Abstract. The Harary matrix of a graph $G$ is defined as $RD(G) = [r_{ij}]$ in which $r_{ij} = \frac{1}{d_{ij}}$ if $i \neq j$ and $r_{ii} = 0$ if $i = j$, where $d_{ij}$ is the distance between the vertices $v_i$ and $v_j$ in $G$. The Harary energy of $G$ is defined as the sum of the absolute values of the eigenvalues of Harary matrix. Two graphs are said to be Harary equienergetic if they have same Harary energy. In this paper we show that the Harary matrix of complement of the line graph of certain regular graphs has exactly one positive eigenvalue. Further we obtain the Harary energy of line graphs and of complement of line graphs of certain regular graphs and thus constructs pairs of Harary equienergetic graphs of same order and having different Harary eigenvalues.

1. Introduction

Harary matrix (also called as reciprocal distance matrix [13]) of a graph was introduced by Ivanciuc et al. [12], and which has influence in the study of molecules in QSPR (quantitative structure property relationship) models [12].

Let $G$ be a simple, undirected, connected graph with $n$ vertices and $m$ edges. Let the vertices of $G$ be labeled as $v_1, v_2, \ldots, v_n$. The adjacency matrix of a graph $G$ is the square matrix $A = A(G) = [a_{ij}]$, in which $a_{ij} = 1$ if $v_i$ is adjacent to $v_j$ and $a_{ij} = 0$, otherwise. The eigenvalues of $A(G)$ are the adjacency eigenvalues of $G$, and they are labeled as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. These form the adjacency spectrum of $G$ [5].

The distance between the vertices $v_i$ and $v_j$, denoted by $d_{ij}$, is the length of the shortest path joining $v_i$ and $v_j$. The diameter of a graph $G$, denoted by $diam(G)$, is the maximum distance between any pair of vertices of $G$. A graph $G$ is said to be $r$-regular graph if all of its vertices have same degree equal to $r$.

The Harary matrix [12] of a graph $G$ is a square matrix $RD(G) = [r_{ij}]$ of order $n$, where
The eigenvalues of $RD(G)$ labeled as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are said to be the Harary eigenvalues or $H$-eigenvalues of $G$ and their collection is called Harary spectrum or $H$-spectrum of $G$. Two non-isomorphic graphs are said to be $H$-cospectral if they have same $H$-spectra. The results on $H$-eigenvalues of a graph are obtained in [4, 6, 7, 11, 23].

The Harary energy or $H$-energy of a graph $G$, denoted by $HE(G)$, is defined as [8]

$$HE(G) = \sum_{i=1}^{n} |\mu_i|.$$ (1.1)

The Harary energy is defined in full analogy with the ordinary graph energy $E(G)$, defined as [9]

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$ (1.2)

The details about ordinary graph energy can be found in [14]. Bounds for the Harary energy of a graph are reported in [2, 4, 8].

Two graphs $G_1$ and $G_2$ are said to be equienergetic if $E(G_1) = E(G_2)$. Results on non-cospectral, equienergetic graphs can be found in [1, 3, 17, 18, 22].

Two connected graphs $G_1$ and $G_2$ are said to be Harary equienergetic or $H$-equienergetic if $HE(G_1) = HE(G_2)$. Trivially, the $H$-cospectral graphs are $H$-equienergetic. In this paper we obtain the $H$-energy of line graphs and of complement of line graphs of certain regular graphs and thus construct $H$-equienergetic graphs having different $H$-spectra.

We need following results.

**Theorem 1.1.** [5] If $G$ is an $r$-regular graph, then its maximum adjacency eigenvalue is equal to $r$.

The line graph of $G$, denoted by $L(G)$ is the graph whose vertices corresponds to the edges of $G$ and two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in $G$ [10]. If $G$ is a regular graph of order $n$ and of degree $r$ then the line graph $L(G)$ is a regular graph of order $nr/2$ and of degree $2r - 2$. 

$$r_{ij} = \begin{cases} \frac{1}{d_{ij}}, & \text{if } i \neq j \\ 0, & \text{if } i = j. \end{cases}$$
Theorem 1.2. [15, 16] For a connected graph $G$, $\text{diam}(L(G)) \leq 2$ if and only if none of the three graphs $F_1$, $F_2$ and $F_3$ of Fig. 1 is an induced subgraph of $G$.

