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CF-PERMUTATION MODULES

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Abstract. Let M be a free module of finite rank over a commutative ring with unity R. Let $R[X]$ be the polynomial ring with coefficients in R. For an R-endomorphism f of M and a polynomial $P(X)$ of $R[X]$ and under certain condition, we show that if the $R[X]$-module $M_{P(f)}$ defined via $P(f)$ is a CF-module, then the $R[X]$-module M_f defined via f is also a CF-module. An application to modules over group rings is also given. For a prime number p and a finite group G, we characterize CF-permutation G-modules.

1. Introduction

All rings considered in this paper are supposed to be with unit. Let R be a commutative ring. If f is an R-endomorphism of a finitely generated R-modul M, then f converts M to a module over the polynomials ring $R[X]$ by $X._f\, m = f(m)$ for all $m \in M$. Equiped with this new structure, M is denoted by M_f (see, for example, [3, 9, 10], for more details about these modules). A canonical form for an R-module M is a decomposition $M \simeq \overset{n}{\underset{i=1}{\oplus}} R/I_i$ where the I_i are ideals of R such that $I_1 \subseteq I_2 \subseteq ... \subseteq I_n \neq R$. If M has a canonical form, the ideals I_i are uniquely determined [5, Lemma 15.13]. In this case M is called a CF-module of type $(I_1, I_2, ..., I_n)$. This notion of CF-module appears in [16] for example, under the appellation "canonical form for a module" (see, for example, [5, 14, 16], for more details about these modules).

Let G be a finite group. If G acts linearly on a finite abelian group M, then M is a $\mathbb{Z}[G]$-module or simply called G-module. A G-module M is said to be a CF-permutation G-module, if there exist a prime number p, G-sets $X_1, X_2, ..., X_n$, and nonzero natural numbers $n_1, n_2, ..., n_n$ such that M is isomorphic to $\overset{n}{\underset{i=1}{\oplus}} \mathbb{Z}/p^{n_i}\mathbb{Z} < X_i >$ (as G-modules). Recall that for a commutative ring R a permutation module is a finitely generated R-free $R[G]$-module which has an R-basis permuted by G. Permutation modules had been extensively studied (see, for example, [2, 6, 7, 8, 11]).

This paper is organized in the following way. In section 2, we essentially show that for $P(X) \in R[X]$, if M_f is a CF-module of type $(I_1, I_2, ..., I_n)$ and $P(X) - X \in$

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I_1, then M_g is also a CF-module of type $(I_1, I_2, ..., I_n)$. We also give an application to modules over group rings. In section 3, for a prime number p, we show that if G is a finite p-group, then a summand of CF-permutation G-module is also a CF-permutation G-module. Always in the case where G is a finite p-group, we show that if M is a CF-permutation G-module of order p^r, where r is a nonzero natural number, and p does not divides r, then G acts trivially on a summand of M. Also, for G a finite p-group, we give a result that characterizes a particular class of CF-permutation G-modules. Finally, for any finite group G and in particular when G is a finite p-group, we give some characterisations of CF-permutation G-modules.

2. CF-modules via polynomials of an endomorphism

Throughout this section R will be a commutative ring, and M will be a finitely generated R-module. Also, f will denote an R-endomorphism of M, $P(X)$ an element of $R[X]$ and $P(f)$ will be denoted by g (i.e., $g = P(f)$).

We need the following lemmas.

Lemma 2.1. *If $M_f = \overset{n}{\underset{i=1}{\oplus}} M_i$, where each M_i is an $R[X]$-submodule of M_f, then $M_g = \overset{n}{\underset{i=1}{\oplus}} M_i$, where each M_i is seen as an $R[X]$-submodule of M_g.*

Proof. Proof is trivial. □

Lemma 2.2. *Let I be an ideal of $R[X]$. Assume that M_f is isomorphic to $R[X]/I$ by an isomorphism φ. Then, φ is an isomorphism from M_g to $R[X]/I$ if and only if $P(X) - X \in I$.*

Proof. Assume that φ is an isomorphism from M_g to $R[X]/I$. Let $x_0 \in M_g$ such that $\varphi(x_0) = 1 + I$. We have

$$X + I = X(1 + I) = X\varphi(x_0) = \varphi(X._g\, x_0).$$

We put $H(X) = X$. Then, we have

$$X + I = \varphi(H(X)._g\, x_0) = \varphi(H(g)(x_0)).$$

As $H(X) = X$, then $H(g) = g$. So, $X + I = \varphi(g(x_0)) = \varphi(P(f)(x_0)) = \varphi(P(X)._f\, x_0) = P(X)\varphi(x_0) = P(X)(1+I) = P(X)+I$. Therefore, $P(X)-X \in I$. Reciprocally, assume that $P(X) - X \in I$. Let $L(X) \in R[X]$ and $m \in M_g$. We have $\varphi(L(X)._g\, m) = \varphi(L(g)(m)) = \varphi((LoP)(X)._f\, m) = (LoP)(X)\varphi(m) = (LoP(X) + I)\varphi(m)$. There exists $a_0, a_1, ..., a_n \in R$ such that $L(X) = a_0 + a_1X +$

