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HARARY SPECTRA AND HARARY ENERGY OF LINE GRAPHS OF REGULAR GRAPHS

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Abstract. The Harary matrix of a graph G is defined as $RD(G) = [r_{ij}]$ in which $r_{ij} = \frac{1}{d_{ij}}$ if $i \neq j$ and $r_{ij} = 0$ if $i = j$, where d_{ij} is the distance between the vertices v_i and v_j in G. The Harary energy of G is defined as the sum of the absolute values of the eigenvalues of Harary matrix. Two graphs are said to be Harary equienergetic if they have same Harary energy. In this paper we show that the Harary matrix of complement of the line graph of certain regular graphs has exactly one positive eigenvalue. Further we obtain the Harary energy of line graphs and of complement of line graphs of certain regular graphs and thus constructs pairs of Harary equienergetic graphs of same order and having different Harary eigenvalues.

1. Introduction

Harary matrix (also called as reciprocal distance matrix [13]) of a graph was introduced by Ivanciuc et al. [12], and which has influence in the study of molecules in QSPR (quantitative structure property relationship) models [12].

Let G be a simple, undirected, connected graph with n vertices and m edges. Let the vertices of G be labeled as $v_1, v_2, \ldots, v_n$. The *adjacency matrix* of a graph G is the square matrix $A = A(G) = [a_{ij}]$, in which $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$, otherwise. The eigenvalues of $A(G)$ are the *adjacency eigenvalues* of G , and they are labeled as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. These form the *adjacency spectrum* of G [5].

The *distance* between the vertices v_i and v_j, denoted by d_{ij}, is the length of the shortest path joining v_i and v_j. The *diameter* of a graph G, denoted by $diam(G)$, is the maximum distance between any pair of vertices of G. A graph G is said to be *r-regular graph* if all of its vertices have same degree equal to r.

The *Harary matrix* [12] of a graph G is a square matrix $RD(G) = [r_{ij}]$ of order n, where

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$$r_{ij} = \begin{cases} \frac{1}{d_{ij}}, & \text{if} \quad i \neq j \\ 0, & \text{if} \quad i = j. \end{cases}$$

The eigenvalues of $RD(G)$ labeled as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are said to be the *Harary eigenvalues* or *H-eigenvalues* of G and their collection is called *Harary spectrum* or *H-spectrum* of G. Two non-isomorphic graphs are said to be *H-cospectral* if they have same H-spectra. The results on H-eigenvalues of a graph are obtained in [4, 6, 7, 11, 23].

The *Harary energy* or *H-energy* of a graph G, denoted by $HE(G)$, is defined as [8]

$$HE(G) = \sum_{i=1}^{n} |\mu_i| \ . \tag{1.1}$$

The Harary energy is defined in full analogy with the *ordinary graph energy* $E(G)$, defined as [9]

$$E(G) = \sum_{i=1}^{n} |\lambda_i| \ . \tag{1.2}$$

The details about ordinary graph energy can be found in [14]. Bounds for the Harary energy of a graph are reported in [2, 4, 8].

Two graphs G_1 and G_2 are said to be *equienergetic* if $E(G_1) = E(G_2)$. Results on non-cospectral, equienergetic graphs can be found in [1, 3, 17, 18, 22].

Two connected graphs G_1 and G_2 are said to be *Harary equienergetic* or *H-equienergetic* if $HE(G_1) = HE(G_2)$. Trivially, the H-cospectral graphs are H-equienergetic. In this paper we obtain the H-energy of line graphs and of complement of line graphs of certain regular graphs and thus construct H-equienergetic graphs having different H-spectra.

We need following results.

Theorem 1.1. [5] *If G is an r-regular graph, then its maximum adjacency eigenvalue is equal to r.*

The *line graph* of G, denoted by $L(G)$ is the graph whose vertices corresponds to the edges of G and two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G [10]. If G is a regular graph of order n and of degree r then the line graph $L(G)$ is a regular graph of order $nr/2$ and of degree $2r - 2$.