Theorem 1.3. [20] If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of a regular graph $G$ of order $n$ and of degree $r$, then the adjacency eigenvalues of $L(G)$ are

$$\lambda_i + r - 2, \quad i = 1, 2, \ldots, n,$$

and

$$-2, \quad n(r - 2)/2 \text{ times}.$$ (2.1)

Theorem 1.4. [19] Let $G$ be an $r$-regular graph of order $n$. If $r, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of $G$, then the adjacency eigenvalues of $\overline{G}$, the complement of $G$, are $n - r - 1$ and $-\lambda_i - 1, \ i = 2, 3, \ldots, n$.

Lemma 1.5. [21] If for any two adjacent vertices $u$ and $v$ of a graph $G$, there exists a third vertex $w$ which is not adjacent to either $u$ or $v$, then

(i) $\overline{G}$ is connected and

(ii) $\text{diam}(\overline{G}) = 2$.

2. $H$-Eigenvalues

Theorem 2.1. [4] Let $G$ be an $r$-regular graph of order $n$ and let $\text{diam}(G) \leq 2$. If $r, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of $G$, then its $H$-eigenvalues are

$$\frac{1}{2}(n + r - 1) \quad \text{and} \quad \frac{1}{2}(\lambda_i - 1), \ i = 2, 3, \ldots, n.$$ (2.1)

Theorem 2.2. Let $G$ be an $r$-regular graph of order $n$. Let $L(G)$ be the line graph of $G$ such that for any two adjacent vertices $u$ and $v$ of $L(G)$, there exists a third vertex $w$ in $L(G)$ which is not adjacent to either $u$ or $v$. Then $\overline{L(G)}$, the complement of $L(G)$, has exactly one positive $H$-eigenvalue, equal to $r(n - 2)/2$.

Proof. Let the adjacency eigenvalues of $G$ be $r, \lambda_2, \ldots, \lambda_n$. From Theorem 1.3, the adjacency eigenvalues of $L(G)$ are

$$2r - 2, \quad \text{and}$$

$$\lambda_i + r - 2, \quad i = 2, 3, \ldots, n,$$

and

$$-2, \quad n(r - 2)/2 \text{ times}. \quad (2.1)$$

From Theorem 1.4 and Eq. (2.1), the adjacency eigenvalues of $\overline{L(G)}$ are

$$(nr/2) - 2r + 1, \quad \text{and}$$

$$-\lambda_i - r + 1, \quad i = 2, 3, \ldots, n,$$

and

$$1, \quad n(r - 2)/2 \text{ times}. \quad (2.2)$$

Figure 1: The forbidden induced subgraphs
The graph $\overline{L(G)}$ is a regular graph of order $nr/2$ and of degree $(nr/2) - 2r + 1$. Since for any two adjacent vertices $u$ and $v$ of $L(G)$, there exists a third vertex $w$ which is not adjacent to either $u$ or $v$ in $L(G)$, by Lemma 1.5, $diam(\overline{L(G)}) = 2$. Therefore by Theorem 2.1 and Eq. (2.2), the $H$-eigenvalues of $L(G)$ are

\[
\begin{align*}
&\frac{(nr - 2r)}{2}, \quad \text{and} \\
&-\frac{(\lambda_i + r)}{2}, \quad i = 2, 3, \ldots, n, \quad \text{and} \\
&0, \quad n(r - 2)/2 \text{ times.}
\end{align*}
\]

(2.3)

All adjacency eigenvalues of a regular graph of degree $r$ satisfy the condition $-r \leq \lambda_i \leq r$ [5]. Therefore $\lambda_i + r \geq 0$, $i = 1, 2, \ldots, n$. The theorem follows from Eq. (2.3).