$... + a_nX^n$. We have $LoP(X) = a_0 + a_1P(X) + ... + a_nP(X)^n$. So,

$$\begin{aligned} LoP(X) - L(X) &= a_1(P(X) - X) + a_2(P(X)^2 - X^2) + ... + \\ &\quad a_n(P(X)^n - X^n) \\ &= a_1(P(X) - X) + a_2(P(X) - X)(P(X) + X) + ... + \\ &\quad a_n(P(X) - X) \sum_{i=0}^{n-1} C_n^i P(X)^{n-1-i}X^i \\ &= (P(X) - X)(a_1 + a_2(P(X) + X) + ... + \\ &\quad a_n \sum_{i=0}^{n-1} C_n^i P(X)^{n-1-i}X^i). \end{aligned}$$

As $P(X)-X \in I$, then $LoP(X)-L(X) \in I$. Therefore, $LoP(X)+I = L(X)+I$. So,

$$\varphi(L(X).{}_g\, m) = (L(X) + I))\varphi(m) = L(X)\varphi(m).$$

More, φ is a bijection, since $M_g = M_f$ as sets. Therefore, φ is an isomorphism from M_g to $R[X]/I$. □

Now, we can prove the following interesting results.

Theorem 2.3. *If $M_f \cong \bigoplus_{i=1}^{n} R[X]/I_i$ where $I_1, I_2, ..., I_n$ are ideals of $R[X]$, and if $P(X) - X \in I_i$ for all $i \in \{1, 2, ..., n\}$, then we have $M_g \cong \bigoplus_{i=1}^{n} R[X]/I_i$.*

Proof. Let φ be an isomorphism from M_f to $\bigoplus_{i=1}^{n} R[X]/I_i$. Obviously, $M_f = \bigoplus_{i=1}^{n} M_{i,f}$, such that φ induces an isomorphism from $M_{i,f}$ to $R[X]/I_i$, for all $i \in \{1, 2, ..., n\}$. By Lemma 2.1, we have $M_g = \bigoplus_{i=1}^{n} M_{i,g}$, with $M_{i,g} = M_{i,f}$ as sets. As $P(X) - X \in I_i$ for all $i \in \{1, 2, ..., n\}$, then by Lemma 2.2, φ induces an isomorphism from $M_{i,g}$ to $R[X]/I_i$, and so φ is an isomorphism from M_g to $\bigoplus_{i=1}^{n} R[X]/I_i$. □

Corollary 2.4. *If M_f is a CF-module of type $(I_1, I_2, ..., I_n)$ and $P(X) - X \in I_1$, then M_g is also a CF-module of type $(I_1, I_2, ..., I_n)$.*

Proof. As $I_1 \subseteq I_i$ for all $i \in \{1, 2, ..., n\}$ and $P(X)-X \in I_1$, then $P(X)-X \in I_i$. Now, the proof follows trivially from Theorem 2.3. □

Let K be a commutative field of characteristic $p > 0$ and let $G = G_1 \times G_2$, where G_1 is a finite abelian p-group and G_2 is a finite cyclic p-group. Let $K[G]$ be the group ring of G over K. (See, for example, [12, 13, 17], for more information on these rings and their modules). We have $K[G] \cong R[G_2]$ where $R = K[G_1]$. Assume that G_2 is of order p^s generated by an element σ. We easily see that the homomorphism $\psi : R[X] \longrightarrow R[G_2]$ defined by $\psi(X) = \sigma$ induces an isomorphism of $R[X]/(X^{p^s}-1)$ to $R[G_2]$. Every $K[G]$-module can be regarded as $R[X]$-module annihilated by $X^{p^s} - 1$.

Let M be a finitely generated $R[G_2]$-module. Let $P(X) \in R[X]$ and $\tau = P(\sigma)$. We define another $R[G_2]$-module structure on M by $\sigma.{}_\tau\, m = P(\sigma).m$ for all

$m \in M$. We denote M as $R[G_2]$-module via τ by M_τ. We will keep these notations for the rest of this section.

We need the following lemmas.

Lemma 2.5. *If $M = \overset{n}{\underset{i=1}{\oplus}} M_i$, where each M_i is an $R[G_2]$-submodule of M, then $M_\tau = \overset{n}{\underset{i=1}{\oplus}} M_i$, where each M_i is seen as an $R[G_2]$-submodule of M_τ.*

Proof. Proof is trivial. □

Lemma 2.6. *Let I be an ideal of $R[G_2]$. Assume that M is isomorphic to $R[G_2]/I$ by an isomorphism φ. Then, φ is an isomorphism from M_τ to $R[G_2]/I$ if and only if $P(\sigma) - \sigma \in I$.*

Proof. The application

$$\begin{array}{ccc} f : M & \longrightarrow & M \\ m & \longmapsto & X.m \end{array}$$