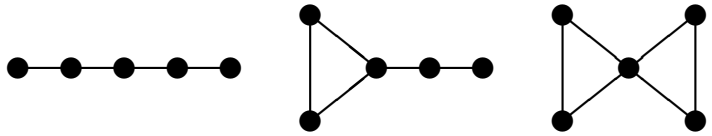


Figure 1: The forbidden induced subgraphs

Theorem 1.2. [15, 16] *For a connected graph G, $diam(L(G)) \leq 2$ if and only if none of the three graphs F_1, F_2 and F_3 of Fig. 1 is an induced subgraph of G.*

Theorem 1.3. [20] *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of a regular graph G of order n and of degree r, then the adjacency eigenvalues of $L(G)$ are*

$$\lambda_i + r - 2, \qquad i = 1, 2, \ldots, n, \qquad \text{and}$$

$$-2, \qquad n(r-2)/2 \text{ times}.$$

Theorem 1.4. [19] *Let G be an r-regular graph of order n. If $r, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of G, then the adjacency eigenvalues of $\overline{G}$, the complement of G, are $n - r - 1$ and $-\lambda_i - 1$, $i = 2, 3, \ldots, n$.*

Lemma 1.5. [21] *If for any two adjacent vertices u and v of a graph G, there exists a third vertex w which is not adjacent to either u or v, then*
(i) *$\overline{G}$ is connected and*
(ii) *$diam\left(\overline{G}\right) = 2$.*

2. H-EIGENVALUES

Theorem 2.1. [4] *Let G be an r-regular graph of order n and let $diam(G) \leq 2$. If $r, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of G, then its H-eigenvalues are $\frac{1}{2}(n + r - 1)$ and $\frac{1}{2}(\lambda_i - 1)$, $i = 2, 3, \ldots, n$.*

Theorem 2.2. *Let G be an r-regular graph of order n. Let $L(G)$ be the line graph of G such that for any two adjacent vertices u and v of $L(G)$, there exists a third vertex w in $L(G)$ which is not adjacent to either u or v. Then $\overline{L(G)}$, the complement of $L(G)$, has exactly one positive H-eigenvalue, equal to $r(n-2)/2$.*

Proof. Let the adjacency eigenvalues of G be $r, \lambda_2, \ldots, \lambda_n$. From Theorem 1.3, the adjacency eigenvalues of $L(G)$ are

$$\left.\begin{array}{rll} 2r-2, & \text{and} & \\ \lambda_i + r - 2, & i = 2, 3, \ldots, n, & \text{and} \\ -2, & n(r-2)/2 \text{ times}. & \end{array}\right\} \tag{2.1}$$

From Theorem 1.4 and Eq. (2.1), the adjacency eigenvalues of $\overline{L(G)}$ are

$$\left.\begin{array}{rll} (nr/2) - 2r + 1, & \text{and} & \\ -\lambda_i - r + 1, & i = 2, 3, \ldots, n, & \text{and} \\ 1, & n(r-2)/2 \text{ times}. & \end{array}\right\} \tag{2.2}$$

The graph $\overline{L(G)}$ is a regular graph of order $nr/2$ and of degree $(nr/2)-2r+1$. Since for any two adjacent vertices u and v of $L(G)$, there exists a third vertex w which is not adjacent to either u or v in $L(G)$, by Lemma 1.5, $diam\left(\overline{L(G)}\right)=2$. Therefore by Theorem 2.1 and Eq. (2.2), the H-eigenvalues of $\overline{L(G)}$ are

$$\left.\begin{array}{lll} (nr-2r)/2, & \text{and} & \\ -(\lambda_i+r)/2, & i=2,3,\ldots,n, & \text{and} \\ 0, & n(r-2)/2 \text{ times.} & \end{array}\right\} \tag{2.3}$$

All adjacency eigenvalues of a regular graph of degree r satisfy the condition $-r\leq\lambda_i\leq r$ [5]. Therefore $\lambda_i+r\geq 0$, $i=1,2,\ldots,n$. The theorem follows from Eq. (2.3). □

