20. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with respect to semi-symmetric non-metric connection with B compact and connected, $dimB = n_1 \geq 2$, $dimF = n_2 \geq 2$, $dimM = n_1 + n_2 = \bar{n}$, $P \in \Gamma(TB)$ and $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$. If $\alpha < 0$, $divP = 0$, and $\pi(P) = 0$, $\tilde{r}_F > 0$, then M is a simply Riemannian product.*

Proof. From the second equation of (5.2) and the equation of theorem 5.1 and the second equation of (4.10) we get

$$f^2\tilde{r}_B - n_1\alpha_1 - \beta f^2 = (n_2 - n_1)f\Delta f + n_1(n_2-1)|\nabla f|^2 + n_1(\bar{n}-1)fPf - n_2f^2divP + n_2f^2\pi(P).$$

Using the conditions we get f is constant. □

Theorem 5.21. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with respect to semi-symmetric non-metric connection with B compact and connected, $dimB = n_1 \geq 2$, $dimF = n_2 \geq 2$, $dimM = n_1 + n_2 = \bar{n}$, $P \in \Gamma(TF)$ and $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$. If $\alpha \leq 0$, $Pf \geq 0$, and $\tilde{r_F} \geq 0$, then M is a simply Riemannian product.*

Proof. From the third equation of (4.10) we have $\tilde{r}_F = n_2\alpha_1 \geq 0$. By the third equation 5.2 and using the condition we can easily shown that f is constant. □

Theorem 5.22. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with respect to semi-symmetric non-metric connection with B compact and connected, $dimB = n_1 \geq 2$, $dimF = n_2 \geq 2$, $dimM = n_1 + n_2 = \bar{n}$, $P \in \Gamma(TF)$ and $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$. If $\alpha \geq 0$, then M is a simply Riemannian product.*

Proof. Let $z \in B$ such that $f(z)$ is maximum of f on B. Then $\mathrm{grad_B f(z)} = 0$ and $\Delta_B f(z) = 0$. From the equation (5.2) in the point z we obtain

$$\alpha f^2(z) + f(z)\Delta_B f(z) = \alpha_2,$$

Again also we have

$$\alpha f^2(z) + f(z)\Delta_B f(z) = \alpha_2 = \alpha f^2 + f\Delta f + (1 - n_2)|\nabla f|^2 f.$$

Hence by using the conditions we get $f\Delta_B f \geq 0$. So, f is constant. □

Similarly for quarter symmetric connection we have the following corollaries.

Corollary 5.23. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with B compact and connected, $dimB = n_1 \geq 1$, $dimF = n_2 \geq 2$, $dimM = n_1+n_2 = \bar{n}$, $P \in \Gamma(TF)$ and $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$. If If $n_1 = 1$, $div_B P = c_1$, $\lambda_2 = n_2\lambda_1 > 0$ where c_1 is constant then M is a simply Riemannian product.*

Corollary 5.24. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with B compact and connected, $dimB = n_1 \geq 2$, $dimF = n_2 \geq 2$, $dimM = n_1+n_2 = \bar{n}$, $P \in \Gamma(TF)$ and $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TB)$. Then M is a simply Riemannian product if any one of the following relation holds.*

a) $\alpha < 0$, $\breve{r}_F \geq 0$, $div_F P = 0$, $\pi(P) = 0$,
b) $\pi(P) = 0$, $div_F P = 0$, $\breve{r}_M \geq \breve{r}_B + \breve{r}_F$,
c) $\alpha \leq 0$, $div_B P \geq 0$, $n_2 = 1$, $\lambda_2 = n_1\lambda_1 > 0$.

When $P \in \Gamma(TF), \xi_1 \in \Gamma(TF), \xi_2 \in \Gamma(TF)$:

Theorem 5.25. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with respect to semi-symmetric non-metric connection with B compact and connected, $dimB = n_1 \geq 1$, $dimF = n_2 \geq 2$, $dimM = n_1 + n_2 = \bar{n}$, $P \in \Gamma(TF)$ and $\xi_1 \in \Gamma(TF), \xi_2 \in \Gamma(TF)$. If $n_1 = 1$, Then M is a simply Riemannian product.*

Proof. Since $n_1 = 1$, $\tilde{r}_B = 0$, by the second equation of (4.11) we get

$$0 = \alpha - n_2\frac{\Delta f}{f},$$

$$\therefore \frac{\Delta f}{f} = \frac{\alpha}{n_2} = c_5,$$

$\therefore \Delta f = c_5 f$. Hence, the Laplacian has constant in sign and so M is a simply Riemannian product. □

Theorem 5.26. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with respect to semi-symmetric non-metric connection with B compact and connected, $dimB = n_1 \geq 2$, $dimF = n_2 \geq 2$, $dimM = n_1 + n_2 = \bar{n}$, $P \in \Gamma(TF)$ and $\xi_1 \in \Gamma(TF), \xi_2 \in \Gamma(TF)$. If $\beta \neq 0$ and P is parallel on F with respect to Levi-Civita connection on F, then M is a simply Riemannian product.*