(where $X.m = \sigma.m$) is an endomorphism of the R-module M. Let $\widetilde{\psi}$ denote the isomorphism induced by ψ. Then,

$$R[G_2]/I \cong \frac{\frac{R[X]}{(X^{p^s}-1)}}{\widetilde{\psi}^{-1}(I)} = \frac{\frac{R[X]}{(X^{p^s}-1)}}{\frac{I'}{(X^{p^s}-1)}} \cong R[X]/I',$$

where $\frac{I'}{(X^{p^s}-1)} = \widetilde{\psi}^{-1}(I)$. Here the isomorphism from $R[G_2]/I$ to $R[X]/I'$ is defined by

$$\begin{array}{ccc} \theta : R[G_2]/I & \longrightarrow & R[X]/I' \\ \sigma + I & \longmapsto & X + I'. \end{array}$$

We have

- M is isomorphic to $R[G_2]/I$ by an isomorphism φ if and only if M_f is isomorphic to $R[X]/I'$ by the isomorphism $\theta o \varphi$;
- φ is an isomorphism from M_τ to $R[G_2]/I$ if and only if $\theta o \varphi$ is an isomorphism from $M_{P(f)}$ to $R[X]/I'$;
- $P(\sigma) - \sigma \in I \Leftrightarrow P(X) - X \in I'$.

Now, a simple application of Lemma 2.2 completes the proof. □

Now, we can prove the following interesting results.

Theorem 2.7. *If $M \cong \overset{n}{\underset{i=1}{\oplus}} R[G_2]/I_i$ where $I_1, I_2, ..., I_n$ are ideals of $R[G_2]$, and if $P(\sigma) - \sigma \in I_i$ for all $i \in \{1, 2, ..., n\}$, then we have $M_\tau \cong \overset{n}{\underset{i=1}{\oplus}} R[G_2]/I_i$.*

Proof. Let φ be an isomorphism from M to $\overset{n}{\underset{i=1}{\oplus}} R[G_2]/I_i$. Obviously, $M = \overset{n}{\underset{i=1}{\oplus}} M_i$, such that φ induces an isomorphism from M_i to $R[G_2]/I_i$, for all $i \in \{1, 2, ..., n\}$. Lemma 2.5, we have $M_\tau = \overset{n}{\underset{i=1}{\oplus}} M_{i,\tau}$, with $M_{i,\tau} = M_i$ as sets. As $P(\sigma) - \sigma \in I_i$ for all $i \in \{1, 2, ..., n\}$, then by Lemma 2.6, φ induces an isomorphism from $M_{i,\tau}$ to $R[G_2]/I_i$, and so φ is an isomorphism from M_τ to $\overset{n}{\underset{i=1}{\oplus}} R[G_2]/I_i$. □

Corollary 2.8. *If M is a finitely generated CF-module of type $(I_1, I_2, ..., I_n)$ and if $P(\sigma) - \sigma \in I_1$, then M_τ is also a CF-module of type $(I_1, I_2, ..., I_n)$.*

Proof. As $I_1 \subseteq I_i$ for all $i \in \{1, 2, ..., n\}$ and $P(\sigma) - \sigma \in I_1$, then $P(\sigma) - \sigma \in I_i$. Now, the proof follows trivially from Theorem 2.7. □

3. CF-permutation G-modules

For a finite group G, it is not difficult to see that a G-module M is a CF-permutation G-module if and only if there exist a prime number p, subgroups $H_1, H_2, ..., H_r$ of G, and nonzero natural numbers $s_1, s_2, ..., s_r$ such that M is isomorphic - as G-module - to $\bigoplus_{i=1}^{r} \mathbb{Z}/p^{s_i}\mathbb{Z} < G/H_i >$.

Let p be a prime number. In the remainder of this section, the prime field $\mathbb{Z}/p\mathbb{Z}$ will be denoted by $\mathbb{F}_p$, and the cardinal number of a finite set S, will be denoted by $|S|$.

Now, we give the following necessary lemma.

Lemma 3.1. *Let G be a finite p-group. If H is a subgroup of G and s is a nonzero natural number, then $\mathbb{Z}/p^s\mathbb{Z} < G/H >$ is an indecomposable G-module.*

Proof. Suppose that there exist a pair of nonzero G-modules M_1 and M_2 such that $M = \mathbb{Z}/p^s\mathbb{Z} < G/H >= M_1 \oplus M_2$. So, $M/pM \cong \mathbb{F}_p[G/H] \cong M_1/pM_1 \oplus M_2/pM_2$ as $\mathbb{F}_p[G]$-modules. We have $pM_1 \neq M_1$ and $pM_2 \neq M_2$. So, it suffices to check that $\mathbb{F}_p < G/H >$ is indecomposable, and this result is shown in the proof of [4, Lemma 0.3]. □

For a particular group G, a summand of a CF-permutation G-module is also a CF-permutation G-module, as shown in the following result.

Proposition 3.2. *Let G be a finite p-group. Then, a summand of a CF-permutation G-module of order a power of p is also a CF-permutation G-module.*

Proof. The proof of this proposition results from the krull-Schmidt's theorem and Lemma 3.1. □

We have the following particular result.