3. H-ENERGY

Theorem 3.1. *Let G be an r-regular graph of order n. Let $L(G)$ be the line graph of G such that for any two adjacent vertices u and v of $L(G)$, there exists a third vertex w in $L(G)$ which is not adjacent to either u or v. Then*

$$HE\left(\overline{L(G)}\right)=r(n-2).$$

Proof. Bearing in mind Theorem 2.2 and Eq. (2.3), the H-energy of $\overline{L(G)}$ is computed as:

$$\begin{aligned} HE\left(\overline{L(G)}\right) &= \frac{nr-2r}{2}+\sum_{i=2}^{n}\frac{(\lambda_i+r)}{2}+|0|\times\frac{n(r-2)}{2} \\ &= r(n-2) \qquad \text{since} \quad \sum_{i=2}^{n}\lambda_i=-r. \end{aligned}$$

□

Theorem 3.2. *Let G be a connected, r-regular graph with n vertices and let none of the three graphs F_1, F_2 and F_3 of Fig. 1 is an induced subgraph of G.*
(i) *If the smallest adjacency eigenvalue of G is greater than or equal to $3-r$, then*

$$HE(L(G))=\frac{3n(r-2)}{2}.$$

(ii) *If the second largest adjacency eigenvalue of G is smaller than $3-r$, then*

$$HE(L(G))=\frac{nr+4r-6}{2}.$$

Proof. Let $r,\lambda_2,\lambda_3,\ldots,\lambda_n$ be the adjacency eigenvalues of a regular graph G. Then from Theorem 1.3, the adjacency eigenvalues of $L(G)$ are

$$\left.\begin{array}{lll} 2r-2 & \text{and} & \\ \lambda_i+r-2, & i=2,3,\ldots,n & \text{and} \\ -2, & n(r-2)/2 \text{ times.} & \end{array}\right\} \tag{3.1}$$

The graph G is regular of degree r and has order n. Therefore $L(G)$ is a regular graph on $nr/2$ vertices and of degree $2r-2$. As none of the three graphs F_1, F_2 and F_3 of Fig. 1 is an induced subgraph of G, from Theorem 1.2, $diam(L(G)) \leq 2$. Therefore from Theorem 2.1 and Eq. (3.1), the H-eigenvalues of $L(G)$ are

$$\left.\begin{array}{lll} (nr+4r-6)/4 & \text{and} & \\ (\lambda_i + r - 3)/2, & i = 2, 3, \ldots, n & \text{and} \\ -3/2, & n(r-2)/2 \text{ times.} & \end{array}\right\} \tag{3.2}$$

Therefore

$$HE(L(G)) = \left|\frac{nr+4r-6}{4}\right| + \sum_{i=2}^{n}\left|\frac{\lambda_i + r - 3}{2}\right| + \left|-\frac{3}{2}\right|\frac{n(r-2)}{2}. \tag{3.3}$$

(i) By assumption, $\lambda_i + r - 3 \geq 0$, $i = 2, 3, \ldots n$, then from Eq. (3.3)

$$\begin{aligned} HE(L(G)) &= \frac{nr+4r-6}{4} + \sum_{i=2}^{n}\left(\frac{\lambda_i + r - 3}{2}\right) + \frac{3n(r-2)}{4} \\ &= \frac{nr+4r-6}{4} + \frac{1}{2}\sum_{i=2}^{n}\lambda_i + (n-1)\left(\frac{r-3}{2}\right) + \frac{3n(r-2)}{4} \\ &= \frac{3n(r-2)}{2} \qquad \text{since} \quad \sum_{i=2}^{n}\lambda_i = -r. \end{aligned}$$

(ii) By assumption, $\lambda_i + r - 3 < 0$, $i = 2, 3, \ldots n$, then from Eq. (3.3)

$$\begin{aligned} HE(L(G)) &= \frac{nr+4r-6}{4} - \sum_{i=2}^{n}\left(\frac{\lambda_i + r - 3}{2}\right) + \frac{3n(r-2)}{4} \\ &= \frac{nr+4r-6}{4} - \frac{1}{2}\sum_{i=2}^{n}\lambda_i - (n-1)\left(\frac{r-3}{2}\right) + \frac{3n(r-2)}{4} \\ &= \frac{nr+4r-6}{2} \qquad \text{since} \quad \sum_{i=2}^{n}\lambda_i = -r. \end{aligned}$$