Proof. Consider the second equation of (4.8) that V, W are orthogonal vector fields tangent to F such that $g_F(V, \xi_1) \neq 0$, $g_F(W, \xi_1) \neq 0$, $D_F(V, W) = 0$, $\pi(W) = 0$ or $g_F(V, \xi_1) \neq 0$, $g_F(W, \xi_2) \neq 0$, $D_F(V, W) = 0$, $\pi(W) = 0$ or $g_F(V, \xi_2) \neq 0$, $g_F(W, \xi_1) \neq 0$, $D_F(V, W) = 0$, $\pi(W) = 0$. Also, P is parallel on F with respect to Levi-Civita connection on F, so we obtain $\nabla_V P = 0$ Then we have

$$\tilde{S}_F(V, W) = \beta f^4 g_F(V, \xi_1) g_F(W, \xi_1),$$

or

$$\tilde{S}_F(V, W) = \gamma f^4 g_F(V, \xi_1) g_F(W, \xi_2),$$

or

$$\tilde{S}_F(V, W) = \gamma f^4 g_F(V, \xi_2) g_F(W, \xi_1).$$

Taking in consideration the different domains of definition of the functions that appear in the above equations, we get f is a constant. □

Similarly for quarter symmetric connection we have the following corollaries.

Corollary 5.27. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with B compact and connected, $dimB = n_1 \geq 1$, $dimF = n_2 \geq 2$, $dimM = n_1 + n_2 = \bar{n}$, $P \in \Gamma(TF)$ and $\xi_1 \in \Gamma(TF), \xi_2 \in \Gamma(TF)$. If $n_1 = 1$, $div_B P = c_1$, $\lambda_2 = n_2 \lambda_1 > 0$ where c_1 is constant then M is a simply Riemannian product.*

Corollary 5.28. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with B compact and connected, $dimB = n_1 \geq 2$, $dimF = n_2 \geq 2$, $dimM = n_1 + n_2 = \bar{n}$, $P \in \Gamma(TF)$ and $\xi_1 \in \Gamma(TF), \xi_2 \in \Gamma(TF)$. If $\beta \neq 0$ and P is parallel on F with respect to Levi-Civita connection on F then M is a simply Riemannian product.*

When $P \in \Gamma(TF), \xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TF)$:

Theorem 5.29. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with respect to semi-symmetric non-metric connection with B compact and connected, $dimB = n_1 \geq 1$, $dimF = n_2 \geq 2$, $dimM = n_1 + n_2 = \bar{n}$, $P \in \Gamma(TF)$ and $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TF)$. If n_1, then M is a simply Riemannian product.*

Proof. Since $n_1 = 1$, $\tilde{r}_B = 0$, by the second equation of (4.12) we get

$$0 = \alpha - n_2 \frac{\Delta f}{f} + \beta,$$

$$\therefore \frac{\Delta f}{f} = \frac{\alpha + \beta}{n_2} = c_5,$$

$\therefore \Delta f = c_5 f$. Hence, the Laplacian has constant in sign and so M is a simply Riemannian product. □

Similarly for quarter symmetric connection we have the following corollary.

Corollary 5.30. *Let $M = B \times_f F$ be a super quasi-Einstein warped product with B compact and connected, $dimB = n_1 \geq 1$, $dimF = n_2 \geq 2$, $dimM = n_1 + n_2 = \bar{n}$, $P \in \Gamma(TF)$ and $\xi_1 \in \Gamma(TB), \xi_2 \in \Gamma(TF)$. If If $n_1 = 1$, $div_B P = c_1$, $\lambda_2 = n_2\lambda_1 > 0$ where c_1 is constant then M is a simply Riemannian product.*

6. Example of Super Quasi-Einstein Space Time :

Example 6.1. Let us consider five dimensional the metric given by

$$ds^2 = dt^2 - (e^t)^2(dx^2 + dy^2 + dz^2) - (e^t)^2 d\psi^2,$$

the fifth co-ordinate is taken to be space-like unlike Wesson [22].
Then, in a local coordinate, the only non-vanishing components of the Christoffel symbols are

$$\Gamma^1_{22} = (e^t)^2 = \Gamma^1_{33} = \Gamma^1_{44},\ \Gamma^1_{55} = (e^t)^2,$$
$$\Gamma^2_{21} = 1 = \Gamma^3_{31} = \Gamma^4_{41},\ \Gamma^5_{51} = 1.$$