Proposition 3.3. *Let G be a finite p-group, and let M be a CF-permutation G-module of order p^r, where r is a nonzero natural number. If p does not divides r, then G acts trivially on a summand of M.*

Proof. If M is a CF-permutation G-module, then there exist $H_1, H_2, ..., H_n$ subgroup of G and nonzero natural numbers $s_1, s_2, ..., s_n$ such that M is isomorphic to $\bigoplus_{i=1}^{n} \mathbb{Z}/p^{s_i}\mathbb{Z} < G/H_i >$. Hence, $|M| = p^r = \prod_{i=1}^{n} = p^{\sum_{i=1}^{n} s_i|G/H_i|}$. As p does not divides $r = \sum_{i=1}^{n} s_i|G/H_i|$, then there exists $1 \leq i \leq n$ such that $|G/H_i| = 1$. That is $\mathbb{Z}/p^{s_i}\mathbb{Z} < G/H_i >= \mathbb{Z}/p^{s_i}\mathbb{Z}$. □

Let G be a finite group. We note by $\tilde{G}$ the element of $\mathbb{Z}[G]$ defined by $\tilde{G} = \sum_{\sigma\in G} \sigma$, and for a G-module M we put $M^G = \{x \in M \mid \sigma x = x,\ \forall \sigma \in G\}$ and ${}_{\tilde{G}}M = \{x \in M \mid \tilde{G}x = \sum_{\sigma\in G} \sigma x = 0\}$. Finally, we note by I_G the augmentation ideal of $\mathbb{Z}[G]$.
The cohomology groups of a G-module M are:

$$\begin{aligned} H^n(G,M) &= Ext^n_{\mathbb{Z}[G]}(Z,M), \quad n \geq 1 \\ H^0(G,M) &= M^G/\tilde{G}M \\ H^{-1}(G,M) &= {}_{\tilde{G}}M/I_G M \\ H^{-n}(G,M) &= Tor^{\mathbb{Z}[G]}_{n-1}(G,M), \quad n \geq 2. \end{aligned}$$

Thus the cohomology groups $H^n(G,M)$ are just the Tate cohomology groups of G with coefficients in M. A G-module M is said to be cohomologically trivial if $H^n(H,M) = 0$ for all n and all subgroups H of G.

Following Serre's terminology [15], recall that a G-module M is said to be induced if it is of the form $M \otimes_{\mathbb{Z}} X$, where X is an abelian group. A G-module is said to be relatively projective if it is a direct summand of an induced G-module. Note that a relatively projective G-module is cohomologically trivial (see [15, p. 148]).

We need the following theorem.

Theorem 3.4. *Let G be a finite p-group, and let M be a G-module satisfying $pM = 0$. The following conditions are equivalent:*

1. *There exists an integer q such that $H^q(G,M) = 0$.*
2. *M is cohomologically trivial.*
3. *M is an induced G-module.*
4. *M is a free $\mathbb{F}_p[G]$-module.*

Proof. See [15, Theorem 5, p. 150]. □

For a particular group G, the following proposition characterizes the CF-permutation G-modules that are cohomologically trivial.

Proposition 3.5. *Let G be a finite p-group, and let $M = \bigoplus_{i=1}^{n} \mathbb{Z}/p^{s_i}\mathbb{Z} < G/H_i >$. Then, M is cohomologically trivial if and only if $H_i = 0$, for every $1 \leq i \leq r$.*

Proof. If $H_i = 0$ for every $1 \leq i \leq r$, then M is a direct sum of relatively projective G-modules. So, it is cohomologically trivial. Conversely, if there exists $1 \leq i \leq r$ such that $H_i \neq 0$, then H_i acts trivially on $\mathbb{Z}/p^{s_i}\mathbb{Z} < G/H_i >$, and so $H^0(H_i, \mathbb{Z}/p^{s_i}\mathbb{Z} < G/H_i >) = \mathbb{Z}/p^{s_i}\mathbb{Z} < G/H_i > /|H_i|\mathbb{Z}/p^{s_i}\mathbb{Z} < G/H_i > \neq 0$. So, M is not cohomologically trivial. □

If M is a finite G-module of order a power of p, then we put

$$\beta_G(M) = |G| dim_{\mathbb{F}_p}(M^G/pM^G) - dim_{\mathbb{F}_p}(M/pM)$$

and

$$\gamma_G(M) = |G| dim_{\mathbb{F}_p}(\tilde{G}M/p\tilde{G}M) - dim_{\mathbb{F}_p}(M/pM).$$

Here we have defined two invariants similar to those defined in [1].

Lemma 3.6. *If M and N are two finite G-modules of order a power of p, then $\beta_G(M \oplus N) = \beta_G(M) + \beta_G(N)$.*

Proof. Proof is trivial. □

Lemma 3.7. *Let s be a nonzero natural number. Then, $\beta_G(\mathbb{Z}/p^s\mathbb{Z}[G]) = 0$.*

Proof. Let $x = \sum_{\sigma \in G} a_\sigma \sigma \in \mathbb{Z}/p^s\mathbb{Z}[G]$. Then, $\tau.x = x$ for all $\tau \in G$ if and only if $a_\sigma = a_1$ for all $\sigma \in G$. So, $(\mathbb{Z}/p^s\mathbb{Z}[G])^G = (\mathbb{Z}/p^s\mathbb{Z})\widetilde{G}$. Therefore, $dim_{\mathbb{F}_p}((\mathbb{Z}/p^s\mathbb{Z}[G])^G/p(\mathbb{Z}/p^s\mathbb{Z}[G])^G) = 1$. Now, it is easy to see that $\beta_G(\mathbb{Z}/p^s\mathbb{Z}[G]) = 0$. □

Always for a particular group G, we have the following interesting results.