3. $H$-Energy

**Theorem 3.1.** Let $G$ be an $r$-regular graph of order $n$. Let $L(G)$ be the line graph of $G$ such that for any two adjacent vertices $u$ and $v$ of $L(G)$, there exists a third vertex $w$ in $L(G)$ which is not adjacent to either $u$ or $v$. Then

$$HE(\overline{L(G)}) = r(n - 2).$$

**Proof.** Bearing in mind Theorem 2.2 and Eq. (2.3), the $H$-energy of $\overline{L(G)}$ is computed as:

$$HE(\overline{L(G)}) = \frac{nr - 2r}{2} + \sum_{i=2}^{n} \frac{(\lambda_i + r)}{2} + |0| \times \frac{n(r - 2)}{2}$$

$$= r(n - 2) \quad \text{since} \quad \sum_{i=2}^{n} \lambda_i = -r.$$
The graph $G$ is regular of degree $r$ and has order $n$. Therefore $L(G)$ is a regular graph on $nr/2$ vertices and of degree $2r-2$. As none of the three graphs $F_1$, $F_2$ and $F_3$ of Fig. 1 is an induced subgraph of $G$, from Theorem 1.2, $\text{diam}(L(G)) \leq 2$. Therefore from Theorem 2.1 and Eq. (3.1), the $H$-eigenvalues of $L(G)$ are

$$
\begin{align*}
\frac{nr + 4r - 6}{4} & \quad \text{and} \\
\frac{\lambda_i + r - 3}{2}, & \quad i = 2, 3, \ldots, n \quad \text{and} \\
-3/2, & \quad n(r-2)/2 \ \text{times}.
\end{align*}
$$

(3.2)

Therefore

$$
HE(L(G)) = \left| \frac{nr + 4r - 6}{4} \right| + \sum_{i=2}^{n} \left| \frac{\lambda_i + r - 3}{2} \right| + \left| -\frac{3}{2} \frac{n(r-2)}{4} \right|.
$$

(3.3)

(i) By assumption, $\lambda_i + r - 3 \geq 0$, $i = 2, 3, \ldots, n$, then from Eq. (3.3)

$$
HE(L(G)) = \frac{nr + 4r - 6}{4} + \sum_{i=2}^{n} \left( \frac{\lambda_i + r - 3}{2} \right) + \frac{3n(r-2)}{4} \\
= \frac{nr + 4r - 6}{4} + \frac{1}{2} \sum_{i=2}^{n} \lambda_i + (n-1) \left( \frac{r-3}{2} \right) + \frac{3n(r-2)}{4} \\
= \frac{3n(r-2)}{2} \quad \text{since} \quad \sum_{i=2}^{n} \lambda_i = -r.
$$

(ii) By assumption, $\lambda_i + r - 3 < 0$, $i = 2, 3, \ldots, n$, then from Eq. (3.3)

$$
HE(L(G)) = \frac{nr + 4r - 6}{4} - \sum_{i=2}^{n} \left( \frac{\lambda_i + r - 3}{2} \right) + \frac{3n(r-2)}{4} \\
= \frac{nr + 4r - 6}{4} - \frac{1}{2} \sum_{i=2}^{n} \lambda_i - (n-1) \left( \frac{r-3}{2} \right) + \frac{3n(r-2)}{4} \\
= \frac{nr + 4r - 6}{2} \quad \text{since} \quad \sum_{i=2}^{n} \lambda_i = -r.
$$

Corollary 3.3. Let $G$ be a connected cubic graph with $n$ vertices and let none of the three graphs $F_1$, $F_2$ and $F_3$ of Fig. 1 is an induced subgraph of $G$. Then

$$
HE(L(G)) = \frac{3n + E(G)}{2}.
$$

Proof. Substituting $r = 3$ in Eq. (3.3) we get
\[ HE(L(G)) = \left| \frac{3n + 6}{4} \right| + \sum_{i=2}^{n} \left| \frac{\lambda_i}{2} \right| + \left| -\frac{3}{2} \right| \frac{n}{2} \]
\[ = \frac{3n + 6}{4} + \frac{1}{2}(E(G) - 3) + \frac{3n}{4} \]
\[ = \frac{3n + E(G)}{2}. \]

4. H-EQUIENERGETIC GRAPHS

**Lemma 4.1.** Let \( G_1 \) and \( G_2 \) be regular graphs of the same order and of the same degree. Then following holds:

(i) \( L(G_1) \) and \( L(G_2) \) are of the same order, same degree and have the same number of edges.