The non-vanishing curvature tensors and the Ricci tensors are

$$R_{1221} = -(e^t)^2 = R_{1331} = R_{1441}, R_{1551} = -(e^t)^2,$$
$$R_{2332} = (e^t)^3 = R_{2442} = R_{3443}, R_{2552} = (e^t)^4 = R_{3553} = R_{4554},$$

and

$$R_{11} = 4, R_{55} = -4(e^t)^2, R_{22} = -4(e^t)^2 = R_{33} = R_{44}.$$

Let us consider the associated scalars α, β, γ and δ and the associated tensor D as follows:

$$\alpha = 4,\ \beta = 2,\ \gamma = -2,\ \delta = -2e^t, \tag{6.1}$$

and

$$D_{i,j} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & e^t & -e^t & 0 & 0 \\ 0 & -e^t & -e^t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6.2}$$

and the 1-forms are given by

$$A_i(x) = \begin{cases} e^t & \text{for} \quad i = 2,3 \\ 0 & \text{otherwise} \end{cases} \text{ and } B_i(x) = \begin{cases} e^t & \text{for} \quad i = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$(i) R_{11} = \alpha g_{11} + \beta A_1 A_1 + \varrho B_1 B_1 + \gamma[A_1 B_1 + A_1 B_1] + \delta D_{11}$$
$$(ii) R_{22} = \alpha g_{22} + \beta A_2 A_2 + \varrho B_2 B_2 + \gamma[A_2 B_2 + A_2 B_2] + \delta D_{22}$$

$$(iii) R_{33} = \alpha g_{33} + \beta A_3 A_3 + \varrho B_3 B_3 + \gamma[A_3 B_3 + A_3 B_3] + \delta D_{44}$$
$$(iv) R_{44} = \alpha g_{44} + \beta A_4 A_4 + \varrho B_4 B_4 + \gamma[A_4 B_4 + A_4 B_4] + \delta D_{44}$$
$$(v) R_{55} = \alpha g_{55} + \beta A_5 A_5 + \varrho B_5 B_5 + \gamma[A_5 B_5 + A_5 B_5] + \delta D_{55}.$$

Since all the cases other than $(i) - (v)$ are trivial, we can say that

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma[A_i B_j + A_j B_i] + \delta D_{ij}, i, j = 1, 2, 3, 4, 5.$$

Then (M^5, g) is a super quasi-Einstein space time.

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References

1. A. L. Besse,*Einstein manifolds*, Springer-Verlag, 1987.
2. J. K. Beem and P. Ehrich, *Lorentzian Geometry* Markel-Deccer Inc.,First Edition, New York, 1981.
3. R. L. Bishop and B. O'Neill,*Geometry of slant Submnaifolds*, Trans. Amer. Math. Soc. **145** (1969), 1-49.
4. M. C. Chaki,*On super quasi Einstein manifolds*, Publ. Math. Debrecen **64** (2004), 481-488.
5. M. C. Chaki and R. K. Maity, *On quasi Einstein manifolds*, Publ Math Debrecen **57**(2000), 297306.
6. P. Debnath and A. Konar, *On super quasi-Einstein manifold*, Publications De LInstitut Mathématique **89**(2011), 95104.
7. R. Deszcez, M. Glogowska, M. Holtos and Z. Senturk,*On certain quasi-Einstein semisymmetric hypersurfaces*, Ann. Univ. Sci. Budapest. Eotovos Sect. Math. **41** (1998), 151-164 (1999).
8. D. Dumitru,*On quasi-Einstein warped products*, Jordon Journal of Mathematics and Statistics **5** (2012), 85-95.
9. D. Dumitru,*On Einstein spaces of odd dimension*, Bul. Transilv. Univ. Brasov Ser. B (N.S.) **14** (2007), 95-97.
10. J. A. Schouten, *Ricci-Calculus-An Introduction to Tensor Analysis and Geometrical Applications*, Springer-Verlag, 1954.
11. S. Golab, *On semi-symmetric and quarter-symmetric linear connections*, Tensor N. S. **29** (1975), 249-254.
12. H. A. Hayden, *Subspace of a space with torsion*, Proc. Lond. Math. Soc. **34** (1932), 27-50.
13. B. O'Neill,*Semi-Riemannian Geometry wih Applications to Relativity*, Academic Press. Inc. New York, 1983.
14. S. Pahan, B. Pal and A. Bhattacharyya *On compact super quasi-Einstein warped product with nonpositive scalar curvature*, Journal of Mathematical Physics, Analysis, Geometry **13** (2017), 353-363.
15. M. Tripathi, *A new connection in a Riemannian manifold*, International Electronic Journal of Geometry **1** (2008), 15-24.
16. C. Özgür,*On some classes of super quasi-Einstein manifolds*, Chaos, Solitons and Fractals **40** (2009), 1156-1161.
17. B. Pal and A. Bhattacharyya,*A Characterization of Warped Product on Mixed Super quasi-Einstein Manifold*, Journal of Dynamical Systems and Geometric Theories **12** (2014), 29-39.
18. Q. Qu and Y. Wang, *Multiply warped products with a quarter-Symmetric connection*, Journal of Mathematical Analysis and Applications **431** (2015), 955-987.

19. Q. Qu, *Quasi-Einstein and Generalized Quasi-Einstein Warped Products with an Affine Connection*, arXiv:1505.03319, 2015.
20. S. Sular and C. Özgür, *Warped product with a semi-symmetric non-metric connection*, Arab. J. Sci. Eng. **36** (2011), 461-473.
21. K. Yano, *On semi-symmetric metric connection*, Rev. Roumaine Math. Pures Appl. **15**(1970), 1579-1586.
22. P. S. Wesson, Astron, Astrophys, 119, 1(1983).

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