Theorem 3.8. *Let G be a finite p-group. Let M be a finite G-module of order a power of p and suppose that $(pM)^G = pM^G$. Then, the following conditions:*

(1) *M is relatively projective,*
(2) *M and pM are cohomologically trivial,*
(3) *$(\beta_G(M), \gamma_G(M)) = (0, 0)$,*

are such that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). If $pM = 0$, then the converses are true.

Proof. (1) $\Rightarrow$ (2): The G-module pM is also relatively projective. So, M and pM are cohomologically trivial.
(2) $\Rightarrow$ (3): Consider the short exact sequence $O \to pM \xrightarrow{i} M \xrightarrow{\pi} M/pM \to O$, where i and π are respectively the canonical injection and projection. So, we have an exact sequence $0 \longrightarrow Hom_{\mathbb{Z}[G]}(\mathbb{Z}, pM) \xrightarrow{F_i} Hom_{\mathbb{Z}[G]}(\mathbb{Z}, M) \longrightarrow Hom_{\mathbb{Z}[G]}(\mathbb{Z}, M/pM) \longrightarrow Ext^1_{\mathbb{Z}[G]}(A, pM)$, where F_i is defined by $f \longmapsto iof$. For each $\mathbb{Z}[G]$-module N, there exists an isomorphism $\phi : Hom_{\mathbb{Z}[G]}(\mathbb{Z}, N) \to N^G$ defined by $f \longmapsto f(1)$, and ϕ^{-1} is defined by $x \longmapsto f$ with $f(1) = x$. We have also $Ext^1_{\mathbb{Z}[G]}(\mathbb{Z}, pM) = H^1(G, pM) = 0$. Then, we have an exact sequence

$$0 \longrightarrow (pM)^G \xrightarrow{i} M^G \longrightarrow (M/pM)^G \longrightarrow 0,$$

where i is the canonical injection. Therefore, $(M/pM)^G$ is isomorphic to $M^G/(pM)^G = M^G/pM^G$. Then, it is not difficult to see that $\beta_G(M) = \beta_G(M/pM)$. From the exact sequence $O \to pM \xrightarrow{i} M \xrightarrow{\pi} M/pM \to O$ we have an exact sequence $Ext^1_{\mathbb{Z}[G]}(\mathbb{Z}, M) \to Ext^1_{\mathbb{Z}[G]}(\mathbb{Z}, M/pM) \to Ext^2_{\mathbb{Z}[G]}(\mathbb{Z}, pM)$. As $Ext^1_{\mathbb{Z}[G]}(\mathbb{Z}, M) = H^1(G, M) = 0$ and $Ext^2_{\mathbb{Z}[G]}(\mathbb{Z}, pM) = H^2(G, pM) = 0$, then $Ext^1_{\mathbb{Z}[G]}(\mathbb{Z}, M/pM) = H^1(G, M/pM) = 0$. From Theorem 3.4, M/pM is free as $\mathbb{F}_p[G]$-module. Hence, from Lemmas 3.6 and 3.7, $0 = \beta_G(M/pM) = \beta_G(M)$. The module M is cohomologically trivial, so $H^0(G, M) \cong M^G/\tilde{G}M = 0$. Hence, $\tilde{G}M = M^G$. Therefore, $\gamma_G(M) = 0$.
If $pM = 0$, then (2) $\Rightarrow$ (1) is trivial from Theorem 3.4.
It is clair that $\tilde{G}M \subseteq M^G$. If $(\beta_G(M), \gamma_G(M)) = (0, 0)$, then $dim_{\mathbb{F}_p}\tilde{G}M = dim_{\mathbb{F}_p}M^G$. So, $\tilde{G}M = M^G$. Therefore, $H^0(G, M) = 0$. Hence, from Theorem 3.4, (3) $\Rightarrow$ (2). □

We have the following interesting result.