□

Corollary 3.3. *Let G be a connected cubic graph with n vertices and let none of the three graphs F_1, F_2 and F_3 of Fig. 1 is an induced subgraph of G. Then*

$$HE(L(G)) = \frac{3n + E(G)}{2}.$$

Proof. Substituting $r = 3$ in Eq. (3.3) we get

$$\begin{aligned} HE(L(G)) &= \left|\frac{3n+6}{4}\right| + \sum_{i=2}^{n}\left|\frac{\lambda_i}{2}\right| + \left|-\frac{3}{2}\right|\frac{n}{2} \\ &= \frac{3n+6}{4} + \frac{1}{2}(E(G)-3) + \frac{3n}{4} \\ &= \frac{3n+E(G)}{2}. \end{aligned}$$

□

4. H-equienergetic graphs

Lemma 4.1. *Let G_1 and G_2 be regular graphs of the same order and of the same degree. Then following holds:*
(i) *$L(G_1)$ and $L(G_2)$ are of the same order, same degree and have the same number of edges.*
(ii) *$\overline{L(G_1)}$ and $\overline{L(G_2)}$ are of the same order, same degree and have the same number of edges.*

Proof. Statement (i) follows from the fact that the line graph of a regular graph is regular and that the number of edges of G is equal to the number of vertices of $L(G)$. Statement (ii) follows from the fact that the complement of a regular graph is regular and that the number of vertices of a graph and its complement is equal. □

Lemma 4.2. *Let G_1 and G_2 be regular graphs of the same order and of the same degree. Let for $i = 1, 2$, $L(G_i)$ be the line graph of G_i such that for any two adjacent vertices u_i and v_i of $L(G_i)$, there exists a third vertex w_i in $L(G_i)$ which is not adjacent to either u_i or v_i. Then $\overline{L(G_1)}$ and $\overline{L(G_2)}$ are H-cospectral if and only if G_1 and G_2 are cospectral.*

Proof. Follows from Eqs. (2.1), (2.2) and (2.3). □

Lemma 4.3. *Let G_1 and G_2 be connected, regular graphs of the same order and of the same degree. Let none of the three graphs F_1, F_2 and F_3 of Fig. 1 be an induced subgraph of G_i, $i = 1, 2$. Then $L(G_1)$ and $L(G_2)$ are H-cospectral if and only if G_1 and G_2 are cospectral.*

Proof. Follows from Eqs. (3.1) and (3.2). □

Theorem 4.4. *Let G_1 and G_2 be regular, non H-cospectral graphs of the same order and of the same degree. Let for $i = 1, 2$, $L(G_i)$ be the line graph of G_i such that for any two adjacent vertices u_i and v_i of $L(G_i)$, there exists a third vertex w_i in $L(G_i)$ which is not adjacent to either u_i or v_i. Then $\overline{L(G_1)}$ and $\overline{L(G_2)}$ form a pair of non H-cospectral, H-equienergetic graphs of equal order and of equal number of edges.*

Proof. Follows from Lemma 4.1, Lemma 4.2 and Theorem 3.1. □

Theorem 4.5. *Let G_1 and G_2 be connected, regular, non H-cospectral graphs of the same order and of the same degree r. Let none of the three graphs F_1, F_2 and F_3 of Fig. 1 be an induced subgraph of G_i, $i = 1, 2$.*
(i) *If the smallest adjacency eigenvalue of G_i, $i = 1, 2$ is greater than or equal to $3 - r$, then line graphs $L(G_1)$ and $L(G_2)$ form a pair of non H-cospectral, H-equienergetic graphs of equal order and of equal number of edges.*
(ii) *If the second largest adjacency eigenvalue of G_i, $i = 1, 2$ is smaller than $3 - r$, then line graphs $L(G_1)$ and $L(G_2)$ form a pair of non H-cospectral, H-equienergetic graphs of equal order and of equal number of edges.*

Proof. Follows from Lemma 4.1, Lemma 4.3 and Theorem 3.2. □

Theorem 4.6. *Let G_1 and G_2 be cubic, connected, non H-cospectral, equienergetic graphs of the same order. Let none of the three graphs F_1, F_2 and F_3 of Fig. 1 be an induced subgraph of G_i, $i = 1, 2$. Then line graphs $L(G_1)$ and $L(G_2)$ form a pair of non H-cospectral, H-equienergetic graphs of equal order and of equal number of edges.*

Proof. Follows from Lemma 4.1, Lemma 4.3 and Corollary 3.3. □

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