(ii) \( \overline{L(G_1)} \) and \( \overline{L(G_2)} \) are of the same order, same degree and have the same number of edges.

*Proof.* Statement (i) follows from the fact that the line graph of a regular graph is regular and that the number of edges of \( G \) is equal to the number of vertices of \( L(G) \). Statement (ii) follows from the fact that the complement of a regular graph is regular and that the number of vertices of a graph and its complement is equal. \( \square \)

**Lemma 4.2.** Let \( G_1 \) and \( G_2 \) be regular graphs of the same order and of the same degree. Let for \( i = 1, 2 \), \( L(G_i) \) be the line graph of \( G_i \) such that for any two adjacent vertices \( u_i \) and \( v_i \) of \( L(G_i) \), there exists a third vertex \( w_i \) in \( L(G_i) \) which is not adjacent to either \( u_i \) or \( v_i \). Then \( \overline{L(G_1)} \) and \( \overline{L(G_2)} \) are H-cospectral if and only if \( G_1 \) and \( G_2 \) are cospectral.

*Proof.* Follows from Eqs. (2.1), (2.2) and (2.3). \( \square \)

**Lemma 4.3.** Let \( G_1 \) and \( G_2 \) be connected, regular graphs of the same order and of the same degree. Let none of the three graphs \( F_1, F_2 \) and \( F_3 \) of Fig. 1 be an induced subgraph of \( G_i, i = 1, 2 \). Then \( L(G_1) \) and \( L(G_2) \) are H-cospectral if and only if \( G_1 \) and \( G_2 \) are cospectral.

*Proof.* Follows from Eqs. (3.1) and (3.2). \( \square \)

**Theorem 4.4.** Let \( G_1 \) and \( G_2 \) be regular, non H-cospectral graphs of the same order and of the same degree. Let for \( i = 1, 2 \), \( L(G_i) \) be the line graph of \( G_i \) such that for any two adjacent vertices \( u_i \) and \( v_i \) of \( L(G_i) \), there exists a third vertex \( w_i \) in \( L(G_i) \) which is not adjacent to either \( u_i \) or \( v_i \). Then \( \overline{L(G_1)} \) and \( \overline{L(G_2)} \) form a pair of non H-cospectral, H-equienenergetic graphs of equal order and of equal number of edges.

*Proof.* Follows from Lemma 4.1, Lemma 4.2 and Theorem 3.1. \( \square \)
**Theorem 4.5.** Let $G_1$ and $G_2$ be connected, regular, non $H$-cospectral graphs of the same order and of the same degree $r$. Let none of the three graphs $F_1$, $F_2$ and $F_3$ of Fig. 1 be an induced subgraph of $G_i$, $i = 1, 2$.

(i) If the smallest adjacency eigenvalue of $G_i$, $i = 1, 2$ is greater than or equal to $3 - r$, then line graphs $L(G_1)$ and $L(G_2)$ form a pair of non $H$-cospectral, $H$-equienergetic graphs of equal order and of equal number of edges.

(ii) If the second largest adjacency eigenvalue of $G_i$, $i = 1, 2$ is smaller than $3 - r$, then line graphs $L(G_1)$ and $L(G_2)$ form a pair of non $H$-cospectral, $H$-equienergetic graphs of equal order and of equal number of edges.

*Proof.* Follows from Lemma 4.1, Lemma 4.3 and Theorem 3.2. □

**Theorem 4.6.** Let $G_1$ and $G_2$ be cubic, connected, non $H$-cospectral, equienergetic graphs of the same order. Let none of the three graphs $F_1$, $F_2$ and $F_3$ of Fig. 1 be an induced subgraph of $G_i$, $i = 1, 2$. Then line graphs $L(G_1)$ and $L(G_2)$ form a pair of non $H$-cospectral, $H$-equienergetic graphs of equal order and of equal number of edges.

*Proof.* Follows from Lemma 4.1, Lemma 4.3 and Corollary 3.3. □

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**References**


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