Theorem 3.9. *Let G be a finite p-group. If $M = \bigoplus_{i=1}^{r} \mathbb{Z}/p^{s_i}\mathbb{Z} < G/H_i >$ is a CF-permutation G-module, then the following conditions are equivalent:*

(1) *$H_i = 0$ for every $1 \leq i \leq r$.*
(2) *M is an induced G-module.*
(3) *M is a relatively projective G-module.*
(4) *M is a cohomologically trivial G-module.*
(5) *$\beta_G(M) = 0$.*
(6) *M/pM is a free $\mathbb{F}_p[G]$-module.*

Proof. (1) $\Rightarrow$ (2): It suffice to show that for a nonzero natural number s, $\mathbb{Z}/p^s\mathbb{Z}[G]$ is an induced G-module. For this, it suffice to see that $\mathbb{Z}/p^s\mathbb{Z}[G] \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}/p^s\mathbb{Z}$ and apply [15, Proposition 1, p. 149].
(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are trivial.
(4) $\Rightarrow$ (5): Suppose that M is cohomologically trivial. Then, by Proposition 3.5, $H_i = 0$ for all $1 \leq i \leq r$. Therefore, from Lemmas 3.6 and 3.7, $\beta_G(M) = 0$.
(5) $\Rightarrow$ (6): Suppose that $\beta_G(M) = 0$. Then, from Lemma 3.6, $\beta_G(\mathbb{Z}/p^{s_i}\mathbb{Z} < G/H_i >) = 0$ for all $1 \leq i \leq r$. As $\beta_G(\mathbb{Z}/p^{s_i}\mathbb{Z} < G/H_i >) = |G| dim_{\mathbb{F}_p}((\mathbb{Z}/p^{s_i}\mathbb{Z} < G/H_i >)^G/p(\mathbb{Z}/p^{s_i}\mathbb{Z} < G/H_i >)^G) - |G|/|H_i|$, then necessarily $H_i = 0$ for all $1 \leq i \leq r$. So, M/pM is a free $\mathbb{F}_p[G]$-module.
(6) $\Rightarrow$ (1): Suppose that M/pM is a free $\mathbb{F}_p[G]$-module. As $M/pM \cong \bigoplus_{i=1}^{r} \mathbb{F}_p < G/H_i >$, then by krull-Schmidt's theorem and Lemma 3.1, $H_i = 0$ for every $1 \leq i \leq r$. □

Remark 3.10. It is possible to have a CF-permutation G-module M such that $\gamma_G(M) = 0$, but M is not relatively projective. Indeed, take G such that $|G| = 2^2$, and $M = (\mathbb{Z}/2\mathbb{Z})^3 \times \mathbb{Z}/2^3\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})[G]$. We have $\tilde{G}M \cong \tilde{G}(\mathbb{Z}/2^3\mathbb{Z}) \times \tilde{G}(\mathbb{Z}/2\mathbb{Z}[G]) \cong 4Z/2^3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Hence, $\gamma_G(M) = 0$. We have also $M^G = (\mathbb{Z}/2\mathbb{Z})^3 \times \mathbb{Z}/2^3\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})\tilde{G}$. So, $H^0(G, M) = M^G/\tilde{G}M \neq 0$. Therefore, M is not relatively projective.

For a particular finite G-module we have the following proposition.

Proposition 3.11. *Let G be a finite group, and let M be a finite G-module such that $M \cong \bigoplus_{i=1}^{r} (\mathbb{Z}/p^{n_i}\mathbb{Z})^{m_i}$ (as abelian groups), where $m_1, m_2, ..., m_r$ and $n_1, n_2, ..., n_r$ are nonzero natural numbers whith $m_i < p$ for every $1 \leq i \leq r$ and $n_1 < n_2 < ... < n_r$. Then the following are equivalent:*

(1) *M is a CF-permutation G-module.*
(2) *G acts trivially on M.*

Proof. (1) $\Rightarrow$ (2): If M is a CF-permutation G-module, then there exist subgroups $H_1, H_2, ..., H_n$ of G and nonzero natural numbers $s_1, s_2, ..., s_n$ such that M is isomorphic to $\bigoplus_{i=1}^{n} \mathbb{Z}/p^{s_i}\mathbb{Z} < G/H_i >$ (as G-modules). Since every abelian p-group is isomorphic to $\bigoplus_{i=1}^{m} \mathbb{Z}/p^{v_i}\mathbb{Z}$, where m and $v_1, v_2, ..., v_m$ are nonzero natural numbers, and $(v_1, v_2, ..., v_m)$ is unique, then we can say that $H_i = G$ for every $1 \leq i \leq n$.
(2) $\Rightarrow$ (1): Trivial. □

A $\mathbb{Z}[G]$-lattice is by definition a $\mathbb{Z}[G]$-module which is free and finitely generated as a $\mathbb{Z}$-module.

Now, we give a characterization of the CF-permutation G-modules using $\mathbb{Z}[G]$-lattices.

Proposition 3.12. *Let M be a finite G-module such that $M \cong (\mathbb{Z}/p^s\mathbb{Z})^m$ (as abelian groups), where m and s are nonzero natural numbers. Then, M is a CF-permutation G-module if and only if there exists an exact sequence of G-modules $O \to N \overset{u}{\to} L \to M \to O$, where L is a permutation $\mathbb{Z}[G]$-lattice of rank m as ($\mathbb{Z}$-module) and u is defined by $x \mapsto p^s x$.*

Proof. If M is a CF-permutation G-module, then there exist subgroups $H_1, H_2, ..., H_n$ of G, such that $M \cong \overset{n}{\underset{i=1}{\oplus}} \mathbb{Z}/p^s\mathbb{Z} < G/H_i >$. Hence, $M \cong \overset{n}{\underset{i=1}{\oplus}} \mathbb{Z} < G/H_i > / \overset{n}{\underset{i=1}{\oplus}} p^s\mathbb{Z} < G/H_i >$. So, we have an exact sequence of G-modules $O \to \overset{n}{\underset{i=1}{\oplus}} \mathbb{Z} < G/H_i > \overset{u}{\to} \overset{n}{\underset{i=1}{\oplus}} \mathbb{Z} < G/H_i > \to M \to O$ where u is defined by $x \mapsto p^s x$. Conversely, let $(e_1, e_2, ..., e_m)$ be a $\mathbb{Z}$-basis of L permutated by G, and let $N' = u(N)$. Then, $p^s e_i \in N'$, for every $1 \leq i \leq m$. So $N'' := \overset{m}{\underset{i=1}{\oplus}} p^s e_i\mathbb{Z}$ is a submodule of N'. $L/N'' \cong L/N' \cong M$. Therefore, $N' = N''$ because $(L/N'')/(N'/N'') \cong L/N''$ (as finite abelian groups). So, $N' = p^s L$ and it is trivial to see that M ($\cong L/p^s L$) is a CF-permutation G-module. □

Proposition 3.12 gives in a trivial way the following corollary.

Corollary 3.13. *Let M be a finite G-module such that $M \cong \overset{n}{\underset{i=1}{\oplus}} (\mathbb{Z}/p^{s_i}\mathbb{Z})^{m_i}$ (as abelian groups), where $m_1, m_2, ..., m_n$ and $s_1, s_2, ..., s_n$ are nonzero natural numbers such that $s_1 < s_2 < ... < s_n$. Then, M is a CF-permutation G-module if and only if for each $1 \leq i \leq n$, there exist an exact sequence of G-modules $O \to N_i \overset{u_i}{\to} L_i \to (\mathbb{Z}/p^{s_i}\mathbb{Z})^{m_i} \to O$, where L_i is a permutation $\mathbb{Z}[G]$-lattice of rank m_i (as $\mathbb{Z}$-module) and u_i is defined by $x_i \mapsto p^s x_i$.*

Now, from Theorem 5.10 of [5] we have the following useful lemma.

Lemma 3.14. *Let R be a commutative ring (not necessarily local), and M a free R-module of finite rank. Any minimal generator system of M is a basis of this module.*

If M is a CF-permutation G-module, then it is not difficult to see that M/pM is also a CF-permutation G-module. But in general the converse is not true.

Example 3.15. Take $p = 2$, $G = \mathbb{Z}/2\mathbb{Z}$ and $M = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^2\mathbb{Z}$. Then M is G-module under the action:

$$\begin{cases} \bar{1}.(\bar{1}, \bar{0}) = (\bar{1}, \bar{2}) \\ \bar{1}.(\bar{0}, \bar{1}) = (\bar{0}, \bar{1}). \end{cases}$$

This action is not trivial, so M can not be a CF-permutation G-module.
The action of G on $M/2M$ is given by

$$\begin{cases} \bar{1}.\overline{(\bar{1},\bar{0})} = \overline{\bar{1}.(\bar{1},\bar{0})} = \overline{(\bar{1},\bar{2})} = \overline{(\bar{1},\bar{0})} \\ \bar{1}.\overline{(\bar{0},\bar{1})} = \overline{\bar{1}.(\bar{0},\bar{1})} = \overline{(\bar{0},\bar{1})}. \end{cases}$$

This action is trivial, so $M/2M$ is CF-permutation G-module.

However, in the following we can prove some interesting results.

Theorem 3.16. *Let G be a finite p-group, and let M be a G-module of order a power of p. The module M is a CF-permutation G-module if and only if it satisfies the three following conditions:*

(1) *The module M/pM is a CF-permutation G-module.*
(2) *If M_i is an indecomposable summand of M, then M_i/pM_i is indecomposable.*
(3) *For every indecomposable summand M_i of M, there exist nonzero natural numbers m_i and s_i such that $M_i \cong (\mathbb{Z}/p^{s_i}\mathbb{Z})^{m_i}$ (as abelian groups).*

Proof. If M is a CF-permutation G-module, then there exist subgroups $H_1, H_2, ..., H_r$ of G and nonzero natural numbers $s_1, s_2, ..., s_r$ such $M \cong \bigoplus_{i=1}^{r} \mathbb{Z}/p^{s_i}\mathbb{Z} < G/H_i >$. Hence, $M/pM \cong \bigoplus_{i=1}^{r} \mathbb{F}_p < G/H_i >$, and it is not difficult to see that M satisfies (2) and (3), since from Lemma 3.1, for each i, $\mathbb{Z}/p^{s_i}\mathbb{Z} < G/H_i >$ and $\mathbb{F}_p < G/H_i >$ are indecomposable.
Conversely, if M satisfies (1), (2) and (3), then there exist subgroups $H_1, H_2, ..., H_r$ of G such that $M/pM \cong \bigoplus_{i=1}^{r} \mathbb{F}_p < G/H_i >$. Let $M_1, M_2, ..., M_{r'}$ be indecomposable submodules of M such that $M = \bigoplus_{i=1}^{r} M_i$. We have $M/pM \cong \bigoplus_{i=1}^{r'} M_i/pM_i$. As for each i, M_i/pM_i is indecomposable from assumptions and $\mathbb{F}_p < G/H_i >$ is indecomposable from Lemma 3.1, then $r' = r$ and we may take $M_i/pM_i \cong \mathbb{F}_p < G/H_i >$ (as $\mathbb{Z}[G]$-modules). Therefore, we have also $M_i/pM_i \cong \mathbb{F}_p < G/H_i >$ (as $\mathbb{F}_p[G]$-modules). Hence, M_i/pM_i has an $\mathbb{F}_p$-basis X_i' permuted by G. Moreover, the action of G on X_i' is transitive. Let $x_i \in M_i$ such that $x_i + pM_i \in X_i'$. We set $X_i = \{\sigma.x_i \ / \ \sigma \in G\}$. Since the reduction modulo pM of X_i is X_i', then from the Nakayama's lemma and Lemma 3.14, it is a basis of M_i over the local ring $\mathbb{Z}/p^{s_i}\mathbb{Z}$. The set X_i is permuted by G. Hence, M_i is a CF-permutation G-module. Therefore, M is a CF-permutation G-module. □

If m and s are two nonzero natural numbers such that, then $p^m\mathbb{Z}/p^s\mathbb{Z}$ is an ideal of the local ring $\mathbb{Z}/p^s\mathbb{Z}$ contained in the maximal ideal $p\mathbb{Z}/p^s\mathbb{Z}$. Hence, it is not difficult to see that the proof of the precedent theorem can be modified to prove the following more general result.

Theorem 3.17. *Let G be a finite p-group, let M be a finite G-module such that $M \cong \bigoplus_{i=1}^{r} \mathbb{Z}/p^{n_i}\mathbb{Z}$ (as abelian groups), where $n_1, n_2, ..., n_r$ are nonzero natural numbers, and let e be a nonzero natural number such that $e \leq n_i$ for every $1 \leq i \leq r$. Then, M is a CF-permutation G-module if and only if it satisfies the three following conditions:*

(1) *The module M/p^eM is a CF-permutation G-module.*

(2) *If M_i is an indecomposable summand of M, then M_i/p^sM_i is indecomposable.*
(3) *For every indecomposable summand M_i of M, there exist two nonzero natural numbers m_i and s_i, such that $M_i \cong (\mathbb{Z}/p^{s_i}\mathbb{Z})^{m_i}$ (as abelian groups).*

For a finite group which is not necessarily a p-group we have the following results.

Theorem 3.18. *Let G be a finite group, and let M be a finite indecomposable G-module of order a power of p. The module M is a CF-permutation G-module if and only if it satisfies the two following conditions:*

(1) *The module M/pM is a CF-permutation G-module.*
(2) *There exist nonzero natural numbers m and s such that $M \cong (\mathbb{Z}/p^s\mathbb{Z})^m$ (as abelian groups).*

Proof. If M is a CF-permutation G-module, then there exist subgroups H of G and nonzero natural numbers s such $M \cong \mathbb{Z}/p^s\mathbb{Z} < G/H >$. Now, it is not difficult to see that M satisfies (1) and (2).
Conversely, if M satisfies (1) and (2), then there exist subgroups $H_1, H_2, ..., H_r$ of G such that $M/pM \cong \overset{r}{\underset{i=1}{\oplus}} \mathbb{F}_p < G/H_i >$. So, there exist subsets $X'_1, X'_2, ..., X'_r$ of M/pM such that $X' = \underset{1\leq i\leq r}{\cup} X'_i$ is an $\mathbb{F}_p$-basis permuted by G, where the action of G on X'_i is transitive for all $1 \leq i \leq r$. Let $x_i \in M$ such that $x_i + pM_i \in X'_i$. We set $X_i = \{\sigma.x_i \ / \ \sigma \in G\}$. Since the reduction modulo pM of X_i is X'_i, then from the Nakayama's lemma and Lemma 3.14, $X = \underset{1\leq i\leq r}{\cup} X_i$ is a basis of M over the local ring $\mathbb{Z}/p^s\mathbb{Z}$. The set X is permuted by G. Therefore, M is a CF-permutation G-module. □

It is not difficult to see that the proof of the precedent theorem can be modified to prove the following more general result.

Theorem 3.19. *Let G be a finite group, let M be a finite indecomposable G-module such that $M \cong \overset{r}{\underset{i=1}{\oplus}} \mathbb{Z}/p^{s_i}\mathbb{Z}$ (as abelian groups), where $s_1, s_2, ..., s_r$ are nonzero natural numbers, and let e be a nonzero natural number such that $e \leq s_i$ for every $1 \leq i \leq r$. Then, M is a CF-permutation G-module if and only if it satisfies the two following conditions:*

(1) *The module M/p^eM is a CF-permutation G-module.*
(2) *There exist two nonzero natural numbers m and s, such that $M \cong (\mathbb{Z}/p^s\mathbb{Z})^m$ (as abelian groups).